

**Stochastic bifurcation in a driven laser system: Experiment and theory**

Lora Billings

*Department of Mathematical Sciences, Montclair State University, Montclair, New Jersey 07043, USA*

Ira B. Schwartz and David S. Morgan

*Naval Research Laboratory, Code 6792, Plasma Physics Division, Washington, D.C. 20375, USA*

Erik M. Boltt

*Department of Mathematics and Computer Science, and Department of Physics, Clarkson University, Potsdam, New York 13699, USA*

Riccardo Meucci

*Istituto Nazionale di Ottica Applicata, Largo E. Fermi 6, 50125 Firenze, Italy*

Enrico Allaria

*Department of Physics, University of Florence, Florence, Italy*

(Received 7 November 2003; published 31 August 2004)

We analyze the effects of stochastic perturbations in a physical example occurring as a higher-dimensional dynamical system. The physical model is that of a class-*B* laser, which is perturbed stochastically with finite noise. The effect of the noise perturbations on the dynamics is shown to change the qualitative nature of the dynamics experimentally from a stochastic periodic attractor to one of chaoslike behavior, or noise-induced chaos. To analyze the qualitative change, we apply the technique of the stochastic Frobenius-Perron operator [L. Billings *et al.*, Phys. Rev. Lett. **88**, 234101 (2002)] to a model of the experimental system. Our main result is the identification of a global mechanism to induce chaoslike behavior by adding stochastic perturbations in a realistic model system of an optics experiment. In quantifying the stochastic bifurcation, we have computed a transition matrix describing the probability of transport from one region of phase space to another, which approximates the stochastic Frobenius-Perron operator. This mechanism depends on both the standard deviation of the noise and the global topology of the system. Our result pinpoints regions of stochastic transport whereby topological deterministic dynamics subjected to sufficient noise results in noise-induced chaos in both theory and experiment.

DOI: 10.1103/PhysRevE.70.026220

PACS number(s): 82.40.Bj, 02.50.-r, 42.55.-f, 02.40.Vh

**I. INTRODUCTION**

Noise-induced escape, which appears as a form of bifurcation in dynamical systems, is now documented in many areas of science and engineering [1]. It arises in stochastic processes, which we consider to be a composition of deterministic and time-dependent noisy systems. Detecting chaos in noisy systems is still an issue of debate. Efforts have been made to carry over operational definitions of deterministic chaos to stochastic systems, such as proving the existence of a positive Lyapunov exponent [2] and exploring the interaction of noise and a global bifurcation based on underlying unstable structures, such as a chaotic saddle [3]. Many of the underlying deterministic systems in these examples have parameter regimes in which multiple attractors give rise to noise-induced escape from one attractor to another. Such systems may be analyzed globally using the Hamiltonian theory of large fluctuations, or considering escape from attracting potential wells along most probable exit paths [4] using the theory of quasipotentials [5,6] or a variational formulation of optimal escape paths [7]. It is well known that noise can excite unstable chaotic structures while destroying regular periodic dynamics, but most studies consider noise-induced chaos occurring near a bifurcation, such as a saddle-node

point or a crisis of chaotic attractors that leave a chaotic saddle present.

In driven deterministic systems, the existence of chaotic invariant sets, such as chaotic saddles, can be proven by examining the topology of intersecting manifolds [8]. As an example, we cite the Melnikov method [9]. Although it has been extended to stochastic systems [10], it is limited in application since it is a bifurcation result that is perturbed from a global homoclinic or heteroclinic connection in a conservative system. Therefore, in many cases, one must rely on algorithmic methods for the numerical computation of unstable objects and their manifolds [11–13], with the hope that one may extract transverse intersections. We also note that in contrast to the hypothesis that noise-induced chaos is caused by a chaotic saddle excitation, a recent result shows that only partially formed manifold intersections (in which no chaotic saddle exists) may also be found to have a positive Lyapunov exponent [14].

In this paper, we compare a bifurcation observed in a nonequilibrium stochastic class-*B* laser experiment to a corresponding model of the system. We include experimental results, as well as the theoretical explanation of the observations. In particular, experiments support the claim that adding larger stochastic perturbations to the system results in qualitatively different dynamics. Using the model, we pro-

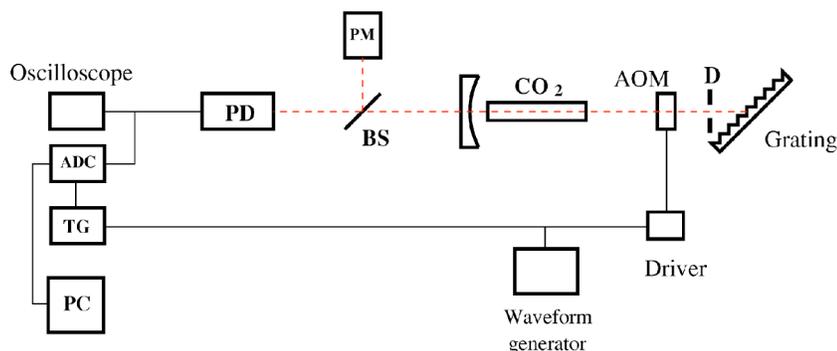


FIG. 1. Experimental apparatus used to perform our measurements. AOM, acousto-optic modulator; BS, beam splitter; PM, power meter; PD, photodetector; TG, trigger generator; ADC, analog-to-digital converter; and PC, personal computer.

vide evidence that this is an example of a bifurcation to noise-induced chaos by explicitly computing the probability transport due to noise. In this way, the interaction of noise and the underlying topology is identified in the emergent dynamics. We present analytic methods that specifically carry out the task of constructing the invariant density and transition probabilities in a rigorous manner to address the problem of this  $P$ -type stochastic bifurcation, as defined in [15]. Since it is a global approach, it is an alternative to using the Hamiltonian theory of large fluctuations, as described in [7,16] for autonomous systems, and in [17] for periodic systems. New tools were developed that are based on discrete approximations to the Frobenius-Perron operator with additive noise, defined as the stochastic Frobenius-Perron operator (SFPO) [18,19].

Using the SFPO, we identify the active regions of stochastic transport, or probability transitions, in the model. The advantage of this method is that we can find the probability density function (PDF) and maximal transport across boundaries in the absence of *a priori* knowledge of manifold structures and without time averaging. From the SFPO method, since one can directly compute the invariant density, spatially averaged Lyapunov spectra may be computed if the linear variation along an orbit is known. For stochastic systems that are sufficiently ergodic, spatial and temporal averages of the Lyapunov spectra are equal, and therefore, a positive Lyapunov exponent averaged spatially is a possible indicator of stochastic, or noise-induced, chaos [15].

