Periodic precursors of nonlinear dynamical transitions

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We study the resonant response of a nonlinear system to external periodic perturbations. We show by numerical simulation that the periodic resonance curve may anticipate the dynamical instability of the unperturbed nonlinear periodic system, at parameter values far away from the bifurcation points. In the presence of noise, the buried intrinsic periodic dynamics can be picked out by analyzing the system's response to periodic modulation of appropriate intensity.

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The effects of periodic perturbation on a nonlinear dynamical system is a long-standing problem and continues to attract much interest in recent years. The main interest in the periodically driven dynamical system has been focused on the response of a nonlinear system, which is near the onset of dynamical instabilities, to small periodic perturbations, small-signal amplification of bifurcating system [1–3], periodic multistability [4,5], control of chaos and spatiotemporal patterns by global or local periodic forcing [6–13], and other periodic driving induced behaviors in excitable or oscillatory systems [14,15]. All those efforts have been dedicated to understanding how the dynamical features of a nonlinear system change as a function of the amplitude and frequency of the periodic modulation. Recently the response of a nonlinear system to stochastic perturbations has become a subject of intense investigation, in particular the stochastic resonance and coherence resonance [16]. It has been shown that the response of a bifurcating system to external noise exhibits characteristic signatures for each class of dynamical instabilities, which is well displayed by the power spectrum [17]. It is interesting to note that the relationship of the power spectrum to the dynamics was discussed in Ref. [18].

This paper addresses the issue of periodic precursors of nonlinear instabilities and studies the response of a nonlinear periodic system to weak periodic modulation in the absence or the presence of noise, over the whole parameter range. We focus our attention to the signature of dynamical instability as revealed by the response of the nonlinear system to very weak periodic perturbation signals, at parameter values that is not near the onset of the bifurcation points of the unperturbed system. It is noted that when the system is near the onset of dynamical instabilities, the theoretical analysis is greatly simplified. Our interest is how to detect the possible dynamical instability by just evaluating the system's response to the very weak sinusoidal perturbations. Our approach involves the direct measurement of the amplitude of the periodically driven system when the control parameter varies, and therefore this method provides real-time evolution of the dynamical features of the unperturbed system and to uncover the key factors that control system dynamics in a real, unknown system.

To quantify the resonant response of a nonlinear system to periodic perturbation we calculate the difference between the maximum and the minimum of the response amplitude as a function of the frequency of the driving sinusoidal signal, for a given signal amplitude. We demonstrate our idea by studying the response of the logistic map and Rossler oscillator under additive periodic driving for period-doubling bifurcations and the coupled logistic maps for Hopf bifurcation. We have also tested other discrete and continuous bifurcating systems and find only the similar results.

First we look at the resonant response of a nonlinear system that undergoes a sequence of period-doubling bifurcations. We consider the logistic map with weak periodic modulation described by

$$x_{n+1} = 1 - a x_n^2 + A \sin(2\pi f n), \tag{1}$$

where a is the control parameter of the map and A and f are the amplitude and frequency of the periodic perturbation, respectively. In the absence of periodic driving the period-doubling bifurcations occur at the parameter values: $a_1 = 0.75$, $a_2 = 1.25$, $a_3 = 1.368\,099, \ldots$. It has been shown that near but before the bifurcation points, noise with appropriate intensity can induce δ-like peaks in the power spectrum that corresponds to the bifurcated dynamics, and thus precludes the occurrence of the dynamical instability. In Fig. 1(a) we show the relative amplitude as a function of the system control parameter a, and the frequency of periodic perturbation f. The relative amplitude is defined by the difference between the maximum and minimum of the dynamical variable evaluated in a period of time. That is, $\Delta x = x_{max} - x_{min}$, where $x_{max} = \max\{x_n, 1 \leq n \leq T\}$, and $x_{min} = \min\{x_n, 1 \leq n \leq T\}$ (here T is an arbitrary large number of iteration steps). The response curves are uniformly shifted up for consecutive values of the control parameters so that of the resonant levels can be appreciated clearly. One of the most interesting observations is that the response curve reveals the dynamical transition in between the successive bifurcation points, where an unperturbed system shows only the simple monotonic oscillation behavior. For example, on the parameter interval $0.75 < a < 1.25$ the resonance amplitude already shows a resonance curve with two peaks at $a = 1.0023$, anticipating the occurrence of period-4 cycle. It is noted that the signature appears at a parameter value that is far from the dynamical instability of the unperturbed system. The dis-

