

# Estimate of blow-up and relaxation time for self-gravitating Brownian particles and bacterial populations

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We determine an exact asymptotic expression of the blow-up time  $t_{coll}$  for self-gravitating Brownian particles or bacterial populations (chemotaxis) close to the critical point in  $d=3$ . We show that  $t_{coll}=t_*(\eta-\eta_c)^{-1/2}$  with  $t_*=0.917\ 677\ 02\dots$ , where  $\eta$  represents the inverse temperature (for Brownian particles) or the mass (for bacterial colonies), and  $\eta_c$  is the critical value of  $\eta$  above which the system blows up. This result is in perfect agreement with the numerical solution of the Smoluchowski-Poisson system. We also determine the exact asymptotic expression of the relaxation time close to but above the critical temperature and derive a large time asymptotic expansion for the density profile exactly at the critical point.

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## I. INTRODUCTION

Recently, several papers have focused on the blow-up properties of a cluster of self-attracting particles [1–13]. This concerns in particular the “isothermal collapse” [14] of self-gravitating Brownian particles and the “chemotactic aggregation” [15] of bacterial populations in biology. These systems have a similar mathematical structure and they are described, in a first approximation, by the Smoluchowski-Poisson system [16]. The Smoluchowski equation is a particular Fokker-Planck equation in physical space involving a diffusion due to Brownian motion and a drift. In the standard Brownian theory developed by Einstein, the drift is due to an external potential [17]. Alternatively, we can consider the more complicated situation in which the potential is produced by the particles themselves. If we assume a Newtonian type of interaction, we have to couple Smoluchowski’s equation to Poisson’s equation.

It can be shown that the Smoluchowski-Poisson system decreases a Lyapunov functional  $F[\rho]$  which can be interpreted as a Boltzmann free energy [18]. Furthermore, the type of evolution depends on the value of a dimensionless parameter  $\eta$ . For  $\eta \leq \eta_c = 2.517\ 551\ 32\dots$ , the Smoluchowski-Poisson system evolves to an equilibrium state which minimizes the free energy at fixed mass. It corresponds to an isothermal distribution of particles similar to isothermal stars and isothermal stellar systems [19]. On the contrary, for  $\eta > \eta_c$ , there is no possible equilibrium state and the system blows up. This is the case for self-gravitating Brownian particles below a critical temperature  $T_c$  and for bacterial populations above a critical mass  $M_c$ . It is found that the collapse is self-similar and that it develops a finite time singularity, i.e., the central density becomes infinite in a finite time  $t_{coll}$  [4,5,8,9]. Then a Dirac peak is formed in the postcollapse regime [12].

It is clear that the collapse time  $t_{coll}$  depends on the distance to the critical point  $\eta_c$  and that it should diverge as

$\eta \rightarrow \eta_c$ . More precisely, by using heuristic arguments, it is argued in [8] that  $t_{coll} \sim t_*(\eta_c - \eta)^{-1/2}$ . This scaling is consistent with numerical simulations of the Smoluchowski-Poisson system. However, the approach of [8] is only qualitative and does not provide the numerical value of  $t_*$ .

After some introductory material presented in Sec. II, we develop a systematic procedure in Sec. III, inspired by [20], which confirms the scaling law  $t_{coll} \sim t_*(\eta_c - \eta)^{-1/2}$  and leads to the explicit value of  $t_*$ . In Sec. IV, we study the relaxation time  $\tau$  to equilibrium below  $\eta_c$  and show that it diverges like  $\tau \sim c_\eta(\eta - \eta_c)^{-1/2}$ , where  $c_\eta$  is given explicitly. In Sec. V, we derive a systematic large time expansion of the density profile exactly at  $\eta_c$ . In particular, we show that the central density approaches its equilibrium value according to  $\rho_0 - \rho(r=0, t) \sim c_\rho/t$ , where  $c_\rho$  is a universal constant which is explicitly computed. Finally, in Sec. VI, we determine the pulsation period of bounded isothermal spheres described by the Euler-Jeans equations close to the critical point.

## II. SELF-GRAVITATING BROWNIAN PARTICLES AND BACTERIAL POPULATIONS

### A. The model equations

In the overdamped regime, a system of self-gravitating Brownian particles is described by the  $N$  coupled stochastic equations

$$\frac{d\mathbf{r}_i}{dt} = -\mu \nabla_i U(\mathbf{r}_1, \dots, \mathbf{r}_N) + \sqrt{2D} \mathbf{R}_i(t), \quad (1)$$

where  $\mu = 1/\xi$  is the mobility ( $\xi$  is the friction coefficient),  $D$  is the diffusion coefficient, and  $\mathbf{R}_i(t)$  is a white noise satisfying  $\langle \mathbf{R}_i(t) \rangle = \mathbf{0}$  and  $\langle R_{a,i}(t) R_{b,j}(t') \rangle = \delta_{ij} \delta_{ab} \delta(t-t')$ , where  $a, b = 1, 2, 3$  refer to the coordinates of space and  $i, j = 1, \dots, N$  to the particles. We define the temperature  $T = 1/\beta$  through the Einstein relation  $\mu = D\beta$ . The particles interact via the potential  $U(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{i < j} \mu(\mathbf{r}_i - \mathbf{r}_j)$ . In this paper,  $u(\mathbf{r}_i - \mathbf{r}_j) = -G/|\mathbf{r}_i - \mathbf{r}_j|$  is the Newtonian binary potential in  $d=3$  dimensions. Starting from the  $N$ -body Fokker-