To contrast our work from previous theories, we note that the bifurcation is far from parameters that would lead to a natural bifurcation to chaos, and large noise levels are included. Many studies in this field have relied on examining small noise limits, such as quasipotential theory [6] and optimal path theory [4], although this work has more recently been extended to the regime of finite noise intensity [20]. Underlying unstable fractal structures and noise-induced basin escape times have also been examined from quasipotential theory [6] for simple maps. The basin boundary in the system we study is a simple structure; i.e., it is nonfractal due to the lack of intersecting stable and unstable manifolds. In fact, only the forward crossings of a heteroclinic tangle could be identified, and no nonattracting chaotic sets are found to exist. The maximum Lyapunov exponent was calculated to increase smoothly through zero at the transition. Although both smooth and discontinuous onset are attributed to noise-induced chaos, the transition, which resembles a noise-induced attractor explosion described in [6], is not ob-

served. In fact, the transition is smooth statistically, evidenced by the smooth transition of a Lyapunov exponent through zero, which may be due to the noise-induced unstable dimension variability [21]. There is also a resemblance to noise-induced switching between multiple attractors as described in [3], but evidence provided by the probability density function supports the fact that trajectories spend as much time (if not more) near the partially formed heteroclinic tangle as the two periodic attractors. We also note that as previously reported in [6] and [3], both explosions and attractor switching are facilitated by fractal basin boundaries and nonattracting chaotic sets.

The layout of the paper is as follows. In Sec. II, we describe the experimental setup of a nonequilibrium stochastic class- $B$  laser. We illustrate the effects of noise on the dynamics of the intensity and show how the structure of the attractor changes. In Sec. III, we briefly review the laser model of the experiment in a reduced form and show that it captures many of the features of the experiment. Section IV illustrates the effect of noise on the laser model and specifically shows how the maximal (or top) Lyapunov exponent depends smoothly on the standard deviation as it transitions from stochastic periodic behavior to stochastic chaos. The global structure of the underlying topology and transport results are presented in Sec. V, and the discussion is presented in Sec. VI.

## II. AN ACOUSTICALLY OPTICAL MODULATED LASER EXPERIMENT WITH NOISE

To examine the effects of external noise in an experiment, we consider an acoustically optical modulated laser system. The experimental apparatus is shown in Fig. 1. It consists of a single-mode CO<sub>2</sub> laser with an intracavity acousto-optic modulator allowing modulation of the cavity losses. The optical cavity is 1.30 m long and the total transmission coefficient  $T$  is 0.10 for a single pass. The intensity decay rate  $k(t)$  can be expressed as follows:

$$k(t) = k(1 + \alpha \sin^2\{B_0[1 + f(t)]\}), \quad (1)$$

where  $k=cT/L$ ,  $c$  is the speed of light in a vacuum,  $L$  is the cavity length,  $\alpha=(1-2T)/2T$ ,  $B_0$  is a bias, and  $f(t)$  is the modulation signal,

$$f(t) = \beta \sin(2\pi\nu t) + \eta(t), \quad (2)$$

with  $\nu=100$  kHz and the modulation amplitude  $\beta$ . The random variable  $\eta$  is considered to be normally distributed with

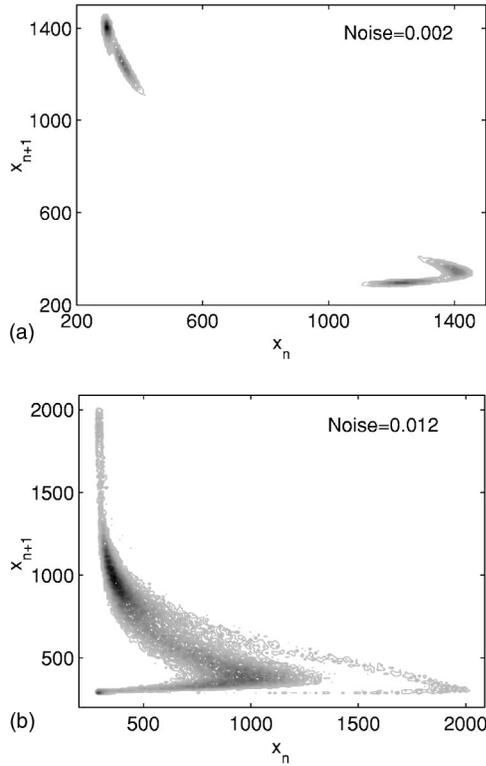


FIG. 2. Contour plots of the embedded intensity data (arbitrary units) under perturbations of  $\eta$ . Darker shades indicate regions visited with higher frequency. Small perturbations are used in the left graph, which result in a noisy periodic orbit. Larger perturbations are used in the right graph. Notice how the emergent dynamics are fundamentally different from the smaller case.

mean zero and standard deviation  $\sigma$ . The noisy signal  $f(t)$  is provided by an arbitrary waveform generator (Tektronix Mod. AWG420), which generates both the sinusoidal signal and the random variable  $\eta$  using an independent internal Gaussian noise generator. Specifically, the noise is added periodically with the period of drive; i.e.,

$$\eta(t) = \eta_n \delta\left(t - \frac{n}{\nu}\right), \quad n = 1, 2, \dots, \quad (3)$$

and  $\delta$  is the Dirac delta function, and  $\eta_n$  is now a discrete random variable.

It is known that by increasing the amplitude modulation, the system undergoes a sequence of subharmonic bifurcations leading to chaos when  $\eta=0$  [22]. However, when noise is added to the system through the driver, the resulting dynamics is highly dependent on the noise amplitude. In Fig. 2, we see two examples of the output of the intensity plotted as a contour map of the embedded data for two values of the noise strength at the same value of the modulation amplitude  $\beta=0.360$ . Note that darker shades indicate regions visited with higher frequency, and  $x_n$  is the local maximum of the measured intensity. The left panel shows the case where small noise results in a two-piece attractor. The deterministic attractor at this parameter value is periodic, located at the dark regions in the middle of the pieces. We describe this behavior as a noisy periodic attractor since its power spec-

trum is slightly broadened around the corresponding frequency. In the right panel, considerably more noise causes a qualitative change in the attractor. The periodic orbit is still located in the darkest regions of the graphs, but notice how there is significant sampling to other parts of the phase space not previously sampled at lower noise amplitudes. To understand the mechanism resulting in the stochastic bifurcation, we consider an accurate model of the experimental process.

