Violation of the Widom scaling law for effective crossover exponents

S. Lübeck
Weizmann Institute of Science, Department of Physics of Complex Systems, 76100 Rehovot, Israel and Theoretische Tieftemperaturphysik, Gerhard-Mercator-Universität, D-47048 Duisburg, Germany
(Received 6 August 2003; revised manuscript received 17 December 2003; published 1 June 2004)

In this work we consider the universal crossover behavior of two nonequilibrium systems exhibiting a continuous phase transition. Focusing on the field driven crossover from mean-field to non-mean-field scaling behavior we show that the well-known Widom scaling law is violated for the effective exponents in the so-called crossover regime.

DOI: 10.1103/PhysRevE.69.066101 PACS number(s): 05.70.Ln, 05.50.+q, 05.65.+b

I. INTRODUCTION

A foundation for the understanding of critical phenomena was provided by Wilson's renormalization group (RG) approach [1,2] which maps the critical point onto a fixed point of a certain transformation of the system's Hamiltonian, Langevin equation, etc. The RG theory presents a powerful tool to estimate the values of the critical exponents, it allows us to predict which parameters determine the universality class and it explains the existence of an upper critical dimension D_c above which the mean-field theory applies. Furthermore, crossover phenomena between two different universality classes are well understood in terms of competing fixed points (see, for instance, Ref. [3]). Nevertheless, there are still some open aspects of crossover phenomena which are discussed in the literature. The question whether the crossover scaling functions are universal was revisited several times [4–10]. For instance it was shown recently that two different models, belonging to the same universality class, are characterized by the same (universal) crossover scaling functions [10]. This is remarkable, since the universal scaling behavior is usually restricted to a small vicinity around the critical point. In the case of a crossover the universal scaling functions span several decades in temperature or conjugated field.

Another question of interest concerns the so-called effective exponents [11] which can be defined as logarithmic derivatives of the corresponding scaling functions. It is still open whether these effective exponents fulfill over the full crossover the well-known scaling laws which connect critical exponents. This question is closely related to the more general and very important question of whether effective exponents obey the scaling laws at all. For instance it is known experimentally [12] as well as theoretically [13] that the asymptotic scaling behavior is often masked by corrections to scaling, so-called confluent singularities. In this case it is useful to analyze the data in terms of effective exponents and the above question naturally arises [14]. Thus the validity of the scaling laws for effective exponents was addressed in experimental works, RG approaches as well as numerical simulations. In particular the violation of the scaling laws for effective exponents was conjectured from a RG approximation [15]. But neither experimental nor numerical work could clearly confirm this conjecture so far. For instance, Binder and Luijten considered numerically a crossover in the Ising model and discussed the validity of the Rushbrook scaling law [6]. The observed nonmonotonic crossover behavior suggests again a violation of the Rushbrook scaling law. However, it cannot be considered as a rigorous proof.

In this work we consider a field driven crossover (in the so-called critical crossover limit [4,7]) in two different models exhibiting a nonequilibrium second order phase transition. Varying the range of interactions we investigate the crossover from mean-field to non-mean-field scaling behavior. The order parameter and the order parameter susceptibility is measured as a function of the conjugated field and we are able to determine the corresponding effective exponents over the full crossover regime. Our results show that the well-known Widom scaling law is clearly violated for the effective crossover exponents. Furthermore, we present a simple analytical argument, suggesting that the scaling laws are valid for the asymptotic scaling regimes (where the systems are characterized by a pure algebraic behavior), whereas the scaling laws do not hold for the crossover regime (characterized by a nonalgebraic behavior).

II. MODELS AND SIMULATIONS

In the following we consider two different cellular automata exhibiting a so-called absorbing phase transition. The first model is the conserved transfer threshold process (CTTP) [16]. In this model lattice sites may be empty, occupied by one particle, or occupied by two particles. Empty and single occupied sites are considered as inactive whereas double occupied lattice sites are considered as active. In the latter case one tries to transfer both particles of a given active site to randomly chosen empty or single occupied nearest neighbor sites.

