Synchronization-based estimation of all parameters of chaotic systems from time series

Debin Huang*
Department of Mathematics, Shanghai University, Shanghai 200436, China
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By a simple combination of adaptive scheme and linear feedback with the updated feedback strength, for a large class of chaotic systems it is proved rigorously by using the invariance principle of differential equations that all unknown model parameters can be estimated dynamically. This approach supplies a systematic and analytical procedure for estimating parameters from time series, and it is simple to implement in practice. In addition, this method is quite robust against the effect of noise and able to respond rapidly to changes in operating parameters of the experimental system. Lorenz and Rössler hyperchaos systems are used to illustrate the validity of this technique.

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Synchronization of unidirectionally coupled chaotic systems and its potential applications in engineering are currently a field of great interest (see [1–7] and references cited therein). An interesting application of chaotic synchronization is to analyze the time series of chaotic systems when partial information about the experimental systems is available [8–12]. Assuming that the number of independent variables and the structure of underlying dynamical equations for a chaotic system are known, we address the problem of estimating all model parameters of the experimental system.

In Refs. [8–10], some schemes such as autosynchronization, error minimization, and the Huberman-Lumer scheme [13] were developed to solve the above problem. However, just as stated in [11], these techniques admit a certain limitation. A new online scheme based on the least-squares approach was recently used to develop a general and robust method for deriving the dynamical system governing the evolution of all model parameters of a chaotic system; see [11]. Note that all methods referred above are almost numerical. For example, the negativity of all conditional Lyapunov exponents of the error system is used to guarantee synchronization between systems in these methods. However, it has recently been reported that the negativity of the conditional Lyapunov exponents is neither a sufficient condition nor a necessary condition for chaotic synchronization; see [14,15] and references cited therein. Due to numerical considerations, some additive parameters (e.g., feedback constants, etc.; see [10,11]) have to be numerically determined.

In this paper, for a large class of chaotic systems we give an analytical and systematic procedure to estimate dynamically all model parameters from time series. By a simple combination of adaptive control and linear feedback with the updated feedback strength, it is proved rigorously by using the invariance principle of differential equations that all unknown parameters can be estimated dynamically from time series of the experimental system. It is crucial for this technique to adapt duly the feedback strength of the linear feedback, which is different from the traditional linear feedback where the feedback constant is fixed. The adaptive controller of parameters—i.e., the dynamical system governing the evolution of all parameters—and the update law of linear feedback strength are given explicitly without determining any additive parameters. When time series for variables of the experimental system are available, a system consisting of only $2n+nm$ equations needs to be solved in order to estimate nm unknown parameters of an n-dimensional chaotic system. Such estimation is quite robust against the effect of noise and able to respond rapidly to changes in operating parameters of the experimental system.

We begin by considering an n-dimensional (experimental) chaotic system in the form of

$$\dot{x} = F(x,p), \tag{1}$$

where $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, $F(x,p) =(F_1(x,p),F_2(x,p),\ldots,F_n(x,p))$, and

$$F_i(x,p) = c_i(x) + \sum_{j=1}^{m} p_{ij}f_{ij}(x), \quad i=1,2,\ldots,n. \tag{2}$$

Here $c_i(x)$ and $f_{ij}(x)$ are some nonlinear functions, and $p=p_{ij}\in\mathbb{U}\subset\mathbb{R}^{nm}$ are nm unknown parameters to be estimated; $\mathbb{U}$ is a bounded set. For the vector functions $F(x,p)$, we give the following assumption.

For any $p\in\mathbb{U}$ and $x=(x_1,x_2,\ldots,x_n)$, $x_0=(x_1^0,x_2^0,\ldots,x_n^0)\in\mathbb{R}^n$, there exists a constant $l>0$ satisfying

$$|F_i(x,p) - F_i(x_0,p)| \leq l \max_j |x_j - x_j^0|, \quad i=1,2,\ldots,n. \tag{3}$$

We call the above condition the uniform Lipschitz condition, and l refers to the uniform Lipschitz constant. Note this condition is very loose; for example, the condition (3) holds as long as $\partial F_i/\partial x_j (i,j=1,2,\ldots,n)$ are bounded. One may check easily that the class of systems in the form of Eqs. (1)–(3) includes almost all well-known chaotic systems such as Lorenz system, Chua's circuit, Rössler hyperchaos system, etc.

