Trapping and steering on lattice strings: Virtual slow waves and directional and nonpropagating excitations

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Using a lattice string model, a number of peculiar excitation situations related to nonpropagating excitations and nonradiating sources are demonstrated. External fields can be used to trap excitations locally but also lead to the ability to steer such excitations dynamically as long as the steering is slower than the field's wave propagation. I present explicit constructions of a number of examples, including temporally limited nonpropagating excitations, directional excitation and virtually slowed propagation. Using these dynamical lattice constructions I demonstrate that neither persistent temporal oscillation nor static localization are necessary for nonpropagating excitations to occur.

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I. INTRODUCTION

Can a local excitation (source) in classical field theories be invisible to observers outside the region of excitation? This question has recently received renewed interest.

Berry *et al.* [1] described a peculiar excitation case for the one-dimensional wave-equation of a perfectly elastic string under tension. They show that the response of the string can be made to be confined to a bounded region by carefully choosing a forced excitation of oscillatory type. This means that the excitation will not propagate away along the string. Denardo gives a simple and intuitive explanation by using a wave interference argument [2]. Gbur, Foley, and Wolf [3] discuss conditions of finite string length and dissipation.

Other recent work investigated nonpropagating excitations include Marengo and Ziolkowski [4–6] who discuss the generalization of nonpropagating conditions of D'Alembertian ($\Box \overset{\text{def}}{=} \nabla^2 - c^{-2}\partial^2/\partial t^2$) operators and its temporally reduced version the Helmholtz operator ($\nabla^2 + k^2$) on various related classical scalar and vector fields. Marengo, Devaney and Ziolkowski [7] give the condition for time-dependent but not necessarily time-harmonic nonradiating sources and for selective directional radiation for the inhomogeneous wave equation in three spatial dimensions. Marengo and Ziolkowski [8] generalize these conditions to more general scalar and vector field dynamics. Marengo, Devaney, and Ziolkowski [9] also give examples in one and three spatial dimension for the time-harmonic case. Hoenders and Ferwerda [10] discuss the relationship of nonradiating and radiating parts of the case of the reduced Helmholtz equation, which can be derived from the string equation by assuming general oscillatory time solutions (see Ref. [1]). Denardo and Miller [11] discuss the related case of leakage from an imperfect nonpropagating excitation on a string. Gbur [12] provides a comprehensive recent review of this topic and the reader is referred to this review for more detailed historical context. Of the earlier work the following contributions are particularly relevent for the discussion here: Schott [13,14] gave the condition for nonradiation of a spherical shell on a circular orbit. Bohm and Weinstein [15] extended this result to more general spherical charge distributions and Goedecke [16] showed how an asymmetrical charge distribution with spin is nonradiating. All of these works are concerned with the case of spatially moving sources. Finally, it is worth noting that nonradiating sources play an important role in inverse problems and have been investigated in a one-dimensional electrodynamic situation by Habashy, Chow, and Dudley [17].

In this paper our purpose is to describe this phenomena in the case of a lattice string in one dimensions by discretizing D'Alembert's solution. This approach is used extensively to simulate vibrating strings and air tubes of musical instruments. See Ref. [18] and references therein.

This leads to explicit dynamical constructions of previously reported nonpropagating excitations. Its simplicity allows for additional insight into the mechanism that allows for the local confinements and the conditions under which they occur. I will show how the basic mechanisms that provide a time-harmonic stationary nonpropagating excitation in one dimension as studied by Berry *et al.* and Gbur, Foley, and Wolf [1,3] allows for a much wider class of excitations. For instance, can such an excitation be relieved from the time-harmonic assumption beyond one period allowing for nonpropagating sources that are short lived. Directional excitations can easily be achieved using very simple bidirectional excitation patterns. These are explicit constructions of such waves in one spatial dimension whose general condition of existence in the three-dimensional case has been derived by Marengo, Devaney and Ziolkowski [7]. Wave propagation can be virtually slowed down. In general I will show that nonpropagating excitations can be extended to steered excitation regions with basic physical restrictions imposed by the underlying field dynamics.

