Solutions for solitons in nonlinear optically induced lattices

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We calculate stationary configurations of superposed states "soliton + cnoidal wave lattice" of the vector nonlinear Schrödinger equation, using the Darboux transformation technique. The obtained expressions contain the Jacobi elliptic and theta functions, and are easily manageable. There are five stationary configurations, in which one of the defocusing media is stable, while those of the focusing medium are classified into two weakly unstable and two unstable. The checking of the solutions as well as the construction of their typical shapes is accomplished with the help of symbolic package MATHEMATICA.

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Recently there has been new interest on optically induced lattices and the localization of light in those gratings [1–3]. It was shown that the vector nonlinear Schrödinger (VNLS) equation

$$\partial_z \psi_j = i\partial_x^2 \psi_j + 2i\sigma(|\psi_1|^2 + |\psi_2|^2)\psi_j, \quad j = 1,2 \tag{1}$$

describes the interaction of light with those gratings in a photorefractive crystal in the limit of weak saturation regime. Strong incoherent interaction of such a grating with a probe beam facilitates the formation of a noble type of a composite optical soliton, where one of the components [described by ψ_1 in Eq. (1)] creates periodic photonic structure, while the other component (ψ_2) experiences Bragg reflection from this structure and forms gap solitons.

The dynamics of the band-gap lattice solitons can be approximately described by coupled nonlinear Schrödinger equations [4], which reduce to VNLS equation (1) under a proper limit [5]. Reference [3] discusses many important physical properties of the solutions of (soliton + lattice) type of these equations through numerical studies. Especially the stability and classifications of the solutions describing these composite states are discussed in depth. Clearly, it will be helpful if we can find analytic expressions of solutions describing these composite states. These general analytical solutions can be used to obtain some important characteristics of lattice solitons in a simple form. Of course the analytic expression is only possible for the integrable nonlinear equation, which in this case corresponds to the so-called desaturable limit.

Quasiperiodic solutions in terms of N-phase θ functions for the VNLS equation are derived in Ref. [6]. The (soliton + lattice) solution can be obtained from these general solutions by taking degenerate limit of the two-phase solution, see Ref. [7] for application of this procedure to the case of single-component nonlinear Schrödinger equation (NLSE). But solutions obtained by above finite-band method have the so-called "effectivization" problem, which is related to extracting the physical solutions by taking proper initial conditions [8,9]. Much more, these solutions have rather complicated form, which makes them difficult to apply to real physical situations.

To get around these difficulties, various direct methods using appropriate ansatz are employed to obtain a series of special periodic solutions [10,11]. For example, Ref. [11] employs the following ansatz:

$$\psi_1 = \sqrt{F\wp^2 + A\wp + B}\,e^{i\theta(z,x)}, \quad \psi_2 = \sqrt{G\wp^2 + D\wp + E}\,e^{i\phi(z,x)}, \tag{2}$$

where $\wp = \wp(x - cz, g_2, g_3)$ is the Weierstrass function. It is easy to see that the absolute values of these solutions (both $|\psi_1|, |\psi_2|$) are periodic over the entire x axis. Much more, to obtain soliton solutions from Eq. (2), it is required to take special values on g_2, g_3. But from this procedure, a pair of solitons emerge in ψ_1, ψ_2 simultaneously. Contrary to these facts, our solutions, for example in Eq. (13), are constituted by a soliton (ψ_2) plus a periodic lattice (ψ_1), the periodicity of which is distorted around the soliton due to their nonlinear coupling. It is clear that the characters of special solutions (2) from ansatz are different from those of ours, and they do not provide the required analytic solutions for solitons in a lattice. In this paper, we employ a simple, but powerful soliton finding technique based on the Darboux transformation (DT) [12] to obtain (soliton +lattice) solutions. The results are compact and easily manageable, at least when we use symbolic packages such as MATHEMATICA. See the related problem in the case of single-component nonlinear Schrödinger equation in Ref. [13].