The second model is a modified version of the Manna sandpile model [17], the fixed-energy Manna model [18]. In contrast to the CTTP the Manna model allows unlimited particle occupation of lattice sites. Lattice sites which are occupied by at least two particles are considered as active and all particles are moved to the neighboring sites selected at random.

In our simulations (see Refs. [19,20] for details) we have used square lattices (with periodic boundaries) of linear size $L \leqslant 2048$. All simulations start from a random distribution of particles. After a transient regime both models reach a steady state characterized by the density of active sites ρ_a. The density ρ_a is the order parameter and the particle density ρ is the control parameter of the absorbing phase transition, i.e., the order parameter vanishes at the critical density ρ_c according to, $\rho_a \propto \delta\rho^{\beta}$, with the reduced control parameter $\delta\rho=\rho/\rho_c - 1$. Below the critical density (in the absorbing phase) the order parameter is zero in the steady state.

Similar to equilibrium phase transitions it is possible in the case of absorbing phase transitions to apply an external field h which is conjugated to the order parameter. The conjugated field has to act as a spontaneous creation of active particles, destroying the absorbing state and therefore the phase transition itself. Furthermore, the associated linear response function $\chi_a=\partial\rho_a/\partial h$ has to diverge at the critical point ($\delta\rho=0$, $h=0$). A realization of the external field for absorbing phase transitions with a conserved field was recently developed in Ref. [19] where the external field triggers movements of inactive particles which may be activated in this way. At the critical density ρ_c the order parameter scales as $\rho_a \propto h^{\beta/\sigma}$. Using the conjugated field it is possible to investigate the equation of state $\rho_a(\rho, h)$, i.e., the order parameter as a function of both the control parameter and the external field. A recently performed scaling analysis reveals that the CTTP and the Manna model are characterized by the same critical exponents as well as by the same universal scaling form of the equation of state, i.e., both models belong to the same universality class [21].

According to the above definition particles of active sites are moved to nearest neighbors only, i.e., the range of interactions is $R=1$. It is straightforward to implement various ranges of interactions into these models [10]. In this modified models particles of active sites are moved (according to the rules of each model) to randomly selected sites within a radius R. Since the dynamics of both considered models is characterized by simple particle hopping processes, various interaction ranges can be easily implemented and high accurate data are available. This is a significant advantage compared to, e.g., equilibrium system like the Ising model where the increasing interaction range causes a slowing down of the dynamics.

For any finite interaction range the phase transition is characterized by non-mean-field scaling behavior which now takes place at the critical density $\rho_{c,R}$. A mean-field phase transition occur for infinite interactions ($R\to\infty$) only. But mean-field behavior could occur away from the critical point if the long range interactions reduce the critical fluctuations sufficiently. The crossover between the mean-field and non-mean-field scaling regimes is described by the famous Ginzburg criterion [22] which states that the mean-field picture is self-consistent in the active phase as long as the fluctuations within a correlation volume are small compared to the order parameter itself. This leads for zero field to the crossover condition $\mathcal{O}(R_{\text{eff}}\,(\rho-\rho_{c,R})^{\varphi})=1$, with the crossover exponent $\varphi=(4-D)/2D$ [10]. In order to avoid lattice effects we use the effective range of interactions [23]

$$R_{\text{eff}}^2 = \frac{1}{z}\sum_{i\neq j} |\underline{r}_i - \underline{r}_j|^2, \quad |\underline{r}_i - \underline{r}_j| \leq R, \tag{1}$$

where z denotes the number of lattice sites within a radius R.

III. UNIVERSAL CROSSOVER SCALING

The crossover scaling function has to incorporate three scaling fields (the control parameter, the external field, and the range of interactions), i.e., we make the phenomenological ansatz

$$\rho_a(\rho,h,R_{\text{eff}}) \sim \lambda^{-\beta_{\text{MF}}}\,\tilde{\mathfrak{R}}(\mathfrak{a}_{\rho}(\rho-\rho_{c,R})\,\lambda, \mathfrak{a}_h h\,\lambda^{\sigma_{\text{MF}}}, \mathfrak{a}_R^{-1}R_{\text{eff}}^{-1}\,\lambda^{\varphi}), \tag{2}$$

