Non-Markovian rotating unstable processes driven by Gaussian colored noise

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In this paper the statistical properties of the mean passage time distribution are used to characterize the decay process of non-Markovian rotating unstable processes driven by Gaussian colored noise and subjected to the influence of a constant external force. The time characterization will be linear and studied in two limiting cases: large and intermediate times. General systems of two variables are studied. In both schemes we show that, for small correlation time of the noise, the non-Markovian effects are taken into account by an effective noise intensity. To compare qualitatively the non-Markovian time scale with respect to the Markovian case, we apply those results to determine the detection bandwidth of a large external signal in a laser system.

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In this Brief Report we extend the study of Ref. [1]. to the case of non-Markovian characterization of the rotating unstable processes driven by Gaussian colored noise (GCN). In the latter, the time characterization will be studied in two limiting cases: large and intermediate times. Explicit results are obtained in the quasi-Markovian approximation—i.e., for small correlation time of the noise. The effects of this non-Markovian contribution are taken into account through a rescaling in the intensity of the noise. We consider that the Markovian characterization of the rotating unstable processes proposed in Ref. [1] has been well established and that it provides the understanding of why the quasideterministic (QD) approach works well in the time characterization of the rotational laser system studied in Refs. [2–4]. Although the nonlinear contributions have also been well characterized, in this paper we only study the linear regime of the decay process because in our theoretical description the stochastic fluctuations are important at the initiation of such a decay process. After this stochastic beginning the dynamics is practically deterministic and the noise plays no important role. The non-Markovian character of the problem makes the mathematics much more difficult to manipulate than that worked out in the Markovian case; in fact, at the stochastic beginning of the decay process the coupling between the rotation parameter of the system and the correlation time of the noise arises in a natural way. However, in the approximation of small correlation time, the mathematics is less complicated and very similar to that given in Ref. [1]. The matrices associated with the initial fluctuations, which in general are not diagonal, satisfy the requirements of a diagonal matrix whose elements contain the non-Markovian effects through a rescaled noise intensity. Our theoretical results are applied to the laser system studied in Ref. [1] using the same experimental data, and the qualitative comparison between both time scales, Markovian and non-Markovian, will be shown when they are used to determine the detection bandwidth of an external large optical signal in the laser.

It is well known that during the 1970s and 1980s the problem of calculating the mean first passage time (MFPT) distribution for continuous Markovian and non-Markovian processes was studied with great interest by several authors for processes such as activation rates, mean lifetime of metastable states, exit problems, optical bistability, and switch-on processes in lasers, to mention some examples. In particular, the problem of non-Markovian passage times can be found in Refs. [5–16]; in some of them [5,6,9], the scheme of Brownian motion was considered. At those dates, another time scale, named nonlinear relaxation time (NLRT), was proposed as an alternative method to characterize the transient stochastic dynamics of unstable states taking into account the dynamical evolution of the system from an initial unstable state to the corresponding stationary state (saturation regime). The study was also formulated for Markovian [17] and non-Markovian [18] processes. In Ref. [16], the MPT was used to characterize the decay of an unstable state driven by GCN and compared with an analog simulation reported in Ref. [19]. The proposal of Ref. [16] has motivated us to extend the study of Ref. [1] to that situation in which the internal fluctuations satisfy the properties of colored noise. Our theory can stimulate to carry out the analog simulation of the decay of rotating unstable systems triggered by colored noise, as well as experiments in lasers, plasmas, or in some other system in which fluctuation can be important. In the case of the laser systems, we would like to make some comments: In Refs. [2–4], the transient statistics, through the MPT distribution, of a certain type of laser, was studied assuming that the spontaneous emission noise, also called quantum noise, which models the internal fluctuations, satisfies the properties of Gaussian white noise (GWN). The validity of this model has been well established after a good comparison with the experimental results [20] and numerical simulation. However, in Ref. [21], experiments on the transient statistics of the growth of dye-laser radiation showed the importance of the random fluctuations of a control parameter, also called the pump parameter, as well as of the quantum noise in the time characterization of that laser radiation. The theory proposed to explain the statistical properties of the growth of dye-laser radiation was given in Ref. [22] assuming the properties of a colored noise for that control parameter. The theory was then compared with those experimental results and the numerical simulation showing good agreement. On the other hand, experiments in the tran-

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sient statistics of those laser systems which involve colored quantum noise have not been reported yet. This paper suggests experiments in that direction. We are looking at the possibility to extend our proposal to the study of electron diffusion in gases when both an electric and a magnetic field is present.