First I will give a quick derivation of the simple lattice model from the wave equation as can also be found in Ref. [18]. Then I will give an argument and construction of the Berry *et al.*, type-nonpropagating excitation purely based on discrete string dynamics. This will then be compared to the

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original approach. Then I will extend the discussion to examples of additional types of nonpropagating waves, including directional and slowed waves. Finally, I will discuss very general constraints on such "steered" localized excitations.

II. LATTICE STRING MODEL

The lattice string model can easily be derived from the wave equation by discretizing the D'Alembert solution. Hence the continuous case will be discussed first.

A. Continuous wave solutions

The one-dimensional homogeneous wave equation of the perfectly elastic string under tension is

$$\mu\frac{\partial^2 y}{\partial t^2} - K\frac{\partial^2 y}{\partial x^2} = 0, \tag{1}$$

where $c^2 = K/\mu$ is derived from mass density μ and tension K. The D'Alembert solution of the homogeneous "free field" case has the well known form [19, p. 596, Eq. (4)]

$$y(x,t) = w^+(x - ct) + w^-(x + ct). \tag{2}$$

Hence the solution of the general of the homogeneous wave equation are two propagating waves whose content is restricted by initial and boundary conditions. As wave equation is linear we have a connection between initial conditions and external driving forces. Driving forces can be seen as infinitesimal time frames that act on the wave dynamics by imposing an initial condition at each point in time. Hence we need to consider the initial value problem to gain insight into both processes at once.

At a give time frame t_i let the following initial conditions hold:

$$y(x,t_i) = f(x,t_i), \tag{3}$$

$$y_t(x,t_i) = g(x,t_i). \tag{4}$$

Equation (3) with Eq. (2) gives a particular solution v^+

$$v^+(x - ct_i) + v^-(x + ct_i) = f(x,t_i). \tag{5}$$

Taking the first temporal derivative of Eq. (2) and satisfying Eq. (4) we get

$$- cv_t^+(x - ct_i) + cv_t^-(x + ct_i) = g(x,t_i). \tag{6}$$

Integrating with respect to x we get [19, Eq. (10) p. 596]

$$- cv^+(x - ct_i) + cv^-(x + ct_i) = k(x_0) + \int_{x_0}^{x} g(s)ds, k(x_0) = - cv^+(x_0) + cv^-(x_0). \tag{7}$$

From Eqs. (5) and (7) we can solve for the traveling wave components

$$v^+(x - ct_i) = \tfrac{1}{2} f(x,t_i) - \frac{1}{2c}\int_{x_0}^{x-ct_i} g(s)ds - \tfrac{1}{2}k(x_0), \tag{8}$$

$$v^-(x + ct_i) = \tfrac{1}{2} f(x,t_i) + \frac{1}{2c}\int_{x_0}^{x+ct_i} g(s)ds + \tfrac{1}{2}k(x_0). \tag{9}$$

We see that forced displacement $f(\cdot)$ splits evenly between left and right traveling waves and the integrated forced velocity $g(\cdot)$ splits with a sign inversion.

For our current discussion I will share the assumption of no initial velocity of Berry *et al.* [1] and hence the integral over $g(\cdot)$ will vanish.

For the infinite string this is already the complete solution for any twice differentiable function of free solutions and external forced displacements.

B. Discrete wave solutions

To arrive at lattice equations we discretize the solution of the wave Eq. (2) in time via the substitution $t \to Tn$ where T is the discrete time-step and n is the discrete time index. This automatically corresponds to a discretization in space as well, because in finite time T a wave will travel $X = cT$ distance according to Eq. (2). The spatial index will be called m. The free-field discrete D'Alembert solution

$$y(mX,nT) = w^+(mX - cnT) + w^-(mX + cnT). \tag{10}$$

In general, we can always express all discrete equations in terms of finite time steps or finite spatial lengths. We chose a temporal expression and substitute $X = cT$ and suppress shared terms in cT to arrive at the index version of the discrete D'Alembert solution [18]