where the universal scaling function $\tilde{\mathfrak{R}}$ is the same for all models belonging to a given universality class whereas all nonuniversal system-dependent features (e.g., the lattice structure, the update scheme, etc.) are contained in the so-called nonuniversal metric factors $\mathfrak{a}_{\rho}$, $\mathfrak{a}_h$, and $\mathfrak{a}_R$ [24]. These factors are determined by three conditions which normalize the scaling function $\tilde{\mathfrak{R}}$. First, the analytically known mean-field scaling function [21,25]

$$\tilde{R}_{\text{MF}}(x,y) = \frac{x}{2} + \sqrt{y + \left(\frac{x}{2}\right)^2} \tag{3}$$

should be recovered for $R\to\infty$, i.e., $\tilde{\mathfrak{R}}(x,y,0)=\tilde{R}_{\text{MF}}(x,y)$. Therefore $\tilde{\mathfrak{R}}(1,0,0)=\tilde{R}_{\text{MF}}(1,0)=1$, $\tilde{\mathfrak{R}}(0,1,0)=\tilde{R}_{\text{MF}}(0,1)=1$, implying $\mathfrak{a}_{\rho}=a_{\rho,R\to\infty}/\rho_{c,R\to\infty}$ and $\mathfrak{a}_h=a_{h,R\to\infty}$. Finally, the nonuniversal metric factor $\mathfrak{a}_R$ can be determined by the condition $\tilde{\mathfrak{R}}(x,0,1)\sim x^{\beta_D}$ for $x\to 0$ yielding [10]

$$\mathfrak{a}_R = \left(\frac{\rho_{c,R=1}}{a_{\rho,R=1}}\frac{a_{\rho,R\to\infty}}{\rho_{c,R\to\infty}}\right)^{\varphi\beta_D/(\beta_{\text{MF}}-\beta_D)} \tag{4}$$

The metric factors were already determined in previous works [10,25], thus no parameter fitting is needed in order to perform the following scaling analysis.

In this work we focus our attention to the field driven crossover, i.e., we consider the CTTP and the Manna model at the critical densities $\rho_{c,R}$ which were determined in Ref. [10]. In Fig. 1 we plot the corresponding data of the CTTP for various values of the interaction range R. As one can see the power law behavior of the order parameter changes with increasing range of interactions.

The scaling form at the critical point is given by (setting $\mathfrak{a}_R^{-1}R_{\text{eff}}^{-1}\lambda^{\varphi}=1$)

$$\rho_a(\rho_{c,R},h,R_{\text{eff}}) \sim (\mathfrak{a}_R R_{\text{eff}})^{-\beta_{\text{MF}}/\varphi}\,\tilde{\mathfrak{R}}(0,\mathfrak{a}_h h\,\mathfrak{a}_R^{\sigma_{\text{MF}}/\varphi}R_{\text{eff}}^{\sigma_{\text{MF}}/\varphi},1), \tag{5}$$

with $\beta_{\text{MF}}=1$ and $\sigma_{\text{MF}}=2$, respectively. For sufficiently small field the universal function scales as

$$\tilde{\mathfrak{R}}(0,x,1) \sim m_{a,h}\,x^{\beta_D/\sigma_D}, \qquad \text{for} \quad x\to 0, \tag{6}$$

with the universal amplitude $m_{a,h}$. The scaling form Eq. (5) has to equal for $R=1$ the D-dimensional scaling ansatz [$\rho_a \sim (a_{h,R=1}h)^{\beta_D/\sigma_D}$] leading to