### III. AN AOM LASER MODEL

In [23], a multifrequency phase control on a two-level, two-dimensional CO<sub>2</sub> laser model produced both experimental and numerical evidence that it was able to preserve periodic behavior within a chaotic window as well as to reexcite chaotic behavior when it is destroyed by a crisis. In the model used, only intensity and population inversion were considered. To retain fidelity between theory and experiment, a more realistic four-level model of a CO<sub>2</sub> laser, which incorporated intensity, two resonant population levels, and two coupled rotational manifolds, was used in [24]. Analysis showed that an approximate reduction to three state variables could be made by examining differences in the resonant and rotational population levels while still retaining many of the global features of the bifurcations. Therefore, we begin our study of the scaled three-dimensional model in a stochastic version, where noise is added to the intensity equation. The variables have already been scaled to be dimensionless [24]. The driven three-dimensional system has the advantages that (i) it is higher dimensional than other models, and (ii) when sampled discretely at the drive frequency, its phase space can be visualized in three dimensions. The model equations are given by

$$\begin{aligned} y_1' &= k_0(y_2 - 1 - \alpha \sin^2\{B[1 + f(t)]\}), \\ y_2' &= -\gamma_1 y_2 - 2k_0 e^{y_1} y_2 + y_3 + P, \\ y_3' &= -\gamma_2 y_3 + z y_2 + z P, \end{aligned} \quad (4)$$

and

$$f(t) = A \sin(\omega t) + \eta(t), \quad (5)$$

where  $\eta(t)$  is discretely modeled as in Eq. (3) with period  $2\pi/\omega$ ,  $y_1$  is the natural logarithm of the intensity,  $y_2$  is the main population difference, and  $y_3$  is the difference in rotational levels. The fixed parameters are  $k_0=32.97$ ,  $\alpha=4$ ,  $B=0.21$ ,  $\omega=0.897\ 597\ 9$ ,  $\gamma_1=10.0643$ ,  $P=0.082$ ,  $\gamma_2=1.0643$ ,  $z=10$ , and we vary  $A$ .

We now describe the topology of Eqs. (4) and (5) without stochastic perturbations, i.e.,  $\eta(t) \equiv 0$ . As shown in the bifurcation diagram in Fig. 3, periodic orbits are represented as a function of  $A$ . As  $A$  is increased, a period-one attractor proceeds through a period-doubling bifurcation. Several saddle-node bifurcations for varying periodic orbits also occur, which will play a role when noise is turned on. We show the first saddle node, which is of period three in the figure. Therefore at  $A=0.214$ , there exists an interval of bistability  $I_A$ , formed by period-four and period-three attractors. Asso-

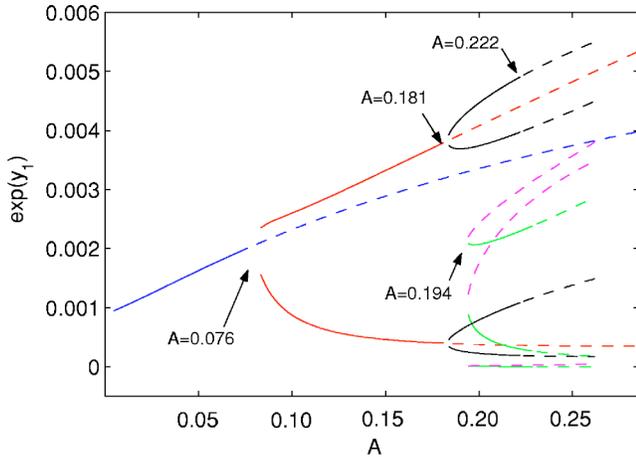


FIG. 3. (Color online) The bifurcation diagram for the laser model as a function of the forcing amplitude. Plotted are branches of both stable and unstable periodic orbits. The y axis is scaled intensity. The parameters are given in the text.

ciated with the period-four attractor is an unstable period-two saddle orbit (which is a flip saddle) and an unstable period-one flip saddle orbit from the period-doubling bifurcation. The period-three attractor has an associated unstable period-three regular saddle orbit arising from a saddle-node bifurcation. We hypothesize that the multi-instability in this system when  $A \in I_A$  has the topological structure needed to induce chaoslike behavior with additive stochastic perturbations. Since the bifurcation diagram contains only stability and amplitude information, we explore the phase space through numerical simulation. (Rigorous analysis of the onset of the saddle-node bifurcation, which leads to bistable regions, is similar to that done in [25] and will be presented elsewhere.)

#### IV. STOCHASTIC DYNAMIC SIMULATIONS

In keeping with the experimental setup, we model the stochastic system as a discrete dynamical system. Since the experimental system was forced periodically with discrete noise using Eq. (3), we can add the perturbations at the same period as that of the drive given by Eq. (5). Consider the periodic sampling as discrete time events of a deterministic system. We add the perturbations to initial conditions, similar to adding noise to a discrete map. In general, we consider stochastically perturbing a function  $F$  with additive noise:  $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $\mathbf{x} \mapsto F(\mathbf{x}) + \eta$ , where  $\eta$  is an identically independently distributed random variable with normal distribution and mean = 0 applied once each iteration. Since we are most interested in the situation where small noise amplitude can have major global consequences, we focus on the case where the random part  $\eta$  is assumed to be independent of state  $\mathbf{x}$  and relatively small, so that the deterministic part  $F$  has primary influence. We add the perturbation to each component independently and set the standard deviation  $\Sigma = \text{diag}(\sigma_1, \sigma_2, \sigma_3)$  as a parameter. This standard deviation is relative to the normalized scaling of the *almost*-compact space we consider. That is, each  $\sigma_n$  is scaled as if the phase

space is a unit box in three dimensions. With no noise, the only observable behavior is asymptotically periodic trajectories converging to the period-three or period-four orbits. By adding noise with increasing standard deviation, a random trajectory changes from a noisy periodic orbit to chaoslike deterministic behavior, visiting the two periodic orbit basins.

We remark that although in the original unscaled model, noise is added multiplicatively, it is approximately equivalent to adding noise additively in the scaled model from Eq. (4). This is due to the fact that the intensity of the original model is represented by the logarithm of the intensity in Eq. (4) [26]. The noise source appears as a term of the form  $\sin^2\{B[1 + A \sin(\omega t) + \eta(t)]\}$ . Taking a Taylor series expansion with respect to  $\eta$  yields a noise term on the order of  $\eta$ , which is independent of the state variables. Since the model is based on the natural logarithm of the intensity, a good approximation to the noise source is that the intensity equation has an additive noise term.

In quantifying underlying complex determinism in stochastic systems, it is inherently difficult to draw a clear line to distinguish between complex oscillations due to significant contributions from deterministic parts influenced by noise and a large noise amplitude effect wherein complex oscillations are primarily due to random Brownian diffusion. One necessary, but not sufficient, condition for the existence of chaos is the calculation of positive Lyapunov exponents. Lyapunov exponents measure the average rate of separation of neighboring initial points. Because we are adding perturbations to this system discretely, we can find a finite-time numerical approximation for the Lyapunov exponents of the map using the linear variational equations of the original system on the Poincaré section. A positive Lyapunov exponent can identify chaotic behavior, but diffusion can yield a positive Lyapunov exponent as well [27]. Since chaos is also associated with the underlying topology of the manifolds of the dynamical system, we examine the unstable structures in the deterministic model and observe how they interact with the stochastic source terms. Specifically, we would like to identify the structures in the original phase space that noise can excite. For example, if noise causes a trajectory to visit a chaotic saddle, then there should be locally unstable contributions to the Lyapunov spectrum. If enough of the unstable contributions are sampled, then the topology underlying the chaotic saddle will be reflected in an increasing maximum exponent.