Our study is developed in the linear approximation of the nonlinear rotating unstable Langevin-type dynamics submitted to the action of an external force, as that proposed in Ref. [1]. In the space of coordinates $\mathbf{x}$ it is written as $\dot{\mathbf{x}}=a\mathbf{x}+W\mathbf{x}+\mathbf{f}_e+\mathbf{z}(t)$ and its corresponding transformed space of coordinates $\mathbf{y}$, obtained by means of the change of variable $\mathbf{y}=e^{-Wt}\mathbf{x}$, reads

$$\dot{\mathbf{y}} = a\mathbf{y} + R^{-1}(t)\mathbf{f}_e + R^{-1}(t)\mathbf{z}(t), \tag{1}$$

where a is real and positive, the matrix W is a real antisymmetric matrix, $R(t)$ is an orthogonal rotation matrix [1], $\mathbf{f}_e$ is the external force, and $\mathbf{z}(t)$ is the fluctuating force whose elements $\xi_i(t)$ satisfy the property of Gaussian colored noise with zero mean value and correlation function

$$\langle \xi_i(t)\xi_j(t')\rangle = \frac{Q_{ij}}{\tau}\delta_{ij}e^{-|t-t'|/\tau}, \tag{2}$$

where Q_{ij} is the matrix which represents the noise intensity and τ the correlation time.

(a) *The MPT for large times*. Following Ref. [1], it is shown in this limiting case that the QD approach tells us that the random passage time required by the system to reach the prescribed reference value R_e^2 is $2at=\ln(R_e^2/h^2)$ whose statistical moments can be calculated from the generating function defined as $G(2a\lambda)=\langle e^{-2a\lambda t}\rangle$ or $G(2a\lambda)=\langle (R_e^2/h^2)^{-\lambda}\rangle$, h being a Gaussian random variable. For two-variable systems and in the approximation of small correlation time, we show that the correlation matrix associated with the effective initial conditions is diagonal with elements $\sigma_{12}=\sigma_{21}=0$ and $\sigma_{11}=\sigma_{22}=\sigma^2$ such that $\sigma^2=Q/a(1+a\tau)=Q'/a$ where $Q'=Q/(1+a\tau)$. So, in this limit of approximation, the rotation parameter ω of the matrix W is decoupled from the correlation time and therefore σ^2 is very similar to that obtained in the GWN case if the noise intensity Q is rescaled by the factor $1/(1+a\tau)$. In this case, both the MPT and its variance are the same as those calculated in the GWN case except that the noise intensity Q is replaced by Q'. Then, the MPT distribution is approximated by

$$\langle 2at\rangle = \langle 2at(\tau=0)\rangle + a\tau, \tag{3}$$

where $\langle 2at(\tau=0)\rangle=\langle 2at\rangle_0-\beta^2+\beta^4/4$ is the passage time in the limit of GWN, $\langle 2at\rangle_0=\ln(aR_e^2/2Q)-\psi(1)$ is the MPT in the absence of external force, $\psi(1)$ is the Euler constant, and $\beta^2=a|\mathbf{f}_e|^2/2Q'(a^2+\omega^2)$. The variance is reduced to

$$\langle (2a\Delta t)^2\rangle = \psi'(1) - \frac{\beta^4}{2}. \tag{4}$$

Both quantities—the MPT and its variance—correspond to the case in which the decay process is dominated by the small noise intensity and a small correlation time—that is, $\beta \leqslant 1$. In Fig. 1 we compare the MPT distribution (3), with