We assume that time series for all variables of Eq. (1), as the experimental output of the system, are available. To estimate all unknown parameters p from these time series, we introduce an auxiliary system of variables $y=(y_1,y_2,\ldots,y_n)$,

*Electronic address: dbhuang@mail.shu.edu.cn

whose evolution equations have identical form to that of x. But the corresponding parameters are not same, which will be set to arbitrary initial values—say, $q=q_{ij}, i=1,2,\ldots,n, j=1,2,\ldots,m$. In contrast to the experimental system (1), the auxiliary system can be controlled in practice, which is also called the receiver system. We consider the linear feedback control, and the receiver system is given by the following equation:

$$\dot{y} = F(y,q) + \epsilon(y-x), \tag{4}$$

where the feedback coupling $\epsilon(y-x)=(\epsilon_1 e_1, \epsilon_2 e_2, \ldots, \epsilon_n e_n)$, $e_i=(y_i-x_i), i=1,2,\ldots,n$, denoting the synchronization error of Eqs. (1) and (4). Instead of the usual linear feedback, the feedback strength $\epsilon=(\epsilon_1,\epsilon_2,\ldots,\epsilon_n)$ here will be adapted duly according to the following update law:

$$\dot{\epsilon}_i = -\gamma_i e_i^2, \quad i=1,2,\ldots,n, \tag{5}$$

where $\gamma_i>0, i=1,2,\ldots,n$, are arbitrary constants. The equations governing the evolution of the parameters q are chosen similar to the adaptive controller used in [10] and quite simply have the form

$$\dot{q}_{ij} = -\delta_{ij} e_i f_{ij}(y), \quad i=1,2,\ldots,n, \quad j=1,2,\ldots,m, \tag{6}$$

where $\delta_{ij}>0, i=1,2,\ldots,n, j=1,2,\ldots,m$, are arbitrary constants. Next we will prove rigorously the main results. We first rewrite system (1) as

$$\dot{x} = F(x,p), \quad \dot{p}=0. \tag{7}$$

For the system consisting of the error equation between Eqs. (4), (6), and (7), and Eq. (5), which is formally called the augment system, we introduce the non-negative function

$$V = \frac{1}{2}\sum_{i=1}^{n} e_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{\delta_{ij}}(q_{ij}-p_{ij})^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\gamma_i}(\epsilon_i+L)^2, \tag{8}$$

where L is a constant bigger than nl—i.e., $L>nl$. By differentiating the function V along the trajectories of the augment system, we obtain

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{n} e_i(\dot{y}_i-\dot{x}_i) + \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{\delta_{ij}}(q_{ij}-p_{ij})\dot{q}_{ij} + \sum_{i=1}^{n}\frac{1}{\gamma_i}(\epsilon_i+L)\dot{\epsilon} \\ &= \sum_{i=1}^{n} e_i[F_i(y,q)-F_i(x,p)+\epsilon_i e_i] - \sum_{i=1}^{n}\sum_{j=1}^{m}(q_{ij}-p_{ij})e_i f_{ij}(y) \\ &\quad - \sum_{i=1}^{n}(\epsilon_i+L)e_i^2 \\ &= \sum_{i=1}^{n} e_i[F_i[y,p]-F_i(x,p)] - L\sum_{i=1}^{n} e_i^2 \leqslant (nl-L)\sum_{i=1}^{n} e_i^2 \leqslant 0,\end{aligned} \tag{9}$$

where we have used the uniform Lipschitz condition (3). It is obvious that $\dot{V}=0$ if and only if $e_i=0, i=1,2,\ldots,n$. Therefore the set $E=\{e=0, q-p=0, \epsilon=\epsilon_0\in\mathbb{R}^n\}$ is the largest invariant set contained in $\dot{V}=0$ for the augment system. Then according to the well-known invariance principle of differential equations [16], starting with arbitrary initial values of the augment system, the orbit converges asymptotically to the set E—i.e., $e\to 0$, $q-p\to 0$, and $\epsilon\to\epsilon_0$ as $t\to\infty$, where the converged strength ϵ_0 depends on the initial values. Namely, the parameters q will approximate asymptotically the correct values of unknown parameters p starting with arbitrary initial values.