We first bring the VNLS equation into a matrix form in terms of 3×3 matrices E, T and $\tilde{E} = [T, E]$,

$$E = \begin{pmatrix} 0 & \sigma\psi_1 & \sigma\psi_2 \\ -\psi_1^* & 0 & 0 \\ -\psi_2^* & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} i/2 & 0 & 0 \\ 0 & -i/2 & 0 \\ 0 & 0 & -i/2 \end{pmatrix}, \tag{3}$$

such that

$$\partial_z E = \partial_x^2 \tilde{E} - 2E^2 \tilde{E}. \tag{4}$$

One can readily check that the components of Eq. (4) are indeed equivalent to the VNLS equation in Eq. (1). The signature σ is either 1 or -1 depending on whether the group velocity dispersion is abnormal ($\sigma=1$) or normal ($\sigma=-1$), or

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the waveguide is self-focusing ($\sigma=1$) or self-defocusing ($\sigma=-1$). One advantage of using matrices is that we can write down the associated linear equation (Lax pair):

$$(\partial_x - E - i\beta T)\Psi = 0, \quad (\partial_z - E\tilde{E} - \partial_x\tilde{E} + i\beta E - \beta^2 T)\Psi = 0, \tag{5}$$

where β is an arbitrary number and $\Psi(z,x,\lambda)$ is a three-component vector.

We now apply the DT to obtain a superposed solution of (soliton + cnoidal wave lattice). When a DT is applied on a given starting solution (cnoidal wave, in this case), it gives a new solution of type (soliton + starting solution) [14,15]. Let us denote a starting solution as $\psi_1=\psi_1^0$, $\psi_2=\psi_2^0$. At this point, we need a solution of the linear equations (5) where ψ_i, $i=1,2$ in E are substituted by the starting solutions, ψ_i^0, $i=1,2$. We denote this solution as a three-component vector

$$\Psi = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix}.$$

The DT now gives the wanted superposed solution ψ_i^s, $i=1,2$ by

$$\psi_i^s(z,x) = \psi_i^0(z,x) - 2\beta\sigma \frac{s_0 s_i^*}{|s_0|^2 + \sigma\sum_{j=1,2}|s_j|^2}, \quad i=1,2. \tag{6}$$

In fact, it can be explicitly checked that ψ_i^s, $i=1,2$ in Eq. (6) are new solutions of the VNLS equation (1) by using the fact that s_i satisfy the associated linear equations (5).

We need stable configurations of superposed states, which avoid modulational instabilities. Numerical studies in Ref. [3] show there exist two weakly (oscillatory) unstable configurations in a focusing medium (denoted by cases I and II) and one (linearly and dynamically) stable configuration in the defocusing case (case III). There are two more stationary configurations, though they are unstable. We explain the corresponding analytic solutions of these three cases (I, II, and III) in sequence, in addition to one unstable configuration. The remaining unstable one can be similarly constructed, but is not presented in this paper. We first describe the DT method with the starting solution of dn-type (A_1 in the notation of Ref. [3]), even though the resulting configuration is unstable. For the starting solution

$$\psi_1^0(z,x) = p \quad \mathrm{dn}(px,k)e^{ip^2(2-k^2)z}, \quad \psi_2^0 = 0, \tag{7}$$

(cn, dn, sn are the standard Jacobi elliptic functions and $k\in(0,1)$ is the modulus of the Jacobi function. As far as elliptic functions are involved we employ terminology and notation of Ref.[16]) the solution of the linear equation becomes

$$s_0 = \exp[ip^2(2-k^2)z/2]\exp[i(\gamma z + \delta px)]\,\frac{\Theta(px-u)}{\Theta(px)},$$

$$s_1 = -\exp[-ip^2(2-k^2)z]\frac{\mathrm{sn}u \quad \mathrm{dn}(px-u)}{\mathrm{cn}u}s_0,$$

$$s_2 = a \quad \exp(\beta x/2 - i\beta^2 z/2). \tag{8}$$

Here, a is an arbitrary constant and the parameter β is related to a real parameter u as

$$\beta = -p\frac{\mathrm{dn}u}{\mathrm{sn}u \quad \mathrm{cn}u}, \tag{9}$$

and γ, δ in Eq. (8) are

$$\gamma = -\frac{p^2}{2}\left[\frac{\mathrm{dn}^2 u}{\mathrm{cn}^2 u} - \frac{1}{\mathrm{sn}^2 u}\right],$$

$$\delta = -i\frac{\Theta'(u)}{\Theta(u)} - \frac{i}{2}\frac{\mathrm{dn}u}{\mathrm{sn}u \quad \mathrm{cn}u} + i\frac{\mathrm{sn}u \quad \mathrm{dn}u}{\mathrm{cn}u}. \tag{10}$$