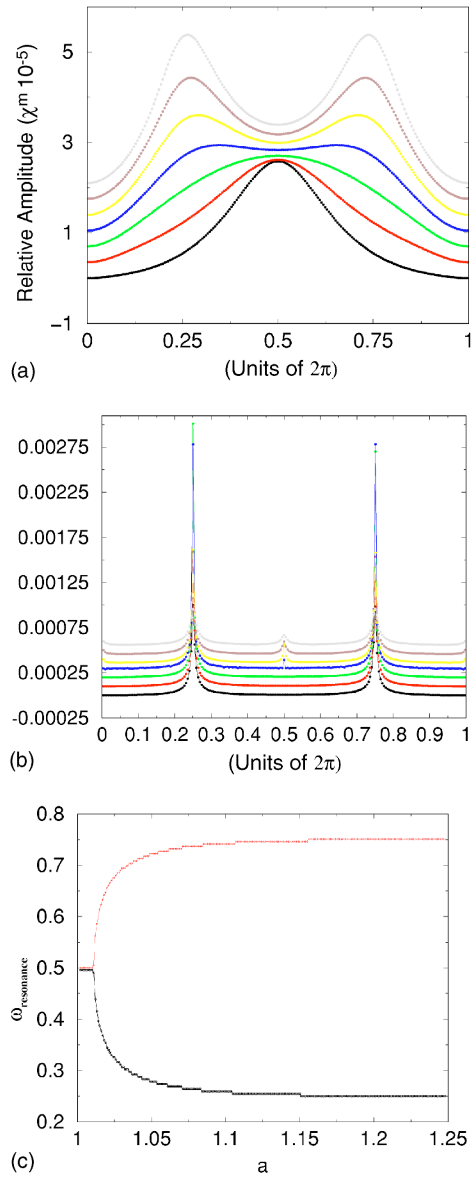

FIG. 1. (Color online) Precursors of period-doubling bifurcation in the logistic map. The variation of resonance response curve is plotted as a function of the frequency of periodic perturbation, at a constant strength of perturbation fixed at $A=10^{-5}$, for different control parameters (from bottom to top): (a) $a=0.96, 0.98, 1.0, 1.02, 1.04, 1.06$, and (b) $a=1.241, 1.144, 1.2471, 1.2501, 1.2531, 1.2561, 1.2591$. (c) The shift of the frequencies corresponding to the resonance peaks as a function of the control parameter a. The amplitude of periodic driving is $A=10^{-5}$.

tance to the bifurcation point may be appreciated by the height and width of the peak because the shape of the periodic precursor becomes a δ function as the parameter approaches the instability point. In Fig. 1(c) we display the change of the positions of the maxima in the resonant curve as the parameter is varied. It is seen that the frequencies corresponding to the resonance peaks start at $\omega=0.5$ and settle down at the frequencies of the bifurcated cycles, that is, $\omega=1/4$ and $\omega=3/4$. It is interesting to note that our approach can also be used to determine the precise value of the