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Planck equation and using a mean-field approximation [16,21,22], we can derive the nonlocal Smoluchowski equation

$$\frac{\partial \rho}{\partial t} = \nabla \left[ \frac{1}{\xi} (T \nabla \rho + \rho \nabla \Phi) \right], \quad (2)$$

where the potential  $\Phi(\mathbf{r}, t)$  is produced by the density  $\rho(\mathbf{r}, t)$  according to the Poisson equation

$$\Delta \Phi = 4\pi G \rho. \quad (3)$$

In [22,23], the mean-field approximation is shown to become exact if the thermodynamical limit  $N \rightarrow +\infty$  is properly taken. The self-gravitating Brownian gas has a rigorous canonical thermodynamic structure [8,16]. Therefore, it can be seen as the canonical counterpart of the usual  $N$ -star problem governed by Newton's equations and possessing a microcanonical structure.

On the other hand, in a biological context, a model of chemotactic aggregation exhibiting blow-up phenomena is provided by the system of equations

$$\frac{\partial \rho}{\partial t} = D \Delta \rho - \chi \nabla (\rho \nabla c), \quad (4)$$

$$\frac{\partial c}{\partial t} = D_0 (\Delta c + \lambda \rho), \quad (5)$$

where  $\rho$  is the concentration of the biological population (amoebae),  $c$  is the concentration of the substance secreted (acrasin), and  $\chi$  measures the strength of the chemotactic drift. Equations (4) and (5) can be viewed as a simplification of the Keller-Segel model [15]. In [24], the approximation  $\partial c / \partial t \approx 0$  is introduced and justified in the regime of strong diffusion. Equation (5) then becomes

$$\Delta c = -\lambda \rho, \quad (6)$$

which is strictly identical to Poisson's equation. This notion of two coupled diffusion fields where one of them responds much faster also occurs in the mean-field theory of diffusion limited aggregation [25]. Both models (self-gravitating Brownian particles and chemotaxis) are thus isomorphic provided that we make the identification  $\Phi \leftrightarrow -(4\pi G / \lambda \chi) c$ ,  $T \leftrightarrow (4\pi G D / \lambda \chi)$ , and  $\xi \leftrightarrow (4\pi G / \lambda \chi)$ . Introducing the mass  $M = \int \rho d^3 \mathbf{r}$  of the system and the radius  $R$  of the domain, we can show that the problem depends on the single dimensionless parameter  $\eta = \beta G M / R$  [8]. A large value of  $\eta$  corresponds to a small temperature  $T$  or a large mass  $M$ . Self-similar blow-up in various dimensions has been studied in [4,5] in the biological context. Related problems have been considered in [8,9,12] in the gravitational context, including a detailed study of the critical dimension  $d=2$  and of the postcollapse regime.

### B. The Smoluchowski-Poisson system

In the following, we shall use the gravitational terminology but we stress that our results are equally valid for the biological problem due to the above analogy. By rescaling

the physical parameters adequately [ $\mathbf{r}' = \mathbf{r} / R$ ,  $\rho' = \rho / (M / R^3)$ ,  $\Phi' = \Phi / (GM / R)$ ,  $T' = T / (GM / R)$ , and  $t' = t / (\xi R^3 / GM)$ ], it is possible to take  $M = R = G = \xi = 1$  without restriction. Then the equations of the problem are

$$\frac{\partial \rho}{\partial t} = \nabla (T \nabla \rho + \rho \nabla \Phi), \quad (7)$$

$$\Delta \Phi = 4\pi \rho, \quad (8)$$

with proper boundary conditions in order to impose a vanishing particle flux on the surface of the confining sphere. If we restrict ourselves to spherically symmetric solutions, the boundary conditions are

$$\frac{\partial \Phi}{\partial r}(0, t) = 0, \quad \Phi(1, t) = -1, \quad T \frac{\partial \rho}{\partial r}(1, t) + \rho(1, t) = 0. \quad (9)$$

With this rescaling, the equations depend only on the temperature  $T$ , or equivalently on the parameter  $\eta = 1 / T$ . As mentioned previously, a small temperature is equivalent (through a rescaling) to a large mass.

Integrating Eq. (8) once, we can rewrite the Smoluchowski-Poisson system in the form of a single integro-differential equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left\{ r^2 \left( T \frac{\partial \rho}{\partial r} + \frac{\rho}{r^2} \int_0^r \rho(r', t) 4\pi r'^2 dr' \right) \right\}. \quad (10)$$