Here the Jacobi theta function Θ is defined by the complete elliptic integral of the first (second) kind $K(E)$,

$$\Theta(u) = \theta_4\left(\frac{\pi u}{2K}\right) = 1 + 2\sum(-)^n q^{n^2}\cos\left(\frac{n\pi u}{K}\right), \tag{11}$$

with $q=\exp(-\pi K'/K)$, $K'\equiv K\sqrt{(1-k^2)}$. To check that s_0, s_1, s_2 in Eq. (8) indeed satisfy the linear equation (5), we use the following identity [16,17];

$$\int_0^u \mathrm{dn}^2 u\,du = \frac{\Theta'(u)}{\Theta(u)} + \frac{E}{K}u. \tag{12}$$

Using Eq. (12) and the addition theorem of Jacobi's elliptic functions, the ∂_x part of the linear equation (5) can be proved. ∂_z-part equation is similarly proved using identities of elliptic functions. This type of solution was first introduced by Sym in a different context (description of vortex motion in hydrodynamics) [18]. It was then applied to an NLSE-related problem in Ref. [13].

Then, the DT in Eq. (6) gives the superposed configuration of "soliton + cnoidal wave":

$$\psi_1^s = \exp[ip^2(2-k^2)z]p\left\{\mathrm{dn}px - 2\frac{\mathrm{dn}u\ \mathrm{dn}(px-u)}{\mathrm{cn}^2 u}\right.$$
$$\left.\times\left(1 + \frac{\mathrm{sn}^2 u\ \mathrm{dn}^2(px-u)}{\mathrm{cn}^2 u} + a^2M^2\right)^{-1}\right\},$$

$$\psi_2^s = \exp\left[ip^2\left(1+\frac{\mathrm{dn}^2 u}{\mathrm{sn}^2 u}\right)z\right]ap\frac{\mathrm{dn}u}{\mathrm{sn}u\ \mathrm{cn}u}$$
$$\times\left[\left(1 + \frac{\mathrm{sn}^2 u\ \mathrm{dn}^2(px-u)}{\mathrm{cn}^2 u}\right)\Big/ M + a^2M\right]^{-1}, \tag{13}$$

where

$$M = \exp\left[-p\left(\frac{\Theta'(u)}{\Theta(u)} + \frac{\mathrm{cn}u \quad \mathrm{dn}u}{\mathrm{sn}u}\right)x\right]\frac{\Theta(px)}{\Theta(px-u)}. \tag{14}$$

Here, ψ_1^s describes a distorted periodic lattice and ψ_2^s describes a soliton. Figure 1 shows the superposed configuration of Eq. (13). The solid line shows ψ_1 component (at $z=0$) while the dashed line is for ψ_2 component with parameters $p=1$, $k=0.9$, $u=2.9$, $\beta=1.87$, $a=1$. Here parameters u and β are related by Eq. (9).

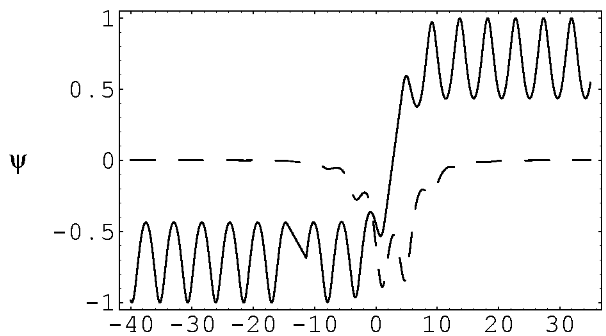

FIG. 1. A bright soliton on a dn background. Solid line: ψ_1; dashed line: ψ_2, with parameters $p=1$, $k=0.9$, $u=2.9$, $\beta=1.87$, $a=1$.

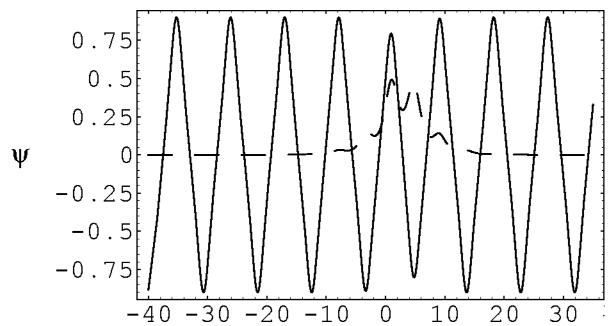

FIG. 2. A bright soliton on a cn background, case I. Solid line: ψ_1; dashed line: ψ_2, with parameters $p=1$, $k=0.9$, $u=2.9$, $\beta=0.58$, $a=1$.