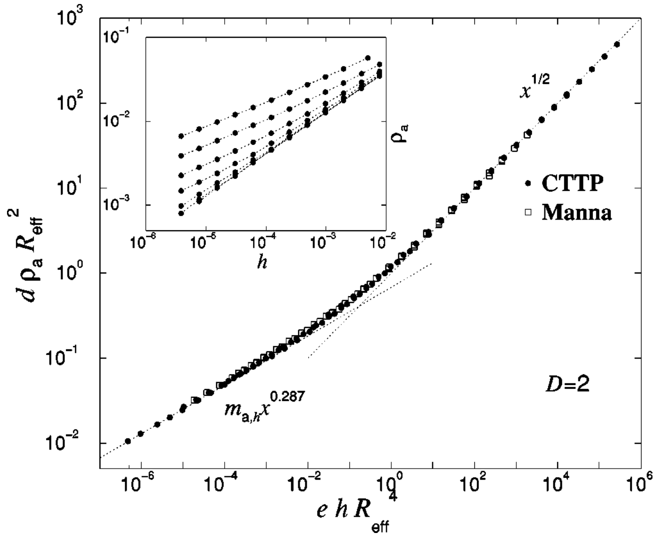

FIG. 1. The universal crossover scaling function $\tilde{\mathfrak{R}}(0,x,1)$ of the order parameter of the CTTP and Manna model at the critical density for $D=2$. The metric factors are given by $e=\mathfrak{a}_h\mathfrak{a}_R^4$ and $d=\mathfrak{a}_R^2$. The dashed lines correspond to the asymptotic behavior of the two-dimensional system ($\beta_{D=2}/\sigma_{D=2}=0.287$) and of the mean-field behavior ($\beta_{\rm MF}/\sigma_{\rm MF}=1/2$). The universal amplitude is given by $m_{a,h}=0.681$. The inset displays the order parameter of the CTTP for various values of the interaction range $R\in\{1,2,4,\ldots,128\}$ (from top to bottom). The dashed lines are just to guide the eyes.

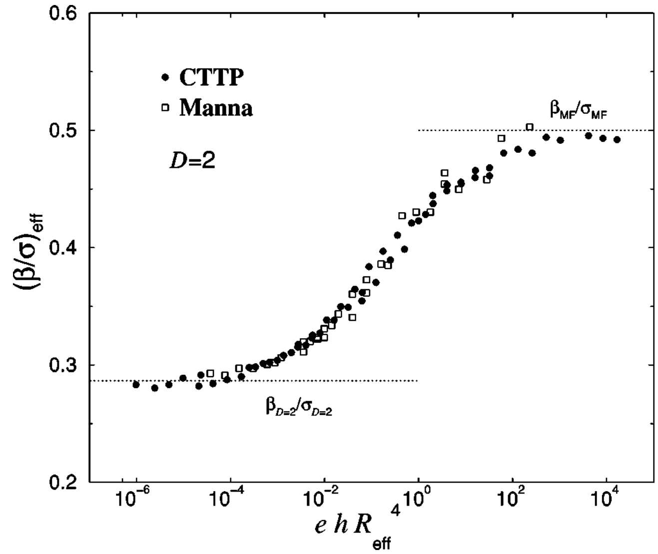

FIG. 2. The effective exponent $(\beta/\sigma)_{\rm eff}$.

$$m_{a,h}=\left(\frac{a_{h,R=1}}{a_{h,R\to\infty}}\right)^{\beta_D/\sigma_D}\mathfrak{a}_R^{\beta_{\rm MF}/\varphi-\sigma_{\rm MF}\beta_D/\sigma_D\varphi}. \tag{7}$$

According to the scaling form Eq. (5) we plot in Fig. 1 the rescaled order parameter $\rho_a(\mathfrak{a}_R R_{\rm eff})^2$ as a function of the rescaled field $\mathfrak{a}_h h(\mathfrak{a}_R R_{\rm eff})^4$. We observe an excellent data collapse for the full crossover behavior confirming the above phenomenological scaling ansatz. However, since the entire crossover region covered several decades it could be difficult to observe small but systematic differences between the scaling functions of both models. Therefore it its customary to scrutinize the crossover via the so-called effective exponent [6,9–11]

$$\left(\frac{\beta}{\sigma}\right)_{\rm eff}=\frac{\partial}{\partial\,\ln x}\ln\tilde{\mathfrak{R}}(0,x,1). \tag{8}$$

The perfect collapse of the corresponding data is shown in Fig. 2 and confirms again the universality of the crossover scaling function $\tilde{\mathfrak{R}}$.