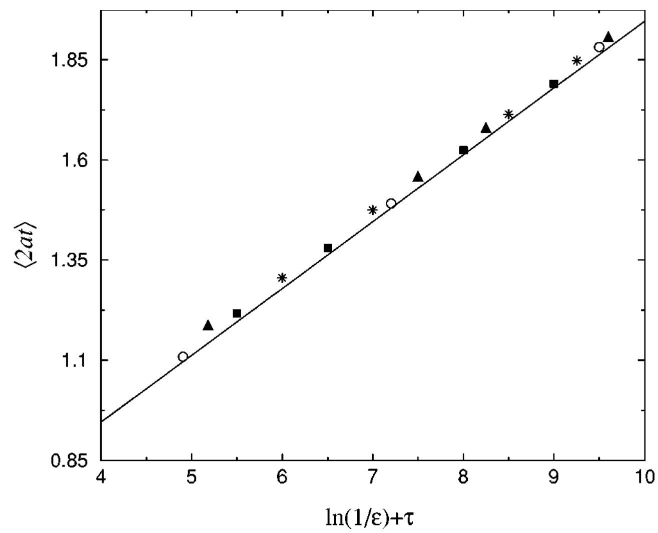

FIG. 1. Comparison between the time scale given by Eq. (3) rescaled with the variable $\ln(1/\epsilon)+\tau$ and numerical simulation for the values $|\mathbf{f}_e|=Q\equiv\varepsilon$, $a=3.0$, $\omega=6.0$, $R_e=1.0$, and different values of τ. The simulation results correspond to values of Q between 10^{-2} and 10^{-5}. The straight line corresponds to theoretical results, Eq. (3); white circles are the simulation results, for $\tau=0$, black squares simulation results for $\tau=0.1$, stars for $\tau=0.2$, and black triangles for $\tau=0.3$.

the numerical simulation for some values of the correlation time. We appreciate the agreement between both results.

(b) *The MPT for intermediate times*. If the amplitude of the external signal is greater than the intensity of the effective noise, then $\beta \gg 1$. In this case the QD in no longer valid and therefore the second alternative approach proposed in Ref. [1] must be used. Again, following the strategy of this reference we calculate the elements of the correlation matrix. In this case they are much more complicated than that calculated in the GWN case. It can be shown after long algebra that, for small correlation time, those elements can be approximated by

$$\langle h_i(t)h_j(t)\rangle = \langle h_i(t)\rangle\langle h_j(t)\rangle + \frac{Q'}{a}\left(1-e^{-2at}\right)\delta_{ij}, \tag{5}$$

where $Q'=Q/(1+a\tau)$, which again defines an effective noise intensity and therefore Eq. (5) is very similar as that given in the GWN case. It coincides with the elements $\sigma_{11}=\sigma_{22}$ of previous the previous section as the time goes to infinity. Therefore, the MPT distribution will be

$$\langle t\rangle = t_p = t_0 - \frac{1}{2a}\ln[1+\phi(t_p)], \tag{6}$$

with $2at_0=\ln[R_e^2(a^2+\omega^2)/|\mathbf{f}_e|^2]$ being the deterministic time scale. For large amplitude of the external force—i.e., $\beta \gg 1$—the variance is now

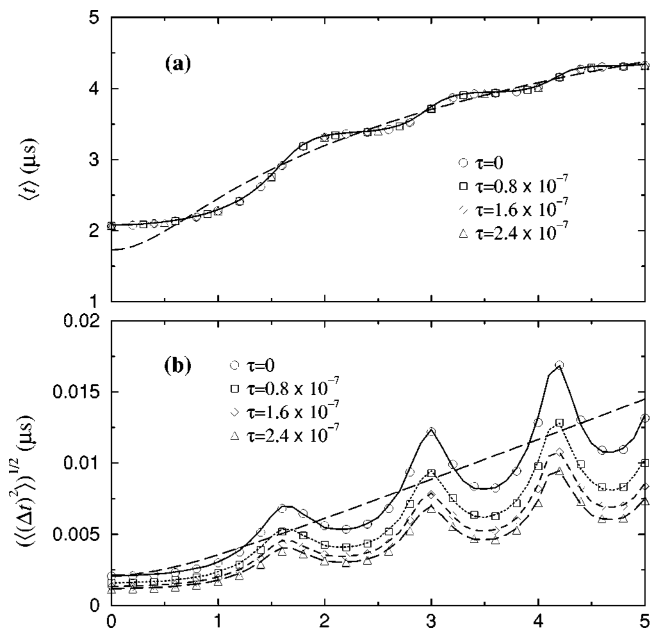

FIG. 2. (a) Linear mean passage time and (b) variance (jitter) as a function of the rotation parameter ω. (a) The dashed line is the deterministic time scale t_0, the solid line corresponds to Eq. (6) and the symbols are the simulation results for some values of τ. (b) The straight dashed line is the variance (jitter) $\langle(\Delta t)^2\rangle \approx 1/\beta^2$. The solid, doted, short dashed, and long dashed curves correspond to the iteration of Eq. (7) for different values of τ. The symbols are the simulation results.