The starting solution (periodic lattice) for the cases I and II is (A_2 in the notation of Ref. [3])

$$\psi_1^0(z,x)=kp \quad \mathrm{cn}(px,k)e^{ip^2(2k^2-1)z}, \quad \psi_2^0=0. \tag{15}$$

This lattice can be obtained from the dn lattice by using an identity $\mathrm{dn}(ku,1/k)=\mathrm{cn}(u,k)$. Similarly, a simple way to obtain the superposed configuration of the case I is by substituting $k\to 1/k$, $p\to kp$ in Eq. (13). Explicit expressions from this procedure are

$$\psi_1^s=\exp[ip^2(2k^2-1)z]kp\left\{\mathrm{cn}px-2\frac{\mathrm{cn}u\ \mathrm{cn}(px-u)}{\mathrm{dn}^2u}\times\left(1+\frac{k^2\mathrm{sn}^2u\ \mathrm{cn}^2(px-u)}{\mathrm{dn}^2u}+a^2|M|^2\right)^{-1}\right\},$$

$$\psi_2^s=\exp\left[ip^2\left(k^2+\frac{\mathrm{cn}^2u}{\mathrm{sn}^2u}\right)z\right]2ap\frac{\mathrm{cn}u}{\mathrm{sn}u\ \mathrm{dn}u}\times\left[\left(1+\frac{k^2\mathrm{sn}^2u\quad \mathrm{cn}^2(px-u)}{\mathrm{dn}^2u}\right)\Big/ M^*+a^2M\right]^{-1}, \tag{16}$$

where

$$M=\exp\left[-p\left(\frac{\Theta_c'(u)}{\Theta_c(u)}+\frac{\mathrm{cn}u\quad \mathrm{dn}u}{\mathrm{sn}u}\right)x\right]\frac{\Theta_c(px)}{\Theta_c(px-u)},$$

$$\Theta_c(x)\equiv 1+2\sum_{n=1}^{\infty}(-)^n\exp\left(-n^2\pi\frac{K'}{K-iK'}\right)\cos\frac{n\pi x}{K-iK'}. \tag{17}$$

Figure 2 shows the superposed configuration in Eq. (16). The solid line shows ψ_1 component (at $z=0$) while the dashed line is for $\exp(-2.31i)\psi_2$(at $z=0$) with parameters $p=1$, $k=0.9$, $u=2.9$, $\beta=0.58$, $a=1$.

The superposed configuration of the case II can be obtained from the case I by substituting $u\to u+iK'$. Explicit expression from this substitution is

$$\psi_1^s=\exp[ip^2(2k^2-1)z]kp\left\{\mathrm{cn}px+2\frac{\mathrm{sn}u\ \mathrm{dn}u\ \mathrm{dn}(px-u)}{k^2\mathrm{cn}^2u\ \mathrm{sn}(px-u)}\times\left(1+\frac{\mathrm{dn}^2(px-u)}{k^2\mathrm{cn}^2u\ \mathrm{sn}^2(px-u)}+a^2|M|^2\right)^{-1}\right\},$$

$$\psi_2^s=\exp[ip^2(k^2-\mathrm{dn}^2u)z]2ap\frac{\mathrm{dn}u\ \mathrm{sn}u}{\mathrm{cn}u}\times\left[\left(1+\frac{\mathrm{dn}^2(px-u)}{k^2\mathrm{cn}^2u\ \mathrm{sn}^2(px-u)}\right)\Big/ M^*+a^2M\right]^{-1}, \tag{18}$$

where

$$M=\exp\left[-p\left(\frac{\Theta_c'(u+iK')}{\Theta_c(u+iK')}-\frac{\mathrm{cn}u\quad \mathrm{dn}u}{\mathrm{sn}u}\right)x\right]\frac{\Theta_c(px)}{\Theta_c(px-u-iK')}. \tag{19}$$

Figure 3 shows the superposed configuration in Eq. (18). The solid line shows ψ_1 component at $z=0$ while the dashed line is for $\exp(-2.84i)\psi_2$ at $z=0$ with parameters $p=1$, $k=0.9$, $u=2.9$, $\beta=1.73$, $a=1$.