In modeling the experiment, we consider additive stochastic perturbations to the first component, setting  $\sigma_2 = \sigma_3 = 0$ . The phase space projection of the attractor changes qualitatively as a function of the standard deviation, as we saw earlier. However, in Fig. 4, we show how the experiment and model both appear to change smoothly as the standard deviation increased. This is reflected in the time-averaged Lyapunov exponent computations. That is, as we increase  $\sigma_1$  away from zero, the Lyapunov exponent increases and has a smooth transition from negative to positive values, as shown in Fig. 5. The crossing is near  $\sigma_1 = 0.064$ . As an example, we graph two trajectories to show the emergent dynamics in the three-dimensional phase space in Fig. 6. Setting  $\sigma_1 = 0.04$ , the largest Lyapunov exponent is negative, predicting noisy periodic behavior, as seen near the period-four orbit. Setting

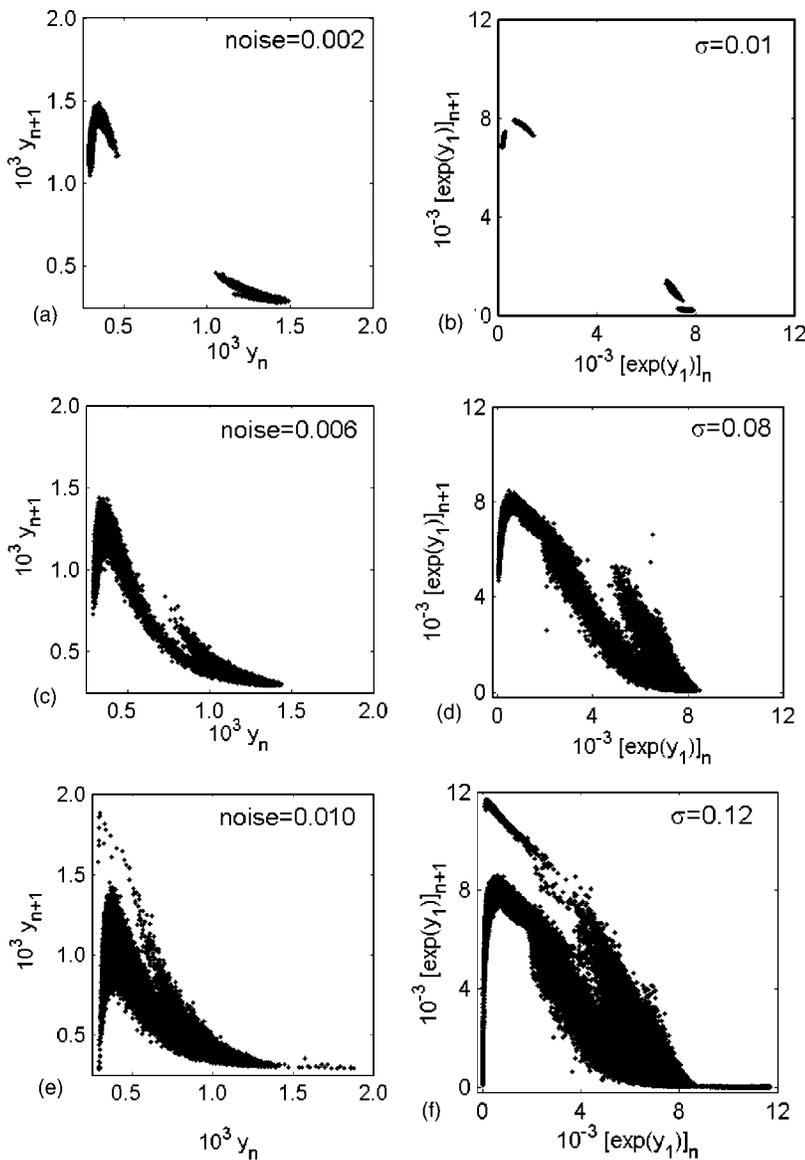


FIG. 4. The left graphs show intensity data (arbitrary units) for increasing perturbations of  $\eta$ . The right graphs show similar results for the model given by Eq. (4). Plotted are successive local maxima of the intensity values. Notice that in both cases, the attractors go from a stochastically perturbed period-four cycle, through a basin hopping attractor, and then to bursting among several basins of attraction from the deterministic case.

$\sigma_1=0.16$ , the largest Lyapunov exponent is positive, predicting chaoslike behavior.

More detail about the dynamics can be obtained by calculating the bursting statistics as a function of the standard deviation of the noise. We approximate the burst rate by

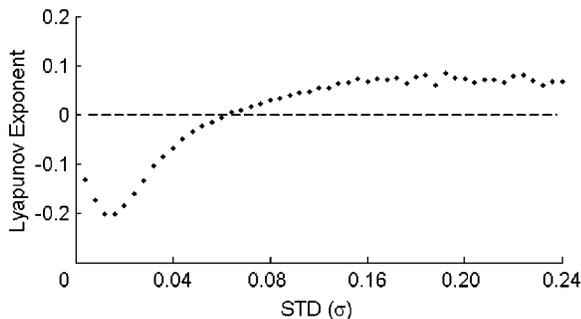


FIG. 5. The largest Lyapunov exponent as a function of the standard deviation of the noise. The transition from negative to positive values is smooth, as predicted in [21].

setting a threshold for  $e^{y_1}$  at 0.009. This value was determined by monitoring a trajectory with no noise. For each standard deviation value, we count the number of points in a random trajectory above the threshold and divide by the total number of points. See Fig. 7 for the results using trajectories 700 000 points long. Notice how bursting occurs for  $\sigma > 0.12$ . This value is different from the bifurcation value predicted by the Lyapunov exponent. Therefore, we will investigate the stochastic dynamical system as the noise parameter  $\sigma$  is varied.

Experimentally, we observe and show in Fig. 6 the changes that occur as  $\sigma$  varies. There exists a two-piece noisy period-four attractor for  $\sigma < 0.064$ . Then, the two pieces join into one attractor for  $0.064 < \sigma < 0.12$ , which is reflected by a positive Lyapunov exponent. Then, the trajectories start to burst and visit the period three. This statistic is not noticeable until  $\sigma=0.12$ . The amount of bursting is reflected in the burst rate. For  $0.064 < \sigma < 0.12$ , the noise provides the transport for the trajectories to visit the stable period-four orbit, the unstable period-two orbit, and the unstable period-one orbit. But for  $\sigma > 0.12$ , the trajectory visits