The Smoluchowski-Poisson system is also equivalent to a single differential equation

$$\frac{\partial M}{\partial t} = T \left( \frac{\partial^2 M}{\partial r^2} - \frac{2}{r} \frac{\partial M}{\partial r} \right) + \frac{1}{r^2} M \frac{\partial M}{\partial r}, \quad (11)$$

for the quantity

$$M(r, t) = \int_0^r \rho(r', t) 4\pi r'^2 dr', \quad (12)$$

which represents the mass contained within the sphere of radius  $r$ . The appropriate boundary conditions are

$$M(0, t) = 0, \quad M(1, t) = 1. \quad (13)$$

## III. ASYMPTOTIC ESTIMATE OF THE COLLAPSE TIME

### A. Systematic expansion

We know that the Smoluchowski-Poisson system blows up for  $T < T_c$ , where  $T_c$  is a critical temperature below which there is no equilibrium state [8]. We shall place ourselves close to the critical point and expand the mass profile as

$$M(r, t) = M_c(r) + \epsilon M_1(r, t) + \epsilon^2 M_2(r, t) + \dots, \quad (14)$$

where  $M_c(r)$  is the equilibrium profile at  $T_c$  and  $\epsilon$  is a small parameter defined as

$$\epsilon = \left( \frac{T_c - T}{T_c} \right)^{1/2} \ll 1. \quad (15)$$

We also rescale the time according to

$$t = \tau/\epsilon. \quad (16)$$

If we substitute the expansion Eq. (14) in Eq. (11) and set equal the terms of the same order, we get to order 0

$$T_c \left( \frac{\partial^2 M_c}{\partial r^2} - \frac{2}{r} \frac{\partial M_c}{\partial r} \right) + \frac{M_c}{r^2} \frac{\partial M_c}{\partial r} = 0, \quad (17)$$

to order 1

$$T_c \left( \frac{\partial^2 M_1}{\partial r^2} - \frac{2}{r} \frac{\partial M_1}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial r} (M_1 M_c) = 0, \quad (18)$$

and to order 2

$$\begin{aligned} \frac{\partial M_1}{\partial \tau} = T_c & \left( \frac{\partial^2 M_2}{\partial r^2} - \frac{2}{r} \frac{\partial M_2}{\partial r} - \frac{\partial^2 M_c}{\partial r^2} + \frac{2}{r} \frac{\partial M_c}{\partial r} \right) \\ & + \frac{1}{r^2} \left( M_c \frac{\partial M_2}{\partial r} + M_1 \frac{\partial M_1}{\partial r} + M_2 \frac{\partial M_c}{\partial r} \right). \end{aligned} \quad (19)$$

The boundary conditions are

$$M_{n \geq 0}(0, \tau) = 0, \quad M'_{n \geq 0}(0, \tau) = 0, \quad (20)$$

$$M_c(1, \tau) = 1, \quad M_{n \geq 1}(1, \tau) = 0. \quad (21)$$

### B. Order 0: The equilibrium state $M_c(\xi)$

At equilibrium, the condition of hydrostatic balance  $T \nabla \rho + \rho \nabla \Phi = 0$  combined with the Gauss theorem  $d\Phi/dr = M(r)/r^2$  leads to the relation

$$M(r) = -Tr^2 \frac{d \ln \rho}{dr}. \quad (22)$$

Since  $M'(r) = 4\pi r^2 \rho$ , we obtain the differential equation

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{T d\rho}{\rho dr} \right) = -4\pi \rho. \quad (23)$$

Introducing the function  $\psi$  through the relation

$$\rho = \rho_0 e^{-\psi}, \quad (24)$$

where  $\rho_0$  is the central density, and using the normalized distance  $\xi = \alpha r$  where  $\alpha = (4\pi\rho_0/T)^{1/2}$ , we can rewrite the relation Eq. (22) in the form

$$M(\xi) = T \frac{\xi^2}{\alpha} \psi'(\xi). \quad (25)$$

Furthermore, according to Eq. (23), the function  $\psi$  is a solution of the Emden equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\psi}{d\xi} \right) = e^{-\psi}, \quad (26)$$

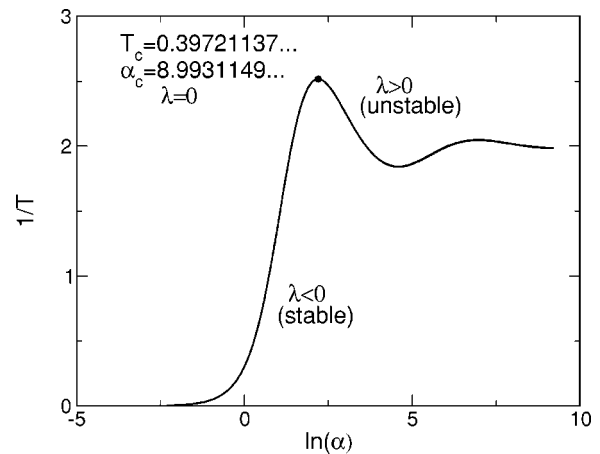


FIG. 1. We plot  $\eta(\alpha) = T^{-1}(\alpha)$  where  $\alpha = (4\pi\rho_0/T)^{1/2}$  parametrizes the series of equilibria. For  $\alpha > \alpha_c$ , unstable modes with  $\lambda > 0$  arise, which are solutions of Eq. (34). The minimum temperature  $T_c$  corresponds precisely to the point of marginal stability  $\lambda = 0$  [8].