Next we consider the order parameter susceptibility. The scaling form of the susceptibility is given by

$$\mathfrak{a}_\chi\,\chi_{\rm a}(\rho,h,R_{\rm eff})\sim\lambda^{\gamma_{\rm MF}}\,\tilde{\mathfrak{C}}(\mathfrak{a}_\rho(\rho-\rho_{{\rm c},R})\,\lambda,\mathfrak{a}_h h\,\lambda^{\sigma_{\rm MF}},\mathfrak{a}_R^{-1}R_{\rm eff}^{-1}\,\lambda^{\varphi}). \tag{9}$$

On the other hand the susceptibility is defined as the derivative of the order parameter with respect to the conjugated field

$$\chi_{\rm a}(\rho,h,R_{\rm eff})=\frac{\partial}{\partial h}\rho_{\rm a}(\delta\rho,h)\sim\mathfrak{a}_h\lambda^{\sigma_{\rm MF}-\beta_{\rm MF}}\tilde{\mathfrak{R}}'(\mathfrak{a}_\rho(\rho-\rho_{{\rm c},R})\lambda,\mathfrak{a}_h h\lambda^{\sigma_{\rm MF}},\mathfrak{a}_R^{-1}R_{\rm eff}^{-1}\lambda^{\varphi}) \tag{10}$$

with $\tilde{\mathfrak{R}}'(x,y,z)=\partial_y\tilde{\mathfrak{R}}(x,y,z)$. By comparing this expression with Eq. (9) we find $\tilde{\mathfrak{C}}(x,y,z)=\partial_y\tilde{\mathfrak{R}}(x,y,z)$, $\mathfrak{a}_\chi=\mathfrak{a}_h^{-1}$, as well as the Widom scaling law

$$\gamma=\sigma-\beta \tag{11}$$

which is well known from equilibrium phase transitions.

Again, the mean-field behavior is recovered for $R\to\infty$, i.e., $\tilde{\mathfrak{C}}(x,y,0)=\tilde{C}_{\rm MF}(x,y)=1/2\,[y+(x/2)^2]^{-1/2}$, implying $\tilde{\mathfrak{C}}(1,0,0)=\tilde{C}_{\rm MF}(1,0)=1$, $\tilde{\mathfrak{C}}(0,1,0)=\tilde{C}_{\rm MF}(0,1)=1/2$, as well as $\gamma_{\rm MF}=1$.

Similar to the order parameter we plot the susceptibility according to the scaling form

$$\mathfrak{a}_h^{-1}\chi_{\rm a}(\rho_{{\rm c},R},h,R_{\rm eff})\sim(\mathfrak{a}_R R_{\rm eff})^{\gamma_{\rm MF}/\varphi}\tilde{\mathfrak{C}}(0,\mathfrak{a}_h h\,(\mathfrak{a}_R R_{\rm eff})^{\sigma_{\rm MF}/\varphi},1). \tag{12}$$

Approaching the transition point the susceptibility is expected to scale as $\tilde{\mathfrak{C}}(0,x,1)\sim m_{\chi,h}x^{-\gamma_D/\sigma_D}$, for $x\to 0$, where the universal power-law amplitude is given by

$$m_{\chi,h}=\left(\frac{a_{h,R=1}}{a_{h,R\to\infty}}\right)^{1-\gamma_D/\sigma_D}\mathfrak{a}_R^{-\gamma_{\rm MF}/\varphi+\sigma_{\rm MF}\gamma_D/\sigma_D\varphi}\frac{\beta_D}{\sigma_D}. \tag{13}$$

The rescaled susceptibility is shown in Fig. 3. Over the entire crossover region we got an excellent data collapse including both asymptotic scaling regimes. The inset displays the effective exponent