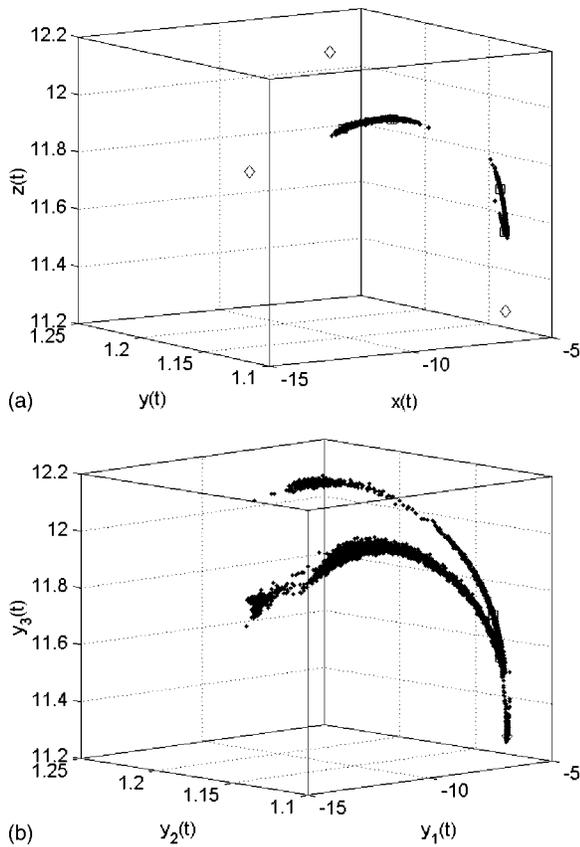


FIG. 6. The left graph shows noisy periodic behavior generated by the system when the standard deviation of the noise is  $\sigma_1 = 0.04$ . This is the behavior predicted by a negative Lyapunov exponent. The right graph shows chaoslike behavior generated by the system when  $\sigma_1 = 0.16$ . This is the behavior predicted by a positive Lyapunov exponent in Fig. 5.

both the period-three and period-four attractors, and the manifolds in between them. See the right graph in Fig. 6 as an example. We will now explore these dynamical changes by a transfer operator-based analysis, and we will compare the results to the topology of the stable and unstable manifolds of the corresponding deterministic system and interpret the influence of the added noise.

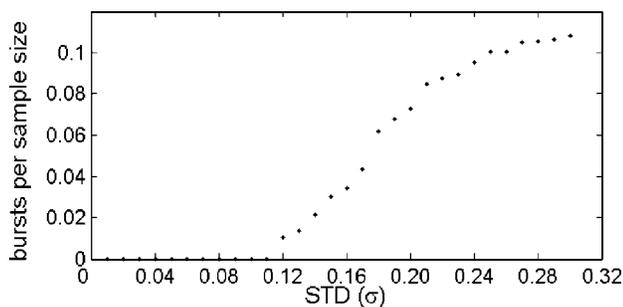


FIG. 7. The burst rate as a function of the standard deviation of the noise. Notice that the bursting increases from zero near  $\sigma = 0.12$ .

### V. PHASE-SPACE ANALYSIS OF STOCHASTIC DYNAMICS

Understanding the interaction between noise and the deterministic topology requires that we examine the structure of the stable and unstable manifolds of the relevant saddles in the prechaotic regime. Locating stable and unstable manifolds can be done in several ways [11,12]. We use the box algorithm from [13] and describe it here briefly. By picking a box containing the unstable saddle with part of its stable and unstable manifolds, we can determine initial conditions that will generate trajectories remaining in the box for a large number of iterations. We then eliminate any points converging to an attractor. The initial conditions remaining in the punctured box approximate the union of the stable manifolds, while the last point of the trajectory that remains in the box approximates the unstable manifolds. This algorithm was used to generate the stable and unstable manifolds in Fig. 8.

As shown in Fig. 8, the two-dimensional stable manifolds of the period-three saddle orbit form the basin boundary between the period-three basin and period-four basin. The one-dimensional unstable manifolds approach the period-four orbit, intersecting the two-dimensional stable manifolds of the period-two and period-one saddles. This forms a forward connection of a heteroclinic tangle in  $\mathbb{R}^3$ . There are no reverse connections or intersections of the stable manifolds of the period-three saddle orbit, which would be necessary for fully developed chaos.

By adding stochastic perturbations with a large enough standard deviation, random trajectories frequently escape their asymptotic limit toward one of the attracting periodic orbits and visit the other. In contrast to basin hopping, the trajectories spend a significant amount of time in between the two attractors, near the forward connections of the heteroclinic tangle. Essentially, short visits to the other basin act like a reverse connection, completing the tangle and enabling chaoslike behavior. Therefore, the trajectory follows the chaoslike dynamics in the time spent in between the two attractors. As the standard deviation of the noise is increased, this reverse jump occurs more frequently and more time is spent in between the attractors. These events can be identified by bursting, and the chaoticlike behavior is captured by both the Lyapunov exponent and burst-rate statistics.

What we wish to identify here is where the noise facilitates the reverse jump in phase space and provide evidence that the phenomenon is similar to a heteroclinic tangle. We begin by analyzing the time-series data. In Fig. 8, the point before-the trajectory switches basins (defined in the noise-free case) is recorded. It is clear that the jumps occur frequently in three regions near the unstable period-three saddles. To quantify these regions, we calculate the Galerkin transport matrix.

The Galerkin transport matrix can be used as a tool to identify transport between the original basins as a function of the standard deviation of the noise added to the system [18]. Let  $\nu(\mathbf{x})$  be the distribution of the random variable  $\eta$ ,

$$\nu(\mathbf{x}) = e^{-(\mathbf{x}^T \Sigma^{-1} \mathbf{x})/2} / \sqrt{(2\pi)^3 \det(\Sigma)}. \tag{6}$$

As a spatial approximation, we use the SFPO in the form

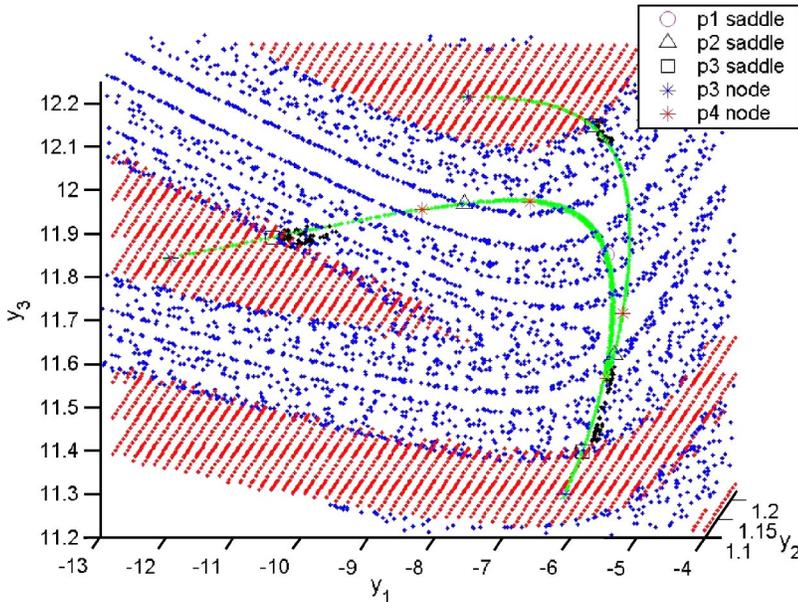


FIG. 8. (Color online) The period-three basin is denoted by stripes and the rest of the space is the period-four basin. The union of the stable manifolds in the phase space is approximated by small dots in the period-four basin. This includes the stable manifolds for the period-one and period-two orbits. The union of the unstable manifolds is the solid curve with the period-three stars at the ends. The boundary between the period-three and period-four basins is formed by the stable manifold of the period-three saddle (squares). Notice how the unstable manifolds of the period-three saddle intersect the stable manifolds inside the period-four basin, forming the forward connections of the heteroclinic tangle. Superimposed is the time-series approximation of the flux from the period-four to the period-three basin in large dots.