$$y(m,n) = w^+(m - n) + w^-(m + n). \tag{11}$$

By Eqs. (8) and (9) we see that at an instance m_i, n_i the discrete contribution of external forced displacements splits evenly between the traveling waves and we arrive at the discrete field equations including external forced displacements

$$W^+(m_i - n_i) = w^+(m_i - n_i) + \tfrac{1}{2} f(m_i,n_i), \tag{12}$$

$$W^-(m_i - n_i) = w^-(m_i - n_i) + \tfrac{1}{2} f(m_i,n_i). \tag{13}$$

III. NONPROPAGATING EXCITATION

Next we will construct the nonpropagating excitation from the lattice string dynamics directly.

For simplicity and without loss of generality, we will assume a region aligning with the discretization domain throughout. We want to construct an excitation which is confined to a length $-L \leqslant x \leqslant L$. For now we will assume that the string should otherwise stay at rest. This implies that there are no incoming waves into the region $\Omega = [-L, L]$ from the outside. We are interested in a nontrivial excitation within the region.

First we consider the contributions to the position $-L$. As there are no incoming external waves we get

$$w^+(- L + n) = 0. \tag{14}$$

We do expect nontrivial wave $w^{-}(-L-n)$ to reach the boundary but we require the total outgoing wave to vanish we have

$$W^{-}(-L-n) = w^{-}(-L-n) + \tfrac{1}{2} f(-L,n) = 0. \qquad (15)$$

The necessary external forced displacement contribution for cancellation needs to be

$$\tfrac{1}{2} f(-L,n) = -w^{-}(-L-n). \qquad (16)$$

The complete incoming wave (12) will see the same forced contribution (16) and with Eq. (14) we get

$$W^{+}(-L+n) = \tfrac{1}{2} f(-L,n) = -w^{-}(-L-n). \qquad (17)$$

Hence, the matched forced displacement leads to a reflection with sign inversion at the region boundary at $-L$.

Following the same line of argument at point L we get the related condition

$$W^{-}(L-n) = \tfrac{1}{2} f(L,n) = -w^{+}(L+n). \qquad (18)$$

With these two conditions we can study the permissible form of excitations. First we assume an initial forced displacement impulse from a position p in the interior of the domain $\Omega\setminus\partial\Omega=(-L,L)$. Hence $-L<p<L$ and $f(p,0)=a_p$ with $a_p \in \mathbb{R}$.

It will take half the impulse $L+p$ steps to reach the left boundary and the other half $L-p$ steps to reach the right one.

At each boundary the respective condition (17) and (18) needs to be satisfied and we get

$$f(-L,L+p) = -f(p,0), \qquad (19)$$

$$f(L,L-p) = -f(p,0). \qquad (20)$$

The impulse will then reflect back and create periodic matching conditions

$$f(-L,L+p+4Lv) = f(p,0), \qquad (21)$$

$$f(-L,L-p+(2v-1)2L) = -f(p,0), \qquad (22)$$

$$f(L,L-p+4Lv) = f(p,0), \qquad (23)$$

$$f(L,L+p+(2v-1)2L) = -f(p,0) \qquad (24)$$

with $v=1,2,\ldots$.

Hence we see that a single impulse will necessitate an infinite periodic series of forced external displacements at the boundaries to trap the impulse inside as each "annihilation" of a half- pulse reaching the boundary leads to a "creation" of a reflected one.

The required impulse response of a boundary forced function $f(\pm L,\cdot)$ can easily be observed from Eqs. (21)–(24) to be spatially periodic in $4L$ with an initial phase factor dictated by the starting position p. Additionally the functional shape of the impulse responses $f(\pm L,\cdot)$ is completely defined for all time steps as $f(\pm L,\cdot)=0$ for all times that Eqs. (21)–(24) do not apply.