$$\langle(\Delta t)^2\rangle = \frac{g^2(t_p)}{a^2\beta^2[1+\phi(t_p)]}\left[1+\frac{\phi'(t_p)}{2a(1+\phi(t_p))}\right]^{-2}, \quad (7)$$

where $\beta^2 = a|\mathbf{f}_e|^2/2Q'(a^2+\omega^2)$ and $\phi(t) = [e^{-2at} - 2e^{-at}\cos\,\omega t]$. Clearly, the MPT distribution is dominated only by the deterministic approximation and therefore does not depend on the type of noise, whereas the variance contains the non-Markovian contribution through the effective intensity of noise Q'. Finally the MPT and its variance are calculated through the iterative procedure $t_p^{(0)} = t_0$ and $t_p^{(n+1)} = t_0 - (1/2a)\ln[1+\phi(t_p^{(n)})]$.

In Figs. 2(a) and 2(b), we plot our analytical predictions given by Eqs. (6) and (7) and the nonoscillating time scale t_0, and compare them with the numerical simulation of the laser system studied in Ref. [1]. According to Refs. [1–4], the control parameter (or pump parameter) of the laser system is $a = F - k$, which is a constant quantity, F and k being the laser parameters.

In Fig. 2(a), the MPT (6) is compared with both the deterministic time scale t_0 and the numerical simulation for the laser system. In Fig. 2(b), we plot some variances (jitters) given by Eq. (7) for different values of the correlation time and compare them with the numerical simulation of the laser system. The straight (dashed) line is the deterministic approximation given by $\langle(\Delta t)^2\rangle \approx 1/\beta^2$ which is not oscillatory. In this plot, to appreciate the non-Markovian effects, we must consider the correlation time as a rescaled quantity given by τ/k, with $\tau = 1.0$, $\tau = 2.0$, and $\tau = 3.0$ divided by k.

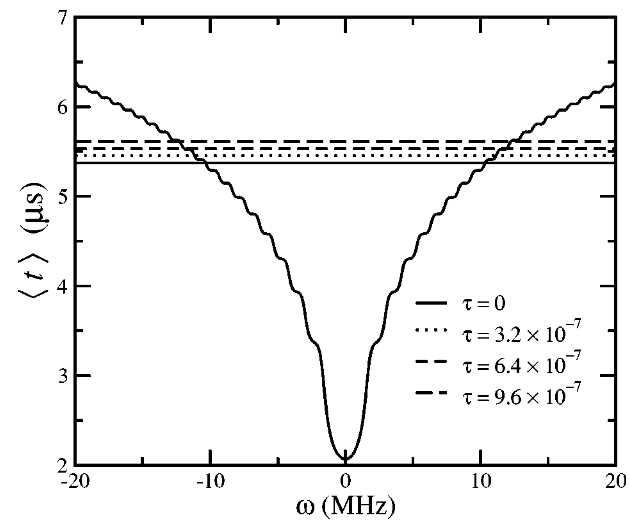

FIG. 3. The effects of the colored noise on the detection bandwidth of a large external signal in a laser. The detection bandwidth is taken as the FWHM of this plot. The curve is the third iteration of Eq. (6). The horizontal lines are the one-half of the switch-on time $\langle t\rangle_0$ given in Eq. (3) when $E_e = 0$. The solid line is the Markovian case for $\tau = 0$. The non-Markovian effects are shown by the dotted, short, and long dashed lines. The detection bandwidth is approximately 20.6 MHz for $\tau = 0$, 22.4 MHz for $\tau = 4.0$, 23.3 MHz for $\tau = 8.0$, and 24.5 MHz for $\tau = 12.0$.

The comparison between the theory and the simulation results in both figures is excellent. In Fig. 3, we show a qualitative comparison between the time scales (3) and (6) when both are used to determine the detection bandwidth of a large injected signal in the laser studied in Ref. [1]. We use the same criterion of that reference, which is defined as that for which the limit of detection on the detuning reduces the initiation time to one-half of that corresponding to the off state—i.e., when the external electric field $E_e = 0$. The detection bandwidth can be taken as the full width at half maximum (FWHM) of the plot in Fig. 3. Again, to appreciate the non-Markovian effects we look at the rescaled value τ/k. With the values of $\tau = 0.0$, $\tau = 4.0$, $\tau = 8.0$, and $\tau = 12.0$ we get the respetive detection bandwidths of 20.6 MHz, 22.4 MHz, 23.3 MHz, and 24.5 MHz, approxiamtely. Therefore the detection bandwidth is amplified as the correlation time increases.