In order to estimate nm model parameters using this method, we need to solve the experimental system (1) (when real experimental data are not available), the receiver system (3), the update law of feedback strength (5), and the adaptive equation of parameters (6). So when time series of the experimental system are available, an extended system consisting of $2n+nm$ equations needs to be solved to estimate nm parameters of an n-dimensional system. Obviously, the synchronization of systems (4), (6), and (7) is global from the above proof, so this estimation approach is quite robust against the effect of noise and able to respond to rapid changes of the operating parameters p of the experimental system (1). In comparison with previous methods for synchronization-based parameter estimation [8–11], the distinguished characteristic of our method is (i) analytical and rigorous because it does not require one to numerically determine any additive parameters (e.g., the feedback constant and stiffness constant introduced in [10,11]); (ii) systematic because the control technique in the form of Eqs. (4)–(6) can be applied to all chaotic systems satisfying the uniform Lipschitz condition (3); (iii) more simple, e.g., the method developed in [11] requires to solve $n+nm+n^2m$ equations for such problem. Therefore the technique developed here is very convenient to implement in practice.

Next we will give two illustrative examples. Our first example is the Lorenz system

$$\dot{x}_1 = p_1(x_2-x_1), \quad \dot{x}_2 = p_2x_1 - x_1x_3 - x_2, \quad \dot{x}_3 = x_1x_2 - p_3x_3, \tag{10}$$

where $x=(x_1,x_2,x_3)$ form the state space and $p=(p_1,p_2,p_3)$ are three parameters to be estimated. Assuming time series of x_1, x_2, and x_3, as experimental output of Eqs. (10), are available. Then according to the method developed above one may easily construct the receiver system [Eq. (4)], the update law [Eq. (5)] of the feedback strength $\epsilon_i, i=1,2,3$, and the adaptive controller [Eq. (6)] of the estimated parameter $q_i, i=1,2,3$. Due to page limits, we do not rewrite these equations.

To estimate the parameters p_1,p_2, and p_3, a system of 12 equations, governing the evolution of the (i) experimental system, (ii) receiver system, (iii) feedback strength, and (iv) parameters, will be solved. We set $p_1=10, p_2=28, p_3=\frac{8}{3}$, $\gamma_i=15, \delta_i=2, i=1,2,3$. Starting with arbitrary initial values of parameters—say, $(q_1(0),q_2(0),q_3(0))=(6,30,10)$—we track quickly the correct values of all model parameters of the experimental system (10). Numerical results are shown in Fig. 1. To consider the robustness against noise, the additive uniformly distributed random noise in the rang $[-2,2]$ (i.e., strength 2) is added to time series x_1, x_2, and x_3, Figure 2

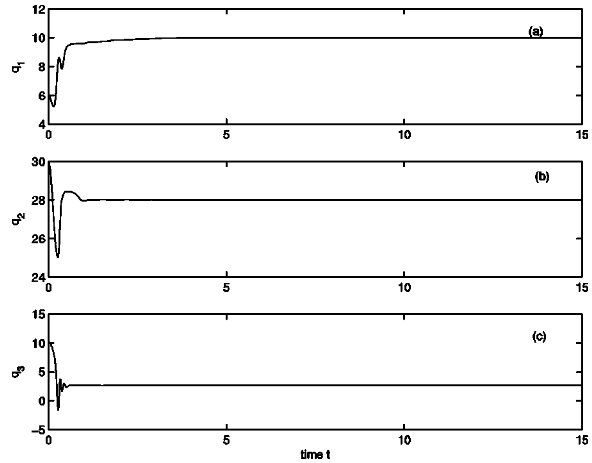

FIG. 1. Temporal evolutions of three parameters q show that the model parameters of the Lorenz system (10) are estimated precisely.

shows that although estimation values of all parameters fluctuate around the corresponding correct values, the fluctuations are very small so that we may estimate all parameters by a simple averaging over these fluctuations. Similar to the consideration in [11], we investigate the above method as to how to respond to rapid changes of the operating parameters. We rapidly change the values of the model parameters from $p_1=10, p_2=28, p_3=\frac{8}{3}$ to $p_1=11, p_2=35, p_3=3$ at $t=5$. Figure 3 shows that the estimation values of parameters converge to the new operating parameters through a rapid, stable transitions.