$$\psi(0) = \psi'(0) = 0. \quad (27)$$

The Taylor expansion of the Emden function near the origin is

$$\psi = \frac{1}{6} \xi^2 - \frac{1}{120} \xi^4 + \dots \quad (28)$$

Using these results, we can check that the Emden equation (26) is equivalent to Eq. (17), as it should be. For a given temperature  $T \geq T_c$ , the Emden equation has to be solved from  $\xi = 0$  to  $\xi = \alpha$  such that

$$\alpha \psi'(\alpha) = \frac{1}{T}. \quad (29)$$

For the profile we are considering, at the critical temperature  $T_c$ , we have to stop the integration at  $\alpha_c$  such that  $T(\alpha_c)$  is minimum. The condition  $[1/T]'(\alpha_c) = 0$  is equivalent to

$$\frac{\alpha_c e^{-\psi(\alpha_c)}}{\psi'(\alpha_c)} = 1. \quad (30)$$

It is found numerically that  $\alpha_c = 8.9931149\dots$  and  $T_c = 0.39721137\dots$ . As an illustration, we plot  $T^{-1}(\alpha)$  as a function of  $\alpha$  in Fig. 1. The critical integrated mass density  $M_c(\xi)$  is shown in Fig. 2.

### C. Order 1: The function $F(\xi)$

Since Eq. (18) is linear, we look for solutions of the form

$$M_1 = A(\tau) F(\xi). \quad (31)$$

The function  $F(\xi)$  satisfies the differential equation

$$\frac{d^2 F}{d\xi^2} - \frac{2}{\xi} \frac{dF}{d\xi} + \frac{1}{\xi^2} \frac{d}{d\xi} (F \xi^2 \psi') = 0, \quad (32)$$

which is equivalent to

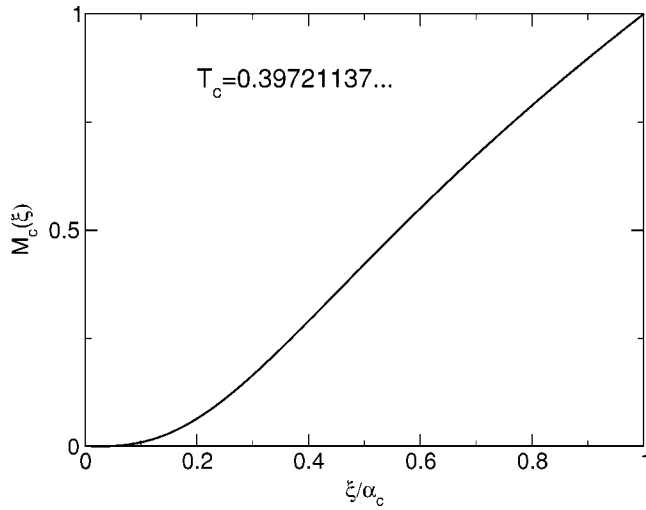


FIG. 2. The integrated mass density is plotted at the critical point.

$$\frac{d}{d\xi} \left( \frac{e^\psi dF}{\xi^2 d\xi} \right) + \frac{F}{\xi^2} = 0, \quad (33)$$

where we have used the Emden equation Eq. (26). Equation (33) is precisely the equation found at the critical point  $T_c$  by analyzing the linear stability of isothermal spheres [8]. Indeed, considering a perturbation  $\delta M \sim e^{\lambda t}$  around a stationary solution  $M(r)$  of Eq. (11), we get

$$T \left( \frac{\partial^2 \delta M}{\partial r^2} - \frac{2}{r} \frac{\partial \delta M}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial r} (M \delta M) = \lambda \delta M. \quad (34)$$

If  $\lambda < 0$ , the stationary solution is stable and, if  $\lambda > 0$ , the stationary solution is unstable. At the critical point  $T_c$  where  $\lambda = 0$  (marginal stability), we recover Eq. (18) with  $M_1 = \delta M$ . Equation (33) has to be solved with the boundary conditions

$$F(0) = F'(0) = 0. \quad (35)$$

We find that  $F(\xi)$  can be simply expressed in terms of  $\psi(\xi)$  as [14]

$$F(\xi) = \xi^3 e^{-\psi} - \xi^2 \psi'. \quad (36)$$

The condition  $F(\alpha_c) = 0$  determines the critical temperature  $T_c$ . Using Eq. (36), this condition is equivalent to

$$\frac{\alpha_c e^{-\psi(\alpha_c)}}{\psi'(\alpha_c)} = 1. \quad (37)$$

Comparing this result with Eq. (30), we find that the criterion of marginal stability ( $\lambda = 0$ ) corresponds to the point  $\alpha_c$  in the series of equilibria  $T = T(\alpha)$  where the temperature is minimum ( $[1/T]'(\alpha_c) = 0$ ) [8,14].