To conclude we can say that, in the quasi-Markovian approximation (small correlation time) and at the beginning of the decay process in both limiting cases, the memory effects of correlation time and the rotation parameter of the system are decoupled from each other. This is shown in the diagonal reduction of the correlation matrix associated with the initial fluctuations, with the non-Markovian contribution taken into account by an effective noise intensity. So when the dynamics is dominated by the effective noise Q' no rotational effects can be appreciated and therefore the MPT distribution appropriate to characterize this dynamical evolution is that given by Eq. (3). It is compared with the numerical simulation as shown in Fig. 1 for arbitrary values of the parameters, showing excellent agreement between both results. If the intensity of the external force dominates over that of the effective noise, the dynamics is rotational and practically deterministic. In this case the appropriate time characterization is given by Eq. (6). The variance (7) contains the non-Markovian effects through an effective noise. Finally the time scales (3) and (6) employed to determine the detection bandwidth of a large injected signal in a laser, used as a prototype model, show that the colored internal noise amplifies the detection bandwidth, as shown in Fig. 3.

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[1] J. I. Jiménez-Aquino and M. Romero-Bastida, Phys. Rev. E **66**, 061101 (2002).
[2] S. Balle, F. De Pasquale, and M. San Miguel, Phys. Rev. A **41**, 5012 (1990).
[3] J. Dellunde, J. M. Sancho, and M. San Miguel, Opt. Commun. **39**, 435 (1994).
[4] J. Dellunde, M. C. Torrent, and J. M. Sancho, Opt. Commun. **102**, 277 (1993).
[5] F. Grote and J. T. Hynes, J. Chem. Phys. **73**, 2715 (1980).
[6] B. Carmeli and A. Nitzan, Phys. Rev. Lett. **49**, 423 (1982).
[7] P. Hanggi and P. Talkner, Phys. Rev. Lett. **51**, 2242 (1983).
[8] P. Hanggi and P. Riseborough, Phys. Rev. A **27**, 3379 (1983).
[9] E. Guardia, F. Marchesoni, and M. San Miguel, Phys. Lett. **100A**, 15 (1984).
[10] P. Hanggi, F. Marchesoni, and P. Grigolini, Z. Phys. B: Condens. Matter **56**, 333 (1984).
[11] C. Van den Broeck and P. Hanggi, Phys. Rev. A **30**, 2730 (1984).
[12] J. M. Sancho, Phys. Rev. A **31**, 3523 (1985).
[13] P. Hanggi, T. J. Mroczkowski, F. Moss, and P. V. E. McClintock, Phys. Rev. A **32**, 695 (1985).
[14] P. Colet, M. San Miguel, J. Casademunt, and J. M. Sancho, Phys. Rev. A **39**, 149 (1989).
[15] F. J. de la Rubia, Enrique Peacock-López, George P. Tsironis, Katja Lindenberg, Laureano Ramirez-Piscina, and J. M. Sancho, Phys. Rev. A **38**, 3827 (1988).
[16] J. M. Sancho and M. San Miguel, Phys. Rev. A **39**, 2722 (1989).
[17] J. I. Jiménez-Aquino, J. Casademunt, and J. M. Sancho, Phys. Lett. A **133**, 364 (1988).
[18] J. Casademunt, J. I. Jiménez-Aquino, and J. M. Sancho, Phys. Rev. A **40**, 5905 (1989).
[19] M. James, F. Moss, P. Hanggi, and C. Van den Broeck, Phys. Rev. A **38**, 4690 (1988).
[20] G. Vemuri and R. Roy, Phys. Rev. A **39**, 2539 (1989).
[21] R. Roy, A. W. Yu, and S. Zhu, Phys. Rev. Lett. **55**, 2794 (1985).
[22] F. de Pasquale, J. M. Sancho, M. San Miguel, and P. Tartaglia, Phys. Rev. Lett. **56**, 2473 (1986).