Our final example is the four-parameter Rössler hyperchaos system

$$\dot{x}_1=-x_2-x_3, \quad \dot{x}_2=x_1+p_1x_2+x_4, \quad \dot{x}_3=p_2+x_1x_3,$$
$$\dot{x}_4=-p_3x_3+p_4x_4, \tag{11}$$

where p_1, p_2, p_3, and p_4 are four parameters to be estimated from time series x_1, x_2, x_3, and x_4. Similarly, let $p_1=0.25$, $p_2=3$, $p_3=0.5$, $p_4=0.05$; the correct values of all

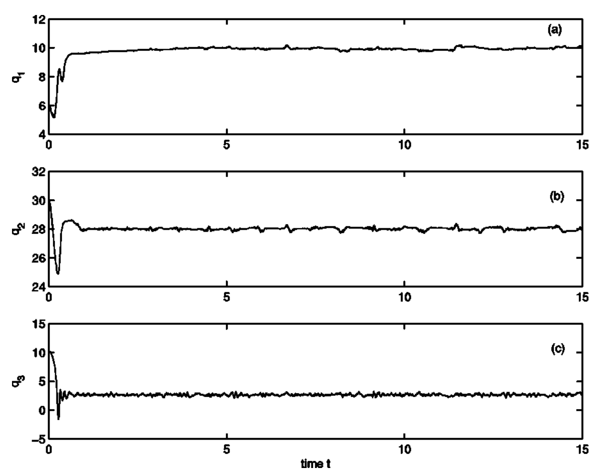

FIG. 2. The estimation values q fluctuate slightly around the correct values of p, respectively, when noise with strength 2 is added to time series x_i, $i=1,2,3$, of the experimental system (10).

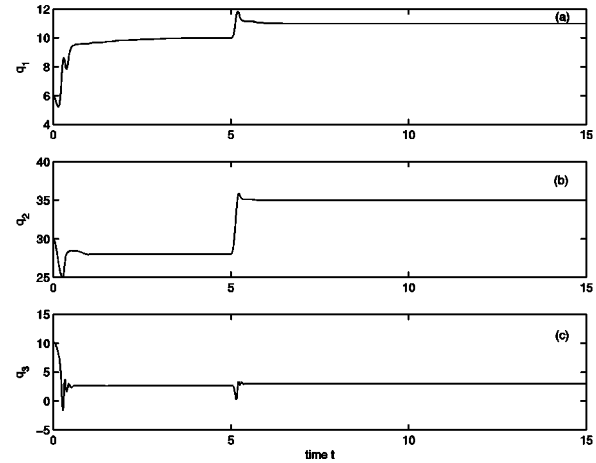

FIG. 3. The temporal evolutions of three estimation values when the operating parameters of system (10) are changed to $p_1=11$, $p_2=35$, $p_3=3$ from $p_1=10$, $p_2=28$, $p_3=\frac{8}{3}$ at $t=5$.

model parameters can be quickly estimated starting with arbitrary initial values of the parameters—say, $q(0)=(0.1,2.8,0.4,1)$; see Fig. 4. Figure 5 shows that when an additive uniformly distributed random noise in the range $[-0.01,0.01]$ is present in the time series x_2, the estimation value q_1 fluctuates slightly round the correct value of p_1, and the values of the other three parameters are estimated more quickly. Figure 6 shows that the estimations are able to track the new parameters through a transition when a perturbation is added to the operating parameters $p_i, i=1,2,3,4$, of the experimental system (11) such that each of them is increased by 10% at $t=10$.

These numerical examples show sufficiently that the developed method is very effective, quite robust against the effect of noise, and able to respond quickly to changes of the operating parameters in the experimental system.