#### D. Order 2: The function $A(\tau)$

Using the previous results, Eq. (19) can be rewritten

$$F(\xi) \dot{A}(\tau) = T_c \alpha_c^2 \left[ \frac{\partial^2 M_2}{\partial \xi^2} - \frac{2}{\xi} \frac{\partial M_2}{\partial \xi} + \frac{1}{\xi^2} \frac{\partial}{\partial \xi} (M_2 \xi^2 \psi') \right] + \alpha_c T_c^2 \xi^2 \psi' e^{-\psi} + \frac{\alpha_c^3}{\xi^2} A^2 F \frac{dF}{d\xi}. \quad (38)$$

We now multiply Eq. (38) by a function  $\chi(\xi)$  and integrate between 0 and  $\alpha_c$ . We choose the function  $\chi$  (see below) so as to eliminate the terms where  $M_2$  appears. We are thus left with

$$\dot{A} = KA^2 + B, \quad (39)$$

where

$$K = \frac{\alpha_c^3 \int_0^{\alpha_c} (\chi/\xi^2) F(dF/d\xi) d\xi}{\int_0^{\alpha_c} \chi F d\xi}, \quad B = \alpha_c T_c^2 \frac{\int_0^{\alpha_c} \chi \xi^2 \psi' e^{-\psi} d\xi}{\int_0^{\alpha_c} \chi F d\xi}. \quad (40)$$

For future convenience, we note that

$$F'(\xi) = \xi^2 e^{-\psi} (2 - \xi \psi'). \quad (41)$$

By construction, the function  $\chi$  satisfies the integral relation

$$\int_0^{\alpha_c} \left[ \frac{\partial^2 M_2}{\partial \xi^2} - \frac{2}{\xi} \frac{\partial M_2}{\partial \xi} + \frac{1}{\xi^2} \frac{\partial}{\partial \xi} (M_2 \xi^2 \psi') \right] \chi(\xi) d\xi = 0. \quad (42)$$

Integrating by parts and using  $M_2(0) = M_2'(0) = M_2(\alpha_c) = 0$ , we obtain

$$\chi(\alpha_c) M_2'(\alpha_c, \tau) + \int_0^{\alpha_c} \left\{ \frac{d^2 \chi}{d\xi^2} + \left( \frac{2}{\xi} - \psi' \right) \frac{d\chi}{d\xi} + \frac{2}{\xi^2} (\xi \psi' - 1) \chi \right\} \times M_2(\xi, \tau) d\xi = 0. \quad (43)$$

Thus, we impose that  $\chi$  is a solution of the differential equation

$$\frac{d^2 \chi}{d\xi^2} + \left( \frac{2}{\xi} - \psi' \right) \frac{d\chi}{d\xi} + \frac{2}{\xi^2} (\xi \psi' - 1) \chi = 0, \quad (44)$$

with the boundary condition

$$\chi(0) = 0. \quad (45)$$

It turns out that the solution of this equation can be expressed in terms of the Emden function as

$$\chi(\xi) = \frac{1}{\xi^2} F(\xi) e^\psi = \xi - \psi' e^\psi, \quad (46)$$

with  $\chi'(0) = 2/3$ . The function  $\chi(\xi)$  is represented in Fig. 3 along with the function  $F(\xi)$ . Using Eq. (37), we readily check that  $\chi(\alpha_c) = 0$ . Therefore, the condition Eq. (42) is satisfied. Accordingly,  $A(\tau)$  is determined by Eq. (39), where  $K$  and  $B$  have to be evaluated using Eq. (40).

#### E. The blow-up time

The solution of Eq. (39) with  $A(\tau) \rightarrow -\infty$  as  $\tau \rightarrow 0$  is

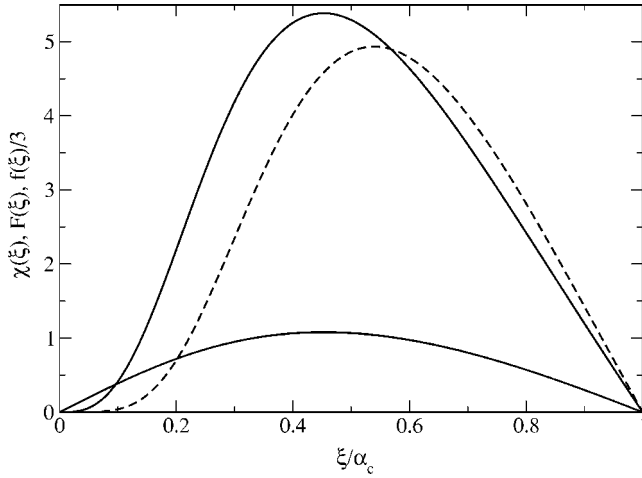


FIG. 3. We plot  $\chi(\xi)$  (lower full curve),  $F(\xi)$  (upper full line), and the eigensolution  $f(\xi)$  of Eq. (57) (dashed line;  $f(\xi)$  has been divided by a factor of 3 for convenience).

$$A(\tau) = \left(\frac{B}{K}\right)^{1/2} \tan\left[\tau(BK)^{1/2} - \frac{\pi}{2}\right]. \quad (47)$$

Returning to the original time variable, we get

$$A(t) = \left(\frac{B}{K}\right)^{1/2} \tan\left[t(1-T/T_c)^{1/2}(BK)^{1/2} - \frac{\pi}{2}\right]. \quad (48)$$

Then, we find that the blow-up occurs for

$$t_{coll} = t_*(\eta - \eta_c)^{-1/2}, \quad (49)$$

with

$$t_* = \pi \left(\frac{\eta_c}{BK}\right)^{1/2}. \quad (50)$$

We find numerically that  $K=62.560\,38\dots$  and  $B=0.471\,627\,4\dots$ . We thus get

$$t_{coll} = t_*(\eta - \eta_c)^{-1/2}, \quad t_* = 0.917\,677\,02\dots \quad (51)$$

This asymptotic result is compared in Fig. 4 with the exact law  $t_{coll}(T)$  obtained by solving the Smoluchowski-Poisson system numerically.