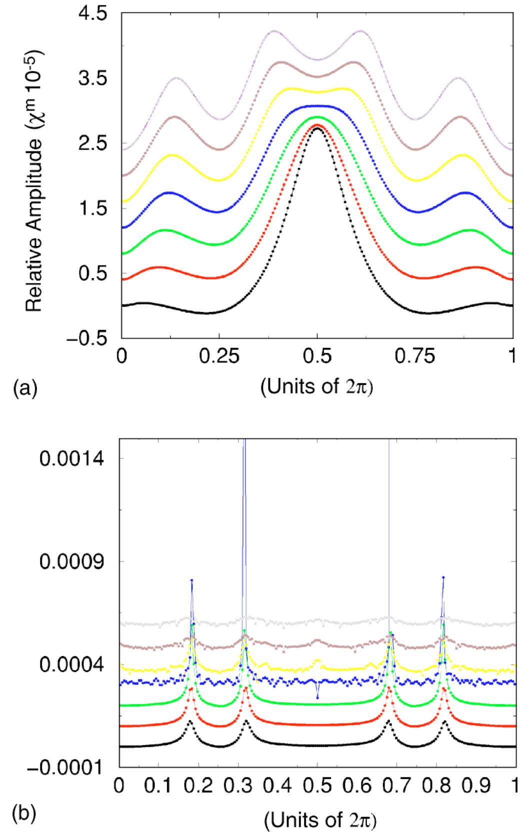

FIG. 2. (Color online) Precursors of Hopf bifurcation in the coupled logistic maps. (a) Signature of dynamical transition from a period-2 cycle to two limit cycles as displayed by the relative amplitude as a function of driving frequency. The amplitude of the periodic modulation is $A=10^{-5}$ (from bottom to top: $a=0.19-0.25$). (b) The variation of the resonant frequencies as a function of the control parameter a. (c) Typical resonance curves as the control parameter crosses the bifurcation point (from bottom to top: $a=0.38-0.44$). The relative amplitude is defined by $\Delta x=x_{max}-x_{min}$.

control parameter of an unknown dynamical system, where the onset of a particular instability occurs. Figure 1(b) shows the variation of the resonant response curve as the control parameter crosses over the period-doubling bifurcation point. It can be seen that the control parameter a is slightly larger than $a=1.25$, small spikes appear at $f=k\pi, k=0,1,2,\ldots$. The resonant response curve shows strong fluctuation when the control parameter is very near to the bifurcation point, which can be regarded as a signal of dynamical instability. As the parameter continues to increase, the δ-like spikes grow into bell-shaped peaks which will be replaced by a smooth curve with slight modulation to signify the next dynamical transition. This precursor scenario repeats itself for each of the consecutive period-doubling bifurcation intervals. It is also interesting to note that the characteristic feature of conventional periodic resonance patterns observed in a linear system that is characterized by the unimodal resonance at the natural frequency is not observed in the nonlinear dynamical system. It should be stressed that the weak periodic perturbation only generates small sinusoidal modulation of the periodic motion of the original system. The time

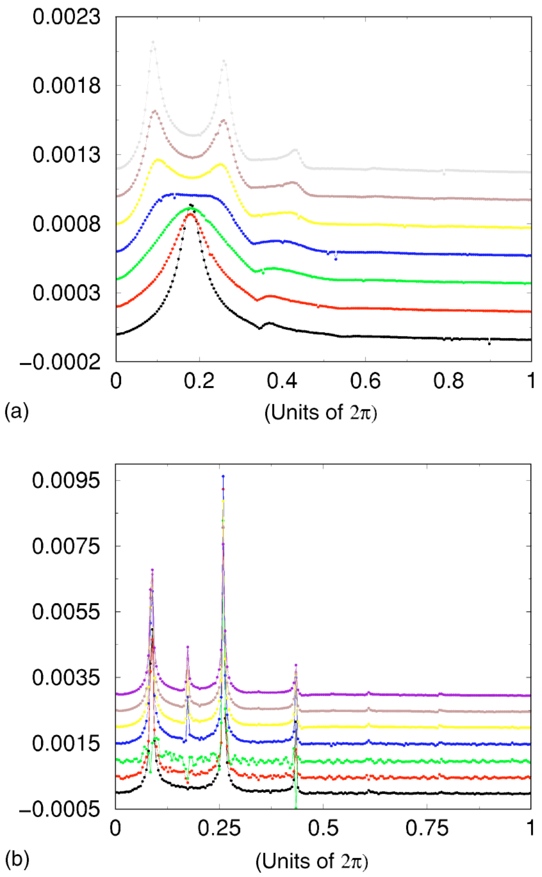

FIG. 3. (Color online) Precursors of period-doubling bifurcation in the Rössler oscillator. The amplitude of sinusoidal modula tion is $A=10^{-4}$. The variation of the resonant response curve as a function of a driving frequency for (a) $c=1.101$, 1.301, 1.501, 1.701, 1.901, 2.101, and 2.301, and (b) $c=2.8101$, 2.8201, 2.8301, 2.8401, 2.8501, 2.8601, and 2.8701.