Finally, the superposed configuration of the case III (sn lattice, defocusing medium, A_0 in the notation of Ref. [3]) is obtained from the dn-lattice result in Eq. (13) by substituting

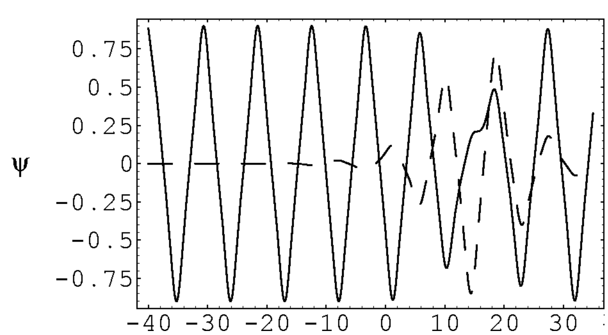

FIG. 3. A bright soliton on a cn background, case II. Solid line: ψ_1; dashed line: ψ_2, with parameters $p=1$, $k=0.9$, $u=2.9$, $\beta=1.73$, $a=1$.

$k \rightarrow ik$, $p \rightarrow ip$, $u \rightarrow iu$. Explicit expressions from this procedure are

$$\psi_1^s = \exp[-ip^2(k^2+1)z]kp\left\{ \mathrm{sn}(px+K) - 2\frac{(1-k^2)\mathrm{sn}u}{k^2\mathrm{cn}^2u\ \mathrm{sn}(px-u)} \times\left(1 - \frac{\mathrm{dn}^2u}{k^2\mathrm{cn}^2u\ \mathrm{sn}^2(px-u)} - a^2|M|^2\right)^{-1}\right\},$$

$$\psi_2^s = \exp\{ip^2[k^2+(1-k^2)/\mathrm{dn}^2u]z\}2ap(1-k^2)\frac{\mathrm{sn}u}{\mathrm{cn}u\ \mathrm{dn}u} \times\left[\left(1-\frac{\mathrm{dn}^2u}{k^2\mathrm{cn}^2u\ \mathrm{sn}^2(px-u)}\right)\Big/ M^* - a^2M\right]^{-1}, \tag{20}$$

where

$$M = \exp\left[-p\left(\frac{\Theta_s'(u-K-iK')}{\Theta_s(u-K-iK')} - \frac{\mathrm{cn}u}{\mathrm{sn}u\quad \mathrm{dn}u}\right)x\right]\frac{\Theta_s(px)}{\Theta_s(px-u+K+iK')},$$

$$\Theta_s(x) \equiv 1 + 2\sum_{n=1}^{\infty}(-)^{n(n+1)}\exp\left(-n^2\pi\frac{K}{K'}\right)\cos\frac{in\pi x}{K'}. \tag{21}$$

Figure 4 shows the superposed configuration in Eq. (20). The solid line shows $-i\psi_1$, while the dashed line is for ψ_2 with parameters $p=1$, $k=0.9$, $u=2.9$, $\beta=1.29$, $a=1$.

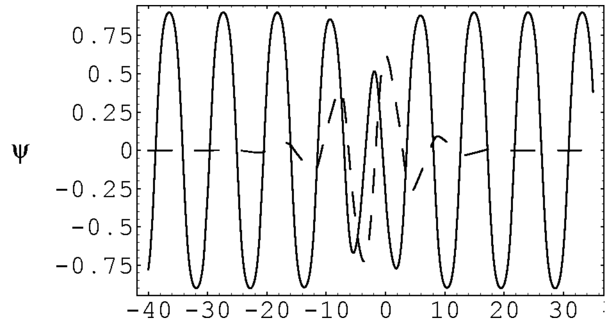

FIG. 4. A bright soliton on a sn background, case III. Solid line: ψ_1; dashed line: ψ_2, with parameters $p=1$, $k=0.9$, $u=2.9$, $\beta=1.29$, $a=1$.

In this paper, we give explicit expressions for four stationary configurations of "soliton + cnoidal wave lattice," each having background lattices of dn, cn (cases I and II), and sn (case III). In the case of cn background, we obtain a new solution (case II) from a given one (case I) by taking a substitution of $u \rightarrow u+iK'$. Similarly, we can obtain another stationary solution from the solution (13) having dn-type cnoidal background. Thus we conclude that there are five stationary configurations of superposed states, where two of them (dn lattice) are unstable, other two (cases I and II of cn lattice) are weakly stable, and final one (case III of sn lattice) is stable. The relation of our solutions with the band-gap structure of the linear spectrum of the periodic structure can be found in Ref. [3].

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