$$P_{F_\nu}[\rho(\mathbf{x})] = \int_{\mathbb{R}^3} K(\mathbf{x}, \mathbf{y}) \rho(\mathbf{y}) d\mathbf{y}, \quad (7)$$

where the stochastic kernel describing the PDF of the noise perturbation is  $K(\mathbf{x}, \mathbf{y}) = \nu(\mathbf{x} - F(\mathbf{y}))$ . Assuming a nonzero external noise is added in each component, Eq. (7) becomes

$$P_{F_\sigma}[\rho(\mathbf{x})] = \frac{1}{\sqrt{(2\pi)^3 \det(\Sigma)}} \int_{\mathbb{R}^3} e^{-(\mathbf{x} - \mathbf{y})^T \Sigma^{-1} (\mathbf{x} - \mathbf{y}) / 2} \rho(\mathbf{y}) d\mathbf{y}. \quad (8)$$

Note that although it is possible to let any of the standard deviations tend to zero in the SFPO where the kernel limits to a delta function, it is more realistic to approximate the zeros by very small values. This is due to the fact that the experiment is always perturbed by small noise. Since we require a finite dimension for computation, we cover the phase space with  $N$  disjoint boxes  $B_i$  and choose a set of basis functions to be the family of characteristic functions

$$\phi_i(x) = \begin{cases} 1 & \text{if } x \in B_i \\ 0 & \text{if } x \notin B_i. \end{cases} \quad (9)$$

In principle, any set of basis functions of  $L^2$  can be used, but we use characteristic functions to help us locate spatial transport, as was motivated historically by Ulam's method. The approximation of the Frobenius-Perron operator projects to a  $N \times N$  matrix, called the Galerkin transport matrix (GTM),

$$A_{i,j} = (P_{F_\sigma}[\phi_i], \phi_j) = \int_{\mathbb{R}^3} P_{F_\sigma}[\phi_i(\mathbf{x})] \phi_j(\mathbf{x}) d\mathbf{x}, \quad 1 \leq i, j \leq N. \quad (10)$$

The GTM describes the mass flow from one box to another over one iteration. That is, the entry for  $A_{i,j}$  approximates the percentage of box  $i$  that iterates to box  $j$  under the stochastic map. Then partition the boxes according to their basin and reorder the GTM by similarity transformations to reflect that

partition. In theory, the case with no noise will result in a block-diagonal matrix, reflecting dynamics in the disjoint basins. Under stochastic perturbations, the GTM approximates three things: (i) the off-diagonal blocks indicate where the transport between basins occurs—this is the mass flux (or simply flux), (ii) the dominant eigenvector having eigenvalue unity approximates the PDF, and (iii) by weighting the mass flux by the PDF, we pinpoint regions in phase space that have the greatest probability of leakage into another basin—this is the area flux. See [19,18] for details.

We show the GTM approximation of the PDF in Fig. 9. Since the noise distribution is assumed to be normal, it is expected that the PDF has nonzero entries everywhere in phase space. However, many of these values are sufficiently small so that when added to unity, they make zero contribution due to the fact they are below machine error. Therefore,

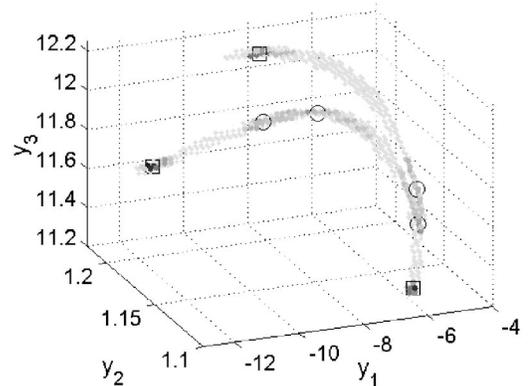


FIG. 9. An approximation by the GTM of the PDF when  $\sigma_1 = 0.16$ . The squares represent the stable period-four orbit. The circles represent the stable period-three orbit. The darker shades indicate regions with the highest probability. Notice that they occur near the stable periodic orbits, but there is structure connecting these regions called the stochastic chaotic saddle. As  $\sigma$  increases, orbits spend more time on the chaotic saddle, indicating the increased frequency of bursting dynamics.

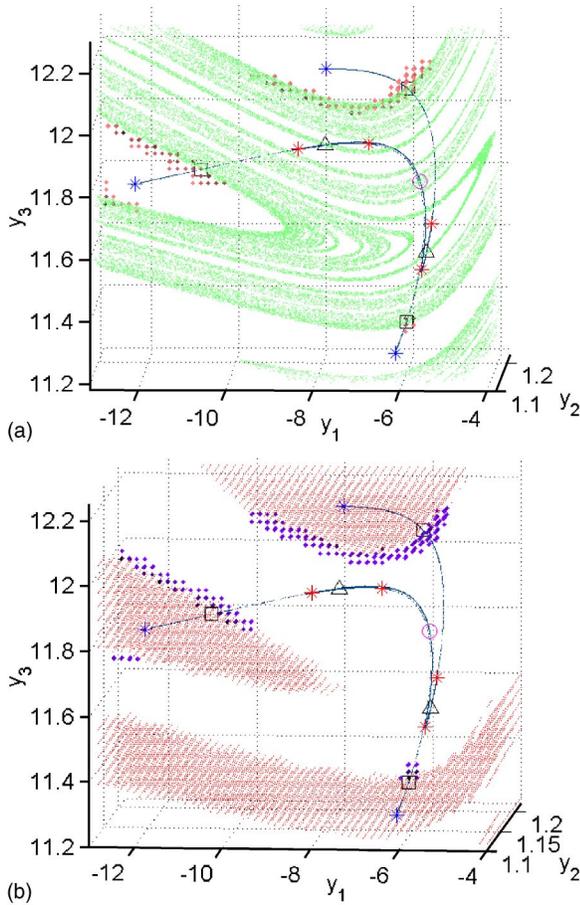


FIG. 10. (Color online) Approximation of the transport from the GTM when  $\sigma_1=0.16$ . (a) shows regions of most active transport from the period-three to the period-four basin in large dots. (b) shows regions of most active transport from the period-four to the period-three basin in large dots. Notice that these regions occur near the period-three saddle orbit represented by the squares. In (a), the union the stable manifolds is displayed in layered sheets and unstable manifolds form the one-dimensional curve with the period-three points (stars) at the ends. They are approximated by the box algorithm from [13]. The stable manifolds of the period-three saddle form the basin boundary between the two basins. In (b), the basin of the period-three orbit is represented by the small dots. Also shown in both (a) and (b) are the stable period-four orbit (stars), the unstable period-two orbit (triangles), and the unstable period-one (circle).

we choose a numerical threshold of machine precision as a lower bound and replace all smaller values to zero in the PDF. Notice that as we add stochastic perturbations to the system, the most frequently visited regions lie near the unstable manifolds of the period-two and period-one saddle orbits from the noiseless case.