A condition for stopping a nonpropagating excitation can be derived from the fact that an impulse will return to its initial position every $4L$ time steps. Additionally it is easy to see that the traveling impulses will occupy the same spatial position every odd multiple of $2L$ with a sign inversion. Hence an impulsive forced displacement $f((-1)^{\mu-1}p,4L\mu) =(-1)^{\mu-1}a_p$ with $\mu=1,2,\ldots$ will cancel an initial impulse $f(p,0)=a_p$. From this we can immediately deduce the following property:

Theorem 1. *The shortest possible single impulse finite nonpropagating excitation takes* $2L$ *time steps.*

and more generally:

Theorem 2. *The time of any single impulse excitation finite nonpropagating excitation has to be* $2\mu L, \mu \in \mathbb{N}$.

More importantly, we observe the property: *Nonpropagating excitations can be finite in duration.*

This is an extension beyond Berry *et al.* [1] which assumes infinitely periodic temporal progressions in their derivations.

The general solution for discrete nonpropagating wave functions can be derived by observing that any initial "phase" p_i is orthogonal to other phases p_j for $i,j \in \Omega\setminus\partial\Omega =(-L,L)$, i.e., $\langle f(p_i,0), f(p_j,0)\rangle=0$ for $i\neq j$. Within a $2L$ period $f(\pm L,\cdot)$ is well defined by $\Sigma_i f(p_i,\cdot)$. Interestingly, though, this provides the only restriction to the forced boundary functions. This can be seen by Theorem 1. After $2L$ each p_i will find constructive interference and can be annihilated or rescaled to an arbitrary other value $a_i(2L)$. Hence any arbitrary succession of $2L-2$ force distributions with a $2L$ termination is permissible. Hence periodicity is not necessary.

The time harmonic case can be derived if the initial force distribution within the domain is not modified over time. Then a configuration will repeat after traveling left and right, being reflected at the domain boundary twice, traversing the length of the region twice. Hence the lowest permissible wavelength is $4L$. By reflecting twice the wave will have gone through a 2π phase shift, but we note that the periodicity condition is also satisfied if any number of additional 2π shifts have been accumulated. Hence we get for permissible wave numbers

$$k = \frac{2\pi n}{4L}, \quad \text{where} \quad n=1,2,\ldots \qquad (25)$$

or

$$kL = \frac{n\pi}{2}. \qquad (26)$$

By allowing only even n we get the Berry *et al.* condition [1] for an even square distribution. The odd n situation corresponds to the odd-harmonic out-of-phase construction proposed by Denardo [2].

Many of these properties can be seen visually in the numerical simulation depicted in Fig. 1.

It is interesting to observe that two synchronous point sources oscillating with the above phase condition will not be completely nonpropagating. They will only be nonpropagating after waves created at the wave onset have escaped.

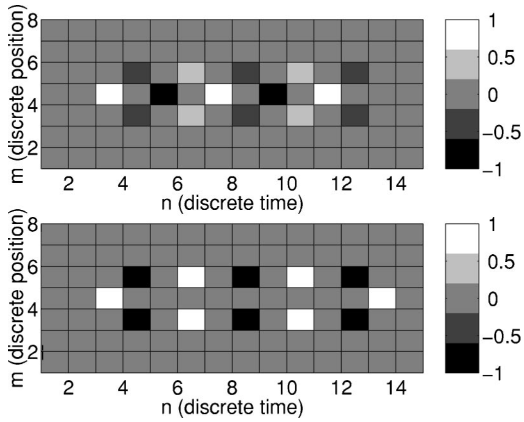

FIG. 1. Simulation of a nonpropagating excitation of width 3 which is annihilated after 3.5 periods. The total temporal length of the excitation is 10. The excitation leaves the string at rest after it is completed. Top: Complete wave pattern. Bottom: Excitation only.

This is a refinement of the argument put forward by Denardo [2] and can intuitively be described as *noninterference of the first trap period*. Hence the first pairs of pulses will have half-amplitude components escaping in either direction but every subsequent period will be trapped. This behavior, which could be called imperfect trapping or trapping with transient radiation, is depicted in Fig. 2. Sources presented by Berry *et al.* and Denardo [1,2] do not display this behavior because the force is assumed to be oscillatory at all times and hence has no onset moment.