We stress again that systems in the form of Eqs. (1)–(3) are so general that they include all well-known chaotic systems—e.g., all examples used in [8–11]. To summarize, for this class of systems we have given a rigorous, systematic

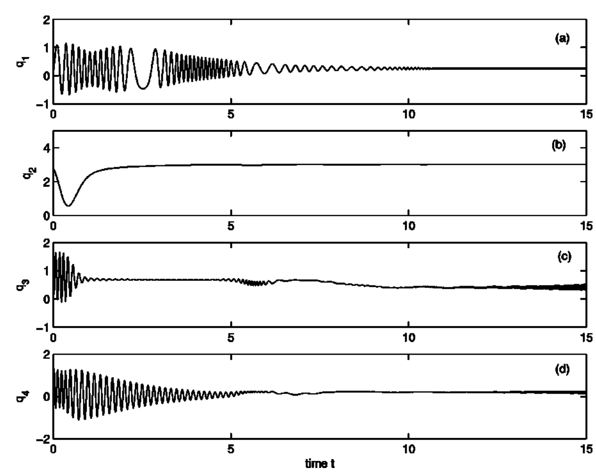

FIG. 4. The correct values of four model parameters of the Rössler hyperchaos system (11) are estimated starting with arbitrary initial values of the parameters—say, $q(0)=(0.1,2.8,0.4,1)$.

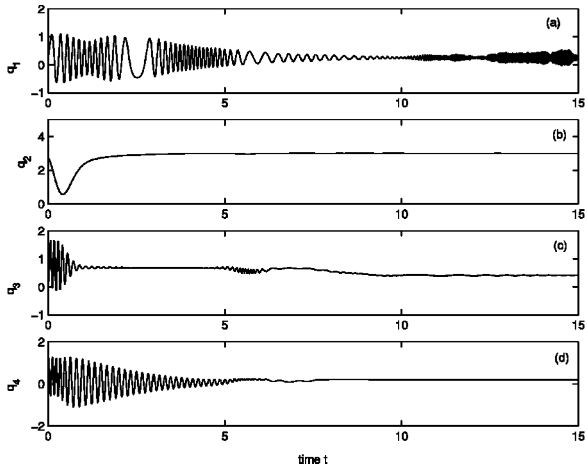

FIG. 5. Only the estimation value q_1 is slightly affected when noise with strength 0.01 is present in the time series x_2 of system (11).

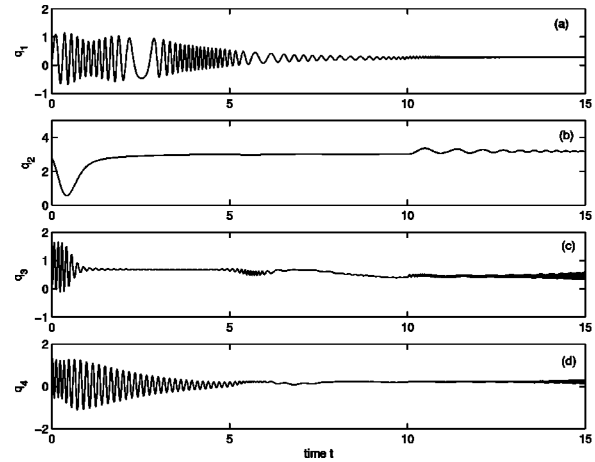

FIG. 6. The estimations are able to respond quickly when each of the operating parameters of the experimental system (11) is increased by 10% from $(0.25, 3, 0.5, 0.05)$ at $t=10$.

procedure to estimate all model parameters from time series by the synchronization based on a simple combination of adaptive control and linear feedback with the updated feedback strength. This approach is able to estimate all unknown parameters of a chaotic system in an online setting, but also is quite robust against the effect of noise and able to respond rapidly to changes of the experimental operating parameters. In comparison with previous methods for synchronization-based parameter estimation [8–11], the distinguished characteristic of our method is systematic, analytical, and even simple to implement in practice. A possible application of this method is to secure message transmission using parameter modulation. We also believe this method can be generalized to the case of discrete dynamical systems by using the invariance principle of difference equations.

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