We observe that as the standard deviation is increased, the PDF spreads and crosses into the period-three basin, and through the stable manifold of the period-three saddle. This is evidence that there is the topology for a trajectory to emulate chaoslike behavior. We now will verify the fact that the trajectory actually uses these regions for transport. This is supported by the area flux, which is shown in Fig. 10. The

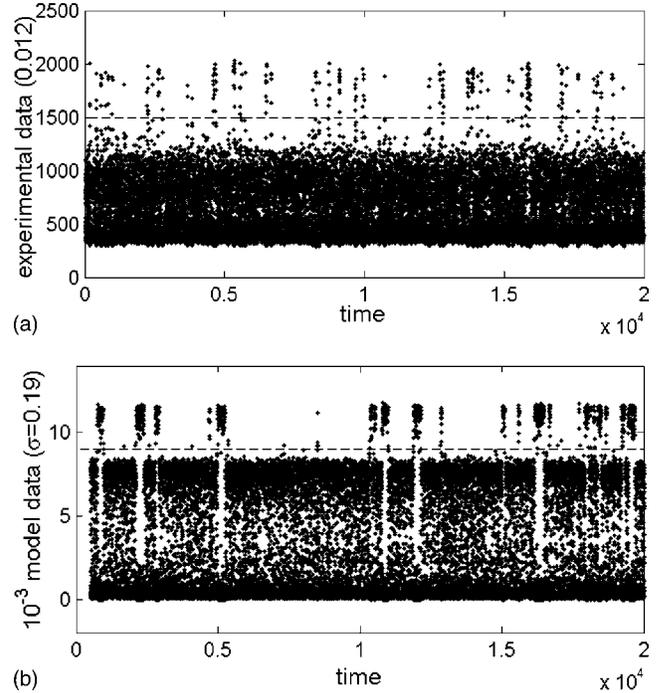


FIG. 11. Bursting time series from the experiment (a) and model (b). The dashed line indicates the threshold used to determine a burst.

regions where the trajectory is most likely to switch basins are found by multiplying the mass flux by the associated PDF value for that region of phase space. Notice the agreement between the transport region predicted by the time series in Fig. 8 and the area flux from the period-four to the period-three basin.

## VI. DISCUSSION

Dynamics with noise is always present in experiments at least at some level. In many cases, noise is sufficiently small so that its role is ignorable with respect to the underlying determinism. However, even relatively low-amplitude noise may play a significant role in which the dynamics takes on a qualitative change that is different from the deterministic structure. In the physical example presented here, we have examined an experiment where noise has been injected into a modulated laser. The amplitude of the noise was adjusted, and the laser was seen to go from stochastically perturbed periodic behavior to one of stochastic-induced chaoslike dynamics. Because discrete control of the random noise amplitude could be achieved, the system was therefore analyzable by a discrete-map approach, thereby revealing explicitly the interaction of noise and the underlying deterministic topology.

In conjunction with the laser experiment, we have examined a quantitative model with additive noise in the intensity. Both exhibit similar bursting behavior, as shown in the time series data in Fig. 11. Although the topology of the experimental dynamics is difficult to ascertain, the quantitative nature of the model does allow an in-depth view of the under-

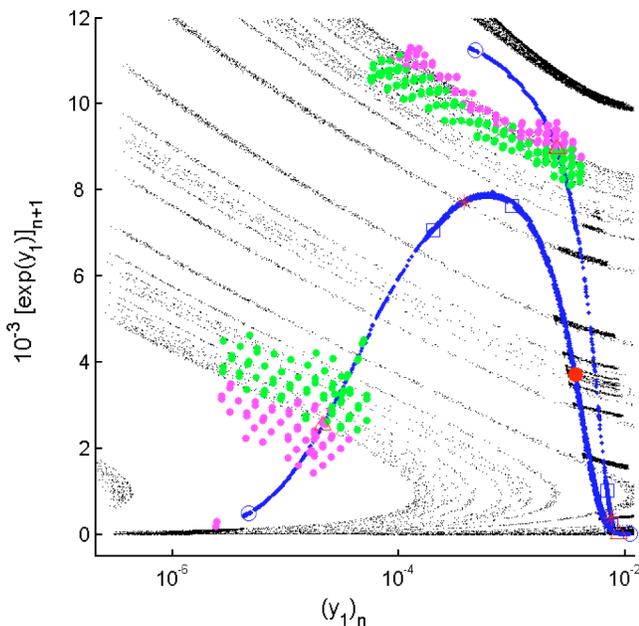


FIG. 12. (Color online) The return map of the fixed points corresponding to the periodic orbits and their projected manifolds. (The  $x$  axis has been plotted logarithmically to show more detail.) The unstable manifolds form the dark solid curve in the middle, while the stable manifolds are approximated by smaller points. In addition, the projected regions of transport are overlaid in large dots. Notice that the transport between the two basins predicted by the GTM in Sec. V lies close to the period-three saddles (triangles) on the stable manifold. The other periodic orbits are labeled as follows: period two, stars; period four, squares; period one, large dot; period-three node, circles; period-three saddle, triangle. The value of the standard deviation used was  $\sigma_1=0.04$ .

lying topology and its relation to noise. In the absence of noise, the topology of the system was determined, and the structure of the stable and unstable manifolds was computed in a prechaotic regime. When noise is added, the structure of the topology interacts with the stochastic fluctuations in such a way to induce chaoslike behavior, which is the emergent structure observed in both theory and experiment as shown in Fig. 4. The stochastic dynamics is the union of local stochastic dynamics within each basin and the dynamics near a partially formed chaotic saddle. For sufficiently large noise amplitudes, local instability near the manifold structure contributes to the time- and space-averaged linear variation so that the Lyapunov exponent becomes positive, which we take as criteria for stochastic bifurcation as defined in [15], and exemplified in [2].

In tying together the dynamics from the model and experiment, we can project the phase portrait of the transport and manifold structure to a lower dimensional return map, as we did in Sec. IV. In Fig. 12, we have depicted the fixed points that correspond to the unstable periodic orbits and their manifolds. Notice that because we used the box algorithm of [13], the manifolds are not grown from a saddle, but reflect the union of all such manifolds in the region we considered. The stable manifold (in black) corresponds to the basin boundary in the original phase space separating the bistable attractors in the deterministic case. In the projection

in Fig. 12, we can see where the maximum probability of transport from one basin to another occurs. Notice that much of it takes place near the basin boundary saddle points. That is, the period-three saddle stable manifold, which forms the basin boundary, intersects the regions of maximal probability transport. The stochastic dynamics fluctuates until it comes near the basin boundary, at which point it is attracted to the saddle point. Noise then takes on a dominant role, where the intensity is either pushed across the basin boundary, or remains in the same basin. The unstable manifold then dominates the noise, by pushing the dynamics further into the respective basin.