Nonpropagating excitations can be used as generic building blocks for other unusual excitation induced behavior on the string. In particular, I will next describe how to construct a uni-directional emitter, and a virtually slowed propagation. In fact, a nonpropagating excitation can be seen as virtually stopping a wave at a particular position.

IV. DIRECTIONAL EXCITATIONS

A one-sided open trap immediately suggests another unusual excitation type, namely the directional excitation. The

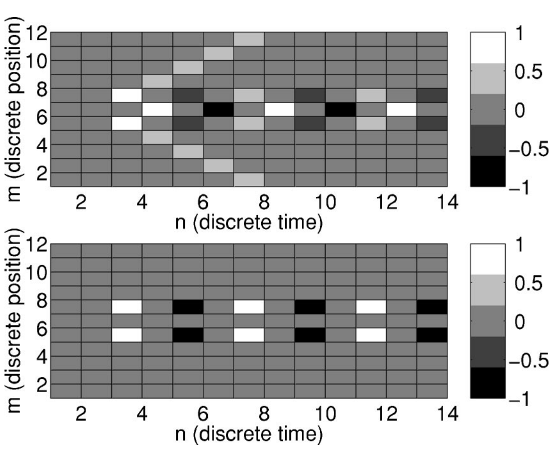

FIG. 2. Simulation of a nonpropagating excitation of width 3 showing escaping waves at the onset transient. Top: Complete wave pattern. Bottom: Excitation only.

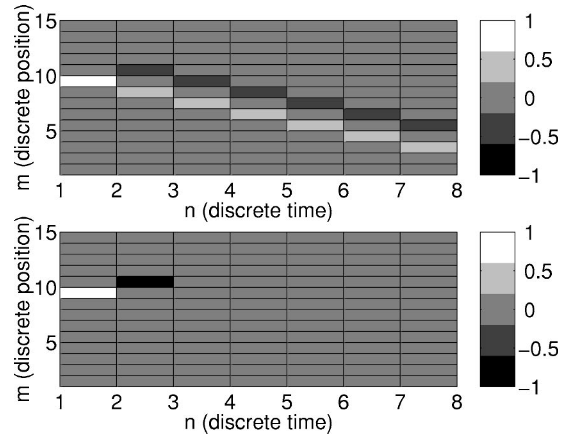

FIG. 3. Simulation of a directional excitation of width 3. The deflected component experienced a sign inversion. The temporal length of the excitation sequence is two, including the initial impulse. Top: Complete wave. Bottom: Excitations only.

string is to be excited in such a way that a traveling wave in only one direction results.

We start with a one-sided open trap. This is a trap that uses a reflection conditions, (17) and (18) only on one side of an initial excitation. Evidently the wave then can only travel in the opposite direction. For the discussion we will describe a right-sided propagator (i.e., a propagator traveling with increasing negative index). The trapping condition then reads

$$f(m+1+p,n+p) = -f(m,n-1). \tag{27}$$

Hence the trapping excitation point is a p time-step lagging negative copy of the original excitation. The emitted wave will have the form

$$\tfrac{1}{2} f(m+1,n+2p) - \tfrac{1}{2} f(m+1,n). \tag{28}$$

The emitting wave will show self-interference at a phase of $2p$ time steps, as can be seen in the simulation depicted in Fig. 3. In general, the self-interference phase can be chosen by the distance p between the wave creation point and the trapping point. It is worth noting that it is possible to eliminate interference by trapping the lagging contribution and hence create a noninterference directional wave left of the trapping region.

V. VIRTUAL SLOW WAVES

Virtual slow waves can be achieved by alternating directional wave propagation with trapping. The slowness of the wave propagation can be controlled by the number and duration times of the traps along a propagation. The propagation characteristics of the dynamic operator have not changed at all, hence we call this state "virtually slow" as opposed to the case where the field itself induces a change in wave propagation speed. This also means that within a slowed or "steered" region the wave propagation is the one prescribed by the dynamic operator $(\partial/\partial x + c(\partial/\partial t))(\partial/\partial x - c(\partial/\partial t))$ on the string $y(x,t)$.