#### IV. RELAXATION TIME ESTIMATE FOR $T > T_c$

We now assume  $T > T_c$ , so that an equilibrium state exists. We look for a time dependent solution for  $M(r, t)$  which converges exponentially to the equilibrium profile

$$M(r, t) = M_T(r) + F_T(r)e^{-t/\tau}. \quad (52)$$

The purpose of this section is to compute the leading asymptotic form of the relaxation time  $\tau(T) \rightarrow +\infty$  as  $T$  goes to  $T_c$ . After inserting this ansatz into the equation of motion for  $M(r, t)$ , we obtain

$$F_T'' + \left(\phi' - \frac{2}{r}\right)F_T' + \alpha^2 e^{-\phi} F_T = -(\tau T)^{-1} F_T, \quad (53)$$

where

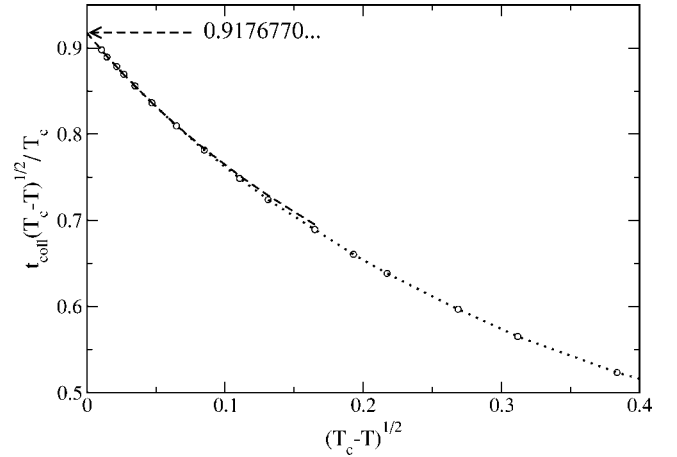


FIG. 4. We plot  $t_{coll} \times (T_c - T)^{1/2} / T_c$  computed numerically as a function of  $(T_c - T)^{1/2}$ , which should converge to  $t_* = 0.917\,977\,02\dots$ . A quadratic fit in the region  $(T_c - T)^{1/2} < 0.05$  retrieves the first four digits of  $t_*$ .

$$\phi(r) = \psi(\alpha r). \quad (54)$$

We have used  $r$  as a coordinate (instead of  $\xi$ ) so as to remain within a fixed interval of space  $0 \leq r \leq 1$  as we make the perturbative expansion in  $\alpha$  (see below). When  $T \rightarrow T_c$ , we have  $\alpha \rightarrow \alpha_c$ ,  $\tau \rightarrow +\infty$ , and  $F_T(r) \rightarrow F(\alpha_c r)$ , where  $F(\xi)$  is given by Eq. (36). We now expand Eq. (53) in power of  $\varepsilon = (\alpha_c - \alpha) / \alpha_c$ , by introducing  $f(\xi)$  and  $\mu$  such that

$$F_T(r) = F(\alpha_c r) + \varepsilon f(\alpha_c r) + O(\varepsilon^2) \quad (55)$$

and

$$(\tau \alpha_c^2 T_c)^{-1} = \mu \varepsilon + O(\varepsilon^2). \quad (56)$$

The relation between  $T - T_c$  and  $\varepsilon$  will be given later. Collecting terms proportional to  $\varepsilon$ , and setting  $\xi = \alpha_c r$ , we obtain

$$f'' + \left(\psi' - \frac{2}{\xi}\right)f' + f e^{-\psi} - \frac{2}{\xi^2} F F' = -\mu F, \quad (57)$$

subject to the boundary condition  $f(0) = f(\alpha_c) = 0$ . This eigenvalue problem selects a unique value for  $\mu$  and hence for  $\tau$ . We were not able to solve this problem analytically. However, we remark that this equation is similar to Eq. (38), without the term leading to  $B$ . We can therefore use the same trick as in Sec. III D. After multiplying Eq. (57) by the function  $\chi$  defined in the previous section, we end up with

$$\mu = \frac{\int_0^{\alpha_c} [2\chi(\xi)/\xi^2] F(\xi) F'(\xi) d\xi}{\int_0^{\alpha_c} \chi(\xi) F(\xi) d\xi} = \frac{2K}{\alpha_c^3} = 0.172\,027\,92\dots \quad (58)$$

The direct numerical solution of Eq. (57) leads of course to the same value. The function  $f(\xi)$  is plotted in Fig. 3.