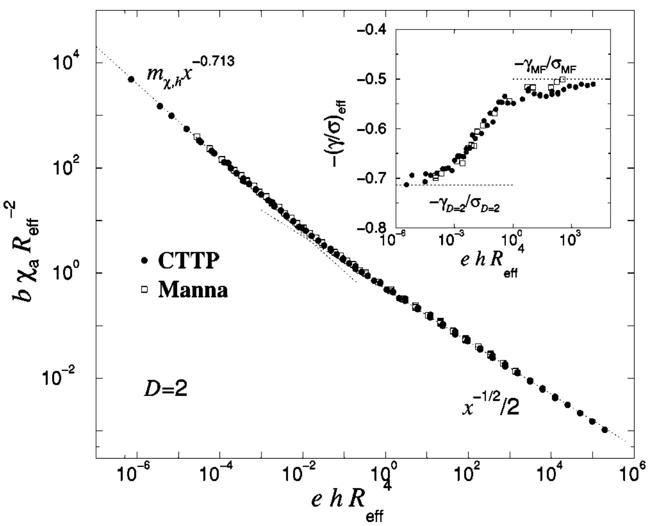

FIG. 3. The universal crossover scaling function $\tilde{\mathfrak{C}}(0,x,1)$ of the susceptibility of the CTTP and the Manna model at the critical density for $D=2$. The metric factors are given by $e=\mathfrak{a}_h\mathfrak{a}_R^4$ and $b=\mathfrak{a}_h^{-1}\mathfrak{a}_R^2$. The dashed lines correspond to the asymptotic behavior of the two-dimensional system ($\gamma_{D=2}/\sigma_{D=2}=0.713$) and of the mean-field behavior ($\gamma_{\rm MF}/\sigma_{\rm MF}=1/2$). The universal amplitude is given by $m_{\chi,h}=0.208$. The inset displays the corresponding effective exponent $(\gamma/\sigma)_{\rm eff}$.

$$\left(\frac{\gamma}{\sigma}\right)_{\rm eff}=-\frac{\partial}{\partial\ \ln x}\ \ln\ \tilde{\mathfrak{C}}(0,x,1) \tag{14}$$

which exhibits again a monotonic crossover from the two-dimensional scaling regime to the mean-field scaling behavior.

IV. WIDOM SCALING LAW

In this way we have obtained the effective exponents $(\beta/\sigma)_{\rm eff}$ and $(\gamma/\sigma)_{\rm eff}$ for the field driven crossover from mean-field to non-mean-field behavior. Thus we are able to check the corresponding Widom scaling law

$$\left(\frac{\gamma}{\sigma}\right)_{\rm eff}=1-\left(\frac{\beta}{\sigma}\right)_{\rm eff}, \tag{15}$$

for the whole crossover region. The corresponding data are shown in Fig. 4. As can be seen the Widom scaling law is fulfilled for the asymptotic regimes ($D=2$ scaling behavior and mean-field scaling) but it is clearly violated for the intermediate crossover region. This result is not surprising if one notices that the above Widom law [Eq. (15)] corresponds to the differential equation [see Eqs. (8) and (14)]

$$-\frac{\partial\ \ln}{\partial\ \ln x}\frac{\partial}{\partial x}\tilde{\mathfrak{R}}(0,x,1)=1-\frac{\partial\ \ln}{\partial\ \ln x}\tilde{\mathfrak{R}}(0,x,1). \tag{16}$$

Using $1=\partial\ \ln\ ax/\partial\ \ln\ x$ we get

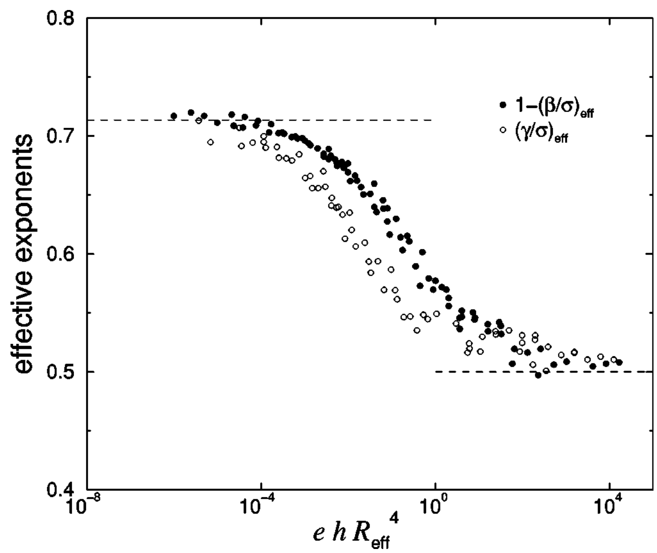

FIG. 4. The violation of Widom scaling law [Eq. (15)] in the crossover regime.