Our model of the class- $B$  laser includes more physics in the problem, which in turn, leads to a more interesting class of dynamical behavior. Most laser models consist of just two population levels, describing the change in the population inversion and intensity. The advantage of such a model is that it may be studied in the plane, having one-dimensional stable and unstable manifolds. The resulting stochastic analysis could be compared directly to manifolds that were built from curve-following methods in the plane. In contrast, the model considered here is based on a four-level model, which agrees quantitatively with the experiment over a large range of values. The model requires two main levels and two rotational levels, resulting in a five-dimensional system of differential equations. Approximating the relaxation rates of the vibrational states by their average allows one to reduce the model to the current three-dimensional driven case [24]. The main difference here is that the stable manifolds are no longer one-dimensional. (The unstable manifolds are one-dimensional, however.) Here, two-dimensional stable manifolds are pierced by one-dimensional unstable manifolds. Therefore, regions of transient behavior may wander over a greater region of phase space in both the deterministic and stochastic models, offering a richer set of dynamical behavior than the two-level laser model.

One of the main conclusions of the current stochastic analysis is that maximal transport from one basin to another may not occur near the basin boundary saddles. Similar examples based on asymptotic properties of problems of escape where the phenomenon of saddle avoidance occurs can be found in [28,29]. We note that the methods used here not only agree with the previous local theories, but is an alternative to describe the global structure of the transport as well [18].

In general, computing stable and unstable manifolds is a difficult task, compounded here by the fact that the manifolds are of different dimensions. The technique used in this paper cannot grow the manifolds from a given saddle. Therefore, the global analysis of the four-level laser system lacks some of the precision of the two-level system. On the other hand, the SFPO tool does not require the manifold constructions *a priori*. Rather, the transport requires a partition of the basins of attraction in the zero noise case. When noise is added, the phase space is reconstructed in terms of transport across the referenced basin boundaries, and thus must contain components of the basin manifolds, regardless of their dimension.

In terms of model development, much work on noise had been done on maps in the plane, or two-dimensional flows.

In this paper, we have presented a stochastic experiment as well as a quantitative model that simulates the stochastic dynamics. The model is itself a reduction of a previously more complicated model [24], but nonetheless, captures the relevant features of the stochastic dynamics. In particular, it captures the interaction of the stochastic dynamics with the underlying topology of the model. This quantifies the smooth change of regular stochastic behavior to a bursting type of behavior between basins, which appears to be chaoslike in

that there exists a smooth transition from negative to positive Lyapunov exponents.

#### ACKNOWLEDGMENTS

L.B. was supported by DARPA under Grant No. DAAD19-03-1-0134. I.B.S. was supported by the Office of Naval Research and the Army Research Office. E.A. was supported by FIRB Contract No. RBAU01B49F\_002.

- 
- [1] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, Lecture Notes in Mathematics Vol. 888 (North-Holland, Amsterdam, 1981).
- [2] N. S. Namachchivaya and N. Ramakrishnan, *J. Sound Vib.* **262**, 613 (2003).
- [3] S. Kraut, U. Feudel, and C. Grebogi, *Phys. Rev. E* **59**, 5253 (1999).
- [4] M. Freidlin and A. Wentzel, *Random Perturbations in Dynamical Systems* (Springer-Verlag, Berlin, 1984).
- [5] R. Graham and T. Tel, *Phys. Rev. Lett.* **52**, 9 (1984).
- [6] A. Hamm, T. Tél, and R. Graham, *Phys. Lett. A* **185**, 313 (1994).
- [7] R. S. Maier and D. L. Stein, *Phys. Rev. E* **48**, 931 (1993).
- [8] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1983).
- [9] S. Wiggins, *Chaotic Transport in Dynamical Systems* (Springer-Verlag, New York, 1992).
- [10] M. Frey and E. Simiu, *Physica D* **63**, 321 (1993).
- [11] B. Krauskopf and K. Green, *J. Comput. Phys.* **186**, 230 (2003).
- [12] B. Krauskopf and H. Osinga, *Chaos* **9**, 768 (1999).
- [13] I. Triandaf, E. M. Bollt, and I. B. Schwartz, *Phys. Rev. E* **67**, 037201 (2003).
- [14] L. Billings and I. B. Schwartz, *J. Math. Biol.* **44**, 31 (2002).
- [15] L. Arnold, *Random Dynamical Systems* (Springer-Verlag, New York, 1998).
- [16] M. I. Dykman, P. V. E. McClintock, V. N. Smelyanski, N. D. Stein, and N. G. Stocks, *Phys. Rev. Lett.* **68**, 2718 (1992).
- [17] J. Lehmann, P. Reimann, and P. Hanggi, *Phys. Rev. Lett.* **84**, 1639 (2000).
- [18] L. Billings, E. M. Bollt, and I. B. Schwartz, *Phys. Rev. Lett.* **88**, 234101 (2002).
- [19] E. M. Bollt, L. Billings, and I. B. Schwartz, *Physica D* **173**, 153 (2002).
- [20] A. Bandrivskyy, S. Beri, and D. G. Luchinsky, *Phys. Lett. A* **314**, 386 (2003).
- [21] Y. C. Lai, Z. Liu, L. Billings, and I. B. Schwartz, *Phys. Rev. E* **67**, 026210 (2003).
- [22] F. T. Arecchi, R. Meucci, G. Puccioni, and J. Tredicce, *Phys. Rev. Lett.* **49**, 1217 (1982).
- [23] I. B. Schwartz, I. Triandaf, R. Meucci, and T. W. Carr, *Phys. Rev. E* **66**, 026213 (2002).
- [24] R. Meucci, D. Cinotti, E. Allaria, L. Billings, I. Triandaf, D. Morgan, and I. B. Schwartz, *Physica D* **189**, 70 (2004).
- [25] T. W. Carr, L. Billings, I. B. Schwartz, and I. Triandaf, *Physica D* **147**, 59 (2000).
- [26] For  $y_1 = \ln x_1$ , the original model defines  $x_1' = k_0(y_2 - 1 - \alpha \sin^2\{B[1 + f(t)]\})x_1$ , where the noise is multiplicative. The change of variables results in  $y_1' = k_0(y_2 - 1 - \alpha \sin^2\{B[1 + f(t)]\})$ , as described in the scaled model in Eq. (4).
- [27] J. B. Gao, C. C. Chen, S. K. Hwang, and J. M. Liu, *Int. J. Mod. Phys. B* **13**, 3283 (1999).
- [28] Z. Schuss and A. Spivak, *Chem. Phys.* **235**, 227 (1998).
- [29] D. G. Luchinsky, R. S. Maier, R. Mannella, P. V. E. McClintock, and D. L. Stein, *Phys. Rev. Lett.* **82**, 1806 (1999).