We now find a simple relation between  $T - T_c$  and  $\varepsilon$  close to  $T_c$  [14]. Using the condition  $\alpha \psi'(\alpha) = 1/T$  and the fact that  $(\xi \psi')'_{\alpha_c} = 0$ , we obtain

$$\frac{1}{T_c} - \frac{1}{T} = \frac{(\alpha_c - \alpha)^2}{2} (\xi \psi')''|_{\alpha_c} + O((\alpha_c - \alpha)^3), \quad (59)$$

which up to leading order in  $\varepsilon$  can be written in the form

$$\varepsilon \sim \sqrt{\frac{2(T - T_c)}{T_c(1 - 2T_c)}}. \quad (60)$$

Finally, we obtain the following diverging behavior for  $\tau$ :

$$\tau^{-1} \sim \mu \sqrt{\frac{2T_c(T - T_c)}{1 - 2T_c}} \alpha_c^2. \quad (61)$$

Numerically, we get

$$\tau \sim c_T (T - T_c)^{-1/2} \sim c_\eta (\eta_c - \eta)^{-1/2} \quad (62)$$

with

$$c_T = 0.036\ 563\ 056 \dots, \quad c_\eta = 0.092\ 049\ 37 \dots \quad (63)$$

### V. THE DECAY OF THE DENSITY PROFILE AT $T = T_c$

Strictly at  $T = T_c$ ,  $\tau = +\infty$ , and one expects a slow convergence to the equilibrium profile  $M_c(r)$ . Let us write

$$M(r, t) = M_c(r) - A(t)M_1(r) - B(t)M_2(r) + \dots \quad (64)$$

We implicitly assume that  $B(t) \ll A(t)$  as  $t \rightarrow +\infty$ . Moreover, the presence of the minus signs in Eq. (64) anticipates the fact that on starting from a flat density the central density increases. In [8], it was argued based on a heuristic argument that  $A(t) \sim t^{-1}$ . It is the purpose of this section to justify this statement as well as to give more quantitative estimates.

Let us postulate for the moment the natural choice  $B(t) \sim A(t)^2$ , which will be justified hereafter. Inserting this ansatz again in the dynamical equation for  $M(r, t)$  and collecting terms of order  $A(t)$  and  $A(t)^2$ , we immediately find that the first equation leads to

$$M_1(r) = \sqrt{\frac{2T_c}{\alpha_c}} F(\alpha_c r), \quad (65)$$

where the constant has been set for later convenience, as multiplying  $M_1$  and dividing  $A$  by the same constant leaves the expansion for  $M$  unchanged. Setting  $M_2(r) = g(\alpha_c r)$ , we find that  $g$  satisfies

$$g'' + \left( \psi' - \frac{2}{\xi} \right) g' + g e^{-\psi} - \frac{2}{\xi^2} F F' = -\lambda F, \quad (66)$$

with

$$\lambda = -\frac{\dot{A}}{A^2} \sqrt{\frac{2\alpha_c}{T_c}} \alpha_c^{-3}, \quad (67)$$

which must be a constant independent of time. Thanks to the proper choice of the constant in Eq. (65), we find that Eq. (66) and Eq. (57) are identical. The only physical solution with  $g(0) = g(\alpha_c) = 0$  corresponds to the eigenvalue  $\lambda = \mu$ . Finally, we find

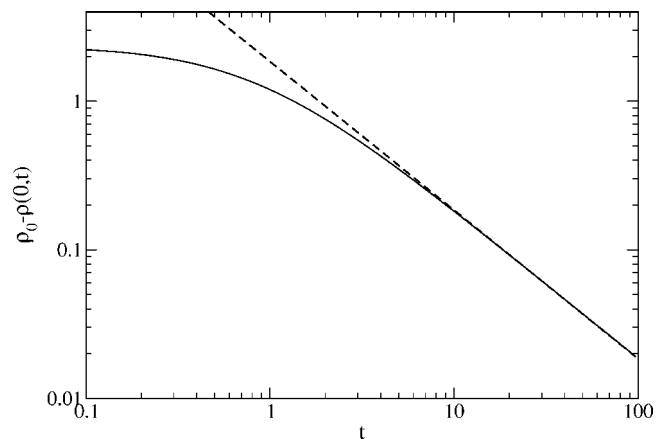


FIG. 5. At  $T = T_c$ , and starting from a uniform mass density, we have computed numerically  $\rho_0 - \rho(r=0, t)$ , where  $\rho_0$  is the equilibrium central density. This is compared with the exact and universal large time estimate  $c_\rho/t$ , with  $c_\rho = (\pi\mu)^{-1} = 1.850\ 338\ 5\dots$  (dashed line).

$$A(t) \sim t^{-1} \mu^{-1} \sqrt{\frac{2\alpha_c}{T_c}} \alpha_c^{-3}. \quad (68)$$

Once the signs have been fixed in Eq. (64), we indeed find that the small time correction to the central density is necessarily negative, as  $M_1$ ,  $M_2$  (or  $g$ ), and  $A$  are found to be positive. Therefore, the equilibrium profile  $M_c(r)$  is approached from below, which is natural since an excess of mass would provoke gravitational collapse.

Let us illustrate this result by calculating the correction to the central density. Using Eq. (65) and Eq. (68), and the fact that  $M_c(r) \sim (4\pi/3)\rho_0 r^3$  and  $F(\xi) \sim (2/3)\xi^3$ , we find that for large time the central density converges to  $\rho_0$  from below in a universal manner,

$$\rho_0 - \rho(r=0, t) = \frac{c_\rho}{t} + O(t^{-2}), \quad (69)$$

with

$$c_\rho = (\pi\mu)^{-1} = 1.850\ 338\ 5 \dots \quad (70)$$

This result is illustrated quantitatively in Fig. 5.