The amount of time spent in traps determines the overall slowness. One example of slow wave consists of an imme-

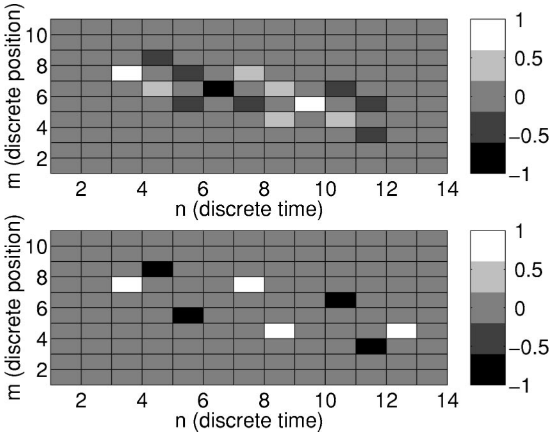

FIG. 4. Simulation of a finite-duration virtual slow wave excitation of width 3. The wave is annihilated after ten steps. Top: Complete wave. Bottom: Excitation only.

diate alteration between one stage of trapping and one step of one-sided propagation illustrated in Fig. 4. The effective propagation speed of the wave can easily be read from the diagram to give $c_{\mathrm{eff}}=c(X/3T)=c/3$. As is evident from Theorem 1, a unit $L=1$ trap will last two time steps and will not propagate spatially and one step of free propagation will last one time step and and make one spatial step, hence resulting in a spatial to temporal ratio of 1:3.

The trapping relations are

$$f(m-2-v,n+1+6v)=f(m+1-v,n+6v)=-f(m,n-1), \tag{29}$$

$$f(m-3-v,n+4+6v)=f(m-v,n+3+6v)=f(m,n-1) \tag{30}$$

with $v=0,2,4,\ldots$.

VI. STEERING

The generalized interpretation of the excitation interaction lead to the general dynamical confinement of waves by external excitation. For instance, following very similar arguments as for virtual slow waves a construction is possible which gives a slowed "cone of influence" by successively widening the trap boundaries at a speed slower than the wave speed c. By this argument it is sufficient for the trap boundaries' change to be less than c for it to be trapping the wave. This is not a necessary condition by the following counter example: Let the trap width be L and change rapidly by some slope $dL>c$ to some new constant width L_2 at which it becomes constant. Obviously the wave will then be able to reach the new boundary even though a local change of the boundary exceeded the dynamical speed c. The necessary condition can be seen from our previous construction. At a trap boundary a wave is reflected and will propagate in the opposite direction of the domain following the linear characteristic c. Only if this characteristic intersects with the dynamic trapping boundary will there be another externally forced reflection as illustrated in Fig. 5. These may in fact

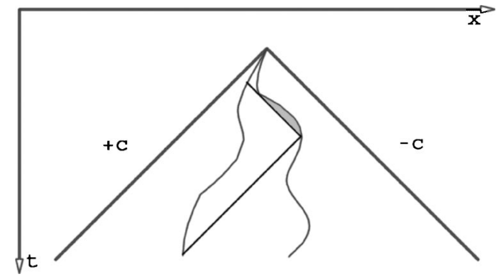

FIG. 5. A grazing propagating wave against a changing trap boundary can create regions (gray) in which no trap affect applies.

have regions where no trapping is necessary and possible.