$$-\ln\ \partial_x\tilde{\mathfrak{R}}(0,x,1)=\ln\ ax-\ln\ \tilde{\mathfrak{R}}(0,x,1)+c, \tag{17}$$

where c is some constant. It is straightforward to show that this differential equation is solved by simple power laws [$\tilde{\mathfrak{R}}(0,x,1)=c_0x^{c_1}$ with $c_1=1/a\ \exp\ c$]. Thus the Widom scaling law is fulfilled in the asymptotic regimes only. In the case that the scaling behavior is affected by crossovers, confluent singularities, etc., no pure power laws occur and the scaling laws do not hold for the corresponding effective exponents.

V. CONCLUSION

In conclusion, the crossover from mean-field to non-mean-field scaling behavior is numerically investigated for two different models exhibiting a second order phase transition. Increasing the range of interactions we are able to cover the full crossover region which spans several decades of the conjugated field. The corresponding data show that the Widom scaling law is violated in the crossover regime. Notice that we focus in our investigations on the particular universality class of absorbing phase transitions only for technical reasons. The demonstrated violation of the Widom scaling can be applied to continuous phase transitions in general.

ACKNOWLEDGMENTS

We would like to thank A. Hucht, K. D. Usadel, and B. Schnurr for helpful discussions. This work was financially supported by the Minerva Foundation (Max Planck Gesellschaft).

[1] K. G. Wilson, Phys. Rev. B **4**, 3174 (1971).
[2] K. G. Wilson, Phys. Rev. B **4**, 3184 (1971).
[3] J. M. Yeomans, *Statistical Mechanics of Phase Transitions* (Clarendon, Oxford, 1992).
[4] E. Luijten, H. W. J. Blöte, and K. Binder, Phys. Rev. Lett. **79**, 561 (1997).
[5] E. Luijten, H. W. J. Blöte, and K. Binder, Phys. Rev. E **54**, 4626 (1996).
[6] E. Luijten and K. Binder, Phys. Rev. E **58**, R4060 (1998).
[7] A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. E **58**, 7146 (1998).
[8] A. Pelissetto, P. Rossi, and E. Vicari, Nucl. Phys. B **554**, 552 (1999).
[9] S. Caracciolo *et al.*, Phys. Rev. E **64**, 046130 (2001).
[10] S. Lübeck, Phys. Rev. Lett. **90**, 210601 (2003).
[11] E. K. Riedel and F. J. Wegner, Phys. Rev. B **9**, 294 (1974).
[12] D. S. Greywall and G. Ahlers, Phys. Rev. Lett. **28**, 1251 (1972).
[13] F. J. Wegner, Phys. Rev. B **5**, 4529 (1972).
[14] G. Ahlers, in *Critical Phenomena and the Superfluid Transition in* ^{4}*He in Quantum Liquids*, edited by J. Ruvalds and T. Regge (North-Holland, Amsterdam, 1978).
[15] M.-C. Chang and A. Houghton, Phys. Rev. Lett. **44**, 785 (1980).
[16] M. Rossi, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. Lett. **85**, 1803 (2000).
[17] S. S. Manna, J. Phys. A **24**, L363 (1991).
[18] A. Vespignani, R. Dickman, M. A. Muñoz, and S. Zapperi, Phys. Rev. E **62**, 4564 (2000).
[19] S. Lübeck, Phys. Rev. E **65**, 046150 (2002).
[20] S. Lübeck, Phys. Rev. E **66**, 046114 (2002).
[21] S. Lübeck and P. C. Heger, Phys. Rev. Lett. **90**, 230601 (2003).
[22] V. L. Ginzburg, Sov. Phys. Solid State **2**, 1824 (1960) [Fiz. Tverd. Tela **2**, 2031 (1960)].
[23] K. K. Mon and K. Binder, Phys. Rev. E **48**, 2498 (1993).
[24] V. Privman and M. E. Fisher, Phys. Rev. B **30**, 322 (1984).
[25] S. Lübeck and A. Hucht, J. Phys. A **35**, 4853 (2002).