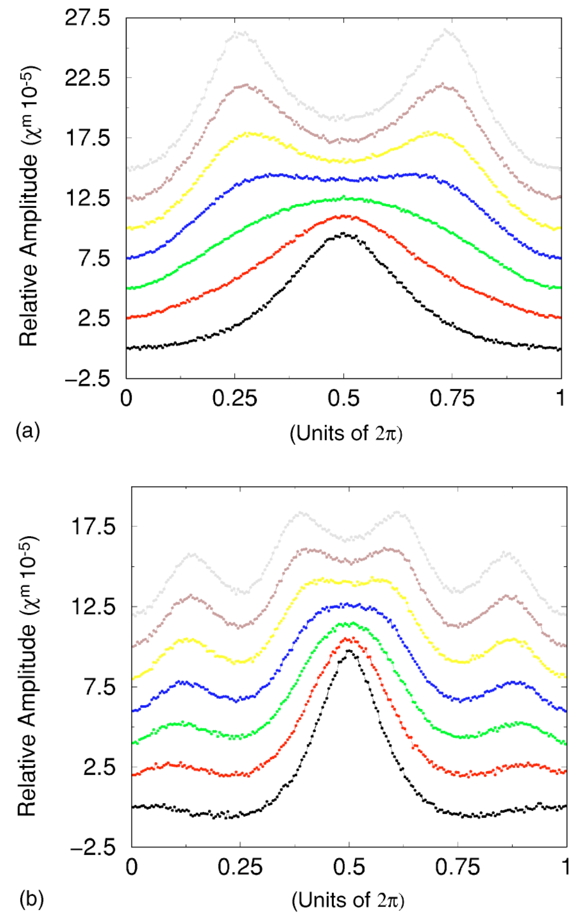

FIG. 4. (Color online) Precursors of period-doubling bifurcation in the logistic map, in the presence of noise. The uniform noise $-1<\xi_n<1$ with strength $D=3\times10^{-5}$ is used. The other parameters are the same as (a) in Fig. 1 and (b) in Fig. 2. The relative amplitude is described by $\Delta x=x_{max}-x_{min}$

evolution of the dynamical variable for parameter $a=0.96$ and $a=1.06$, for instance, shows no appreciable difference in the frequency of their modulated motion although the amplitude of the modulated cycle changes with the control parameter a. This is to say, the direct measurement of the time series gives no indicator of the ongoing dynamical transition.

We now turn to consider the effect of periodic perturbation on a system that is near the onset of Hopf bifurcation. As an example of a discrete system, we consider the coupled logistic maps [19],

$$x_{n+1}=1-ax_n+\epsilon(y_n-x_n)+A\sin(2\pi fn), \qquad (2)$$

$$y_{n+1}=1-ax_n+\epsilon(x_n-y_n), \qquad (3)$$

where ϵ is the coupling strength. It is known that for $\epsilon=0.4$, a Hopf bifurcation occurs at $a_H=0.409\,88$, at which the characteristic multipliers of the period-2 orbit cross the unit circle corresponding to the birth of an invariant curve or torus in the phase space of the system. Figure 2(a)displays the change of resonant response curve with control parameter a. Since the transition is from period 2 to two limit cycles through Hopf bifurcation, the resonance curve is characterized by one peak for the period-2 oscillation. It is seen that at $a=0.225$ the central peak located at $f=1/2$ begins to deform and gives rise to two peaks on each side of the central peak. As the parameter is further increased, those four peaks turn into δ-like spikes, symmetrically distributed with respect to the original peak. As the parameter a approaches the dynamical transition point the resonance peaks become sharper and suddenly disappear when the bifurcation point is crossed, see Fig. 2(b). Slightly after the Hopf bifurcation, small spikes appear at $f=0$, π, and 2π, which is a signature of the onset of the dynamical instability for a general nonlinear system under weak periodic modulation.