VII. INTERACTION WITH BACKGROUND FIELDS

It is important to note that while we assumed that the incoming wave vanishes, see Eq. (14), the outgoing wave condition (15) does not change if there is, in fact, an incoming wave. The "reflection wave" (17) and (18) can be rewritten for a nonzero incoming field without affecting the trapping

$$W^{+}(-L+n)=w^{+}(-L+n)+\frac{1}{2}f(-L,n),$$

$$\frac{1}{2}f(-L,n)=-w^{-}(-L-n) \tag{31}$$

and

$$W^{-}(L-n)=w^{-}(L-n)+\frac{1}{2}f(L,n),$$

$$\frac{1}{2}f(L,n)=-w^{+}(L-n). \tag{32}$$

These conditions are "absorbing" in the sense that an external field entering the trapping region will not leave it.

The "noninteracting" property of a trap defined by the periodic matching conditions (21)–(24) can be seen by assuming a nonzero incoming wave at one point of the trap boundary $\delta\Omega$. Then the total wave entering the trapping region, the sum of the wave created by the trapping condition, and the incoming wave value $\frac{1}{2}f(\delta\Omega^1,\cdot)+w^{\pm}(\delta\Omega^1,\cdot)$, where $\delta\Omega^1$ denotes the first trap boundary reached. When reaching the second trapping boundary $\delta\Omega^2$ the now outgoing wave will see a matching force $f(\delta\Omega^2,\cdot)=-\frac{1}{2}f(\delta\Omega^1,\cdot)$ leaving an outgoing wave contribution $w^{\pm}(\delta\Omega^2,\cdot)=w^{\pm}(\delta\Omega^1,\cdot)$ to escape the trapping region Ω.

In order to achieve selective radiation, only part of the content of a trapped region is trapped at the boundary as can be achieved by using a reduced force at the trapping boundary or by selectively omitting certain phases in the trapping force pattern.

Relationship of traps to nonradiating sources

Marengo and Ziolkowski [4] present ideas very much related to ideas presented here and in Berry *et al.* [1]

However, they arrive at a definition of nonradiating (NR) sources that is not obviously similar to the traps presented here. In particular, they define NR sources as being noninteracting. While Ref. [4] notes that a central property of NR sources is that they store nontrivial field energy, traps described here cannot only store, but accumulate and selectively radiate waves.

The difference can be understood by observing that, for example, Berry *et at*. assume a simple time-harmonic driver [see Eq. (3) of Ref. [1]] throughout their discussion:

$$f(x,t) = \mathrm{Re}\{f(x)e^{-i\omega t}\}. \tag{33}$$

By our earlier discussion we see that the temporal progression of the boundary has to match the content of the interior domain. Hence once the boundary is defined to be oscillatory the interior of the domain needs to be spatially harmonic as derived in [1,4] and has been re-derived here. Hence a NR source as noted in literature, with the exception of the general orthogonality formulation for time-varying sources given by Marengo, Devaney, and Ziolkowski [7], can be thought of as a time-oscillatory trap.

The arguments made here use a formalism that is discrete in nature. However, the discreteness of the arguments is not necessarily restrictive. The continuous case can be imagined with the discrete time step made small $(T\to 0)$ or, alternatively, discrete pulses can be substituted with narrow distributions of compact support. In neither case are the results of interest derived here altered.

As has already been derived in Refs. [1,3] the critical condition for nonpropagating waves lies at the boundary of the domain range that the wave ought not to leave. In the discrete case it is easy to see how this insight can be used and generalized. In fact, the boundaries of the confining domain need not be static, nor need the condition be used in a two-sided fashion.

VIII. CONCLUSION

In summary, this paper presented constructions of a broad class of nonpropagating sources on a string lattice model using trapping conditions. In particular, this includes numerical demonstrations of finite-duration nonpropagating excitations, directional excitations, as well as virtually slowed waves. These examples help explain the extension of nonpropagating sources beyond the time-periodic case and include treatment of onset, annihilation and spatial steering. These properties ought to be observable in experiments well described by the wave equation. This equation often arises in problems in acoustics, elasticity, optics, and electromagnetism. And hence the results presented here apply to these domains of application. While here I discussed the forward problem, these results also relate to the inverse problem of finding source contributions from the one-dimensional field state as occur, for example, in acoustical, optical, and electromagnetic detection problems.

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