Finally, we discuss the *a priori* unjustified choice  $B(t) \sim A^2(t)$ . The case  $B(t) \ll A^2(t)$  is clearly impossible as it amounts to taking  $g=0$  in Eq. (66), leading to an equation that is never satisfied by  $F$ , whatever the choice for  $A(t)$ . The opposite case  $B(t) \gg A^2(t)$  leads to an equation similar to Eq. (66) but where the term proportional to  $FF'$  is absent. For this equation, it is clear that if a solution  $g_0$  exists for a specific  $\lambda_0$ , then  $\lambda/\lambda_0 g_0$  is also a solution associated with  $\lambda$ . This implies that the only solution satisfying the boundary condition  $g(0) = g(\alpha_c) = 0$  is actually the null function associated to  $\lambda=0$ , which is not permitted for  $T > T_c$ . Hence  $B(t) \sim A^2(t)$  is the only possibility leading to a physically sensible solution.

## VI. THE PULSATION PERIOD OF ISOTHERMAL SPHERES

In Sec. IV, we have determined the relaxation time of self-gravitating Brownian particles described by the Smoluchowski-Poisson system close to the critical point  $T_c$ . Writing the perturbation as  $\delta\rho \sim e^{\lambda t}$  where  $\lambda = -1/\tau$ , the eigenvalue equation for  $\lambda$  can be written in dimensional form as [8]

$$\frac{d}{dr} \left( \frac{1}{4\pi\rho r^2} \frac{dq}{dr} \right) + \frac{Gq}{Tr^2} = \frac{\lambda\xi}{4\pi\rho Tr^2} q, \quad (71)$$

where  $q(r) = \delta M(r)$ . This equation is the same as Eq. (53). Using the result of Sec. IV, and returning to dimensional parameters, the inverse relaxation time is given by

$$\lambda = \mp \frac{1}{c_\eta} (\eta_c - \eta)^{1/2} \frac{GM}{\xi R^3}. \quad (72)$$

The sign  $-$  corresponds to  $\alpha < \alpha_c$  and leads to exponentially damped perturbations (stable). The sign  $+$  corresponds to  $\alpha > \alpha_c$  and leads to exponentially growing perturbations (unstable).

We now consider the case of gaseous self-gravitating systems (stars) described by the Euler-Jeans equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (73)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi, \quad (74)$$

$$\Delta \Phi = 4\pi G \rho. \quad (75)$$

We use an isothermal equation of state  $p = \rho T$  and we assume that the system is confined within a region of size  $R$ . These equations provide the usual starting point for the analysis of the gravitational stability of gaseous systems. The case of an infinite homogeneous medium was first investigated by Jeans [26] in his classical study. The case of a finite inhomogeneous isothermal medium was considered by Chavanis [14]. In that case, the equation of pulsation can be written in the form

$$\frac{d}{dr} \left( \frac{1}{4\pi\rho r^2} \frac{dq}{dr} \right) + \frac{Gq}{Tr^2} = \frac{\lambda^2}{4\pi\rho Tr^2} q, \quad (76)$$

where, as before,  $\delta\rho \sim e^{\lambda t}$ . Comparing with Eq. (71), we see that the results of Sec. IV remain valid provided that  $\lambda\xi$  is replaced by  $\lambda^2$ . Therefore, Eq. (72) translates into

$$\lambda^2 = \mp \frac{1}{c_\eta} (\eta_c - \eta)^{1/2} \frac{GM}{R^3}. \quad (77)$$

For  $\alpha > \alpha_c$ , the eigenvalue  $\lambda = \pm \sqrt{\lambda^2}$  can be positive, implying instability. For  $\alpha < \alpha_c$ , the eigenvalue  $\lambda = \pm i\sqrt{-\lambda^2}$  is purely imaginary so that the time evolution of the perturbation has an oscillatory nature. Close to the critical point, the pulsation period is given by

$$\omega = \frac{1}{\sqrt{c_\eta}} (\eta_c - \eta)^{1/4} \left( \frac{GM}{R^3} \right)^{1/2}. \quad (78)$$

## VII. CONCLUSION

In this paper, we have obtained the asymptotic expressions of the blow-up time and relaxation time of self-gravitating Brownian particles and biological populations close to the critical point of collapse in  $d=3$ . An excellent agreement is obtained by numerically solving the Smoluchowski-Poisson system. This study confirms and improves the results obtained in [8] on the basis of heuristic arguments. A possible extension of this work would be to consider the case of polytropic distributions arising in case of anomalous diffusion [11]. More generally, we could consider the generalized Smoluchowski equation (81) proposed in [18] which is valid for an arbitrary equation of state. This equation can provide (among other applications) a generalized chemotactic model valid when the diffusion coefficient or the drift term depends on the density. These extensions will be considered in future works.

It can be noted that the exact determination of the collapse time is specific to our model. As far as we know, in other works on gravitational collapse, the collapse time  $t_{coll}$  is just left as a constant of integration. Our situation is original because, as we work in a bounded domain, we can have either equilibrium (for  $T > T_c$ ) or collapse (for  $T < T_c$ ). Then the collapse time depends on the distance to  $T_c$  at which it diverges. There is no critical temperature in other models of gravitational collapse which consider an infinite domain. Therefore, the collapse time is probably not universal in these models (and depends on the initial conditions), contrary to our situation close to  $T_c$ .

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