To demonstrate the applicability of a periodic precursor for a flow system we studied a periodically modulated Rossler system described by

$$\dot{x}=-y-z+A\sin(2\pi ft), \qquad (4)$$

$$\dot{y}=x+ay, \qquad (5)$$

$$\dot{z}=b+z(x-c), \qquad (6)$$

where the parameters a and b are fixed at $a=b=0.2$, and c is the system control parameter. We calculate the relative amplitude defined by $\Delta x = x_{max} - x_{min}$, with $x_{max} = \max\{x(t), 0 \leqslant t \leqslant T\}$ and $x_{min} = \min\{x(t), 0 \leqslant t \leqslant T\}$. Figure 3 shows the typical resonance curves corresponding to the period-doubling bifurcation from period-2 to period-4 cycles. We find qualitatively equivalent behavior as in the logistic map. As for the Hopf bifurcation we studied a normal form equation for Hopf bifurcation given by

$$\dot{x} = -y + (a - x^2 - y^2)x + A \sin(2\pi f t), \tag{7}$$

$$\dot{y} = x + (a - x^2 - y^2)y, \tag{8}$$

where the Hopf bifurcation takes place at $a=0$. The typical resonance curve is a small precursor bump that grows continuously from zero height for the parameter a, is sufficiently far away from the dynamical instability to a δ function at the parameter value slightly before the Hopf bifurcation point, with the maximum of the peak always located at the frequency of the limit cycle born from the Hopf bifurcation, in contrast to the one-bump resonance curve for period-doubling bifurcation which is centered at $f=\pi$. We have tested our approach on other flow systems such as Lorenz oscillator and the Morris-Lucar neuron model. We find this resonance picture a common precursor of the Hopf bifurcation.

To test the robustness of periodic precursors in the presence of noise, we study the response of the logistic map to the simultaneous additive periodic and stochastic perturbations. We find that since the strength of the noise is not too much stronger than that of the periodic perturbation, the dynamical transition can be detected. Figure 4 shows the evolution of resonance curve under an extra additive of noise in addition to periodic modulation. Here the control parameter is chosen to be in the period-2 to period-4 range of the period-doubling bifurcation in logistic maps as shown in Fig. 4(a) and the Hopf bifurcation from period-2 cycle to two limit cycles in the coupled logistic maps, see Fig. 4(b). The data shown in Fig. 4 are the mean resonance curves averaged over 400 sample runs. As can be seen, the weak periodic driving can still pick out the dynamical transition points, as displayed by the resonant response curves. If we look at the time evolution of the dynamical variables, we find the strongly fluctuated periodic bands, with the periodic signal completely buried by the noise. Nevertheless, it is still possible to extract the noise-contaminated deterministic dynamics from the analysis of the resonance response properties.

In summary, we have studied the resonance response of periodically driven nonlinear dynamical systems. We show that the response of the system to the weak periodic perturbation can be used to predict the dynamical instability long before the system undergoes the dynamical transition. Since the amplitude of periodic modulation is very small the system's dynamical variables do not deviate very much from their nominal values, which may find important applications in physiological systems where the detection of the onset of some pathological events is highly desired by using of some means that does not provoke substantial changes in the original, unperturbed system. We also show that our approach is robust in the presence of weak external noise. We find that for certain level of noisy perturbation, the dynamical transition may be detected by increasing accordingly the strength of periodic modulation. However, caution must be taken because when periodic or stochastic perturbations are strong enough, the external driving induced effects may occur, resulting in novel dynamical properties. Within the limit of small driving, our results are not dependent on the amplitude of the periodic perturbation.

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