Noise-enhanced stability in fluctuating metastable states

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We derive general equations for the nonlinear relaxation time of Brownian diffusion in randomly switching potential with a sink. For piece-wise linear dichotomously fluctuating potential with metastable state, we obtain the exact average lifetime as a function of the potential parameters and the noise intensity. Our result is valid for arbitrary white noise intensity and for arbitrary fluctuation rate of the potential. We find noise enhanced stability phenomenon in the system investigated: The average lifetime of the metastable state is greater than the time obtained in the absence of additive white noise. We obtain the parameter region of the fluctuating potential where the effect can be observed. The system investigated also exhibits a maximum of the lifetime as a function of the fluctuation rate of the potential.

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I. INTRODUCTION

Activated-escape in systems with metastable states underlies many physical, chemical, and biological problems. Examples are crystal growth, tunnel diode, lasers, quantum liquids, spin systems, protein folding, and polymer physics [1,2]. The most interesting and stubborn case of metastable state is one described by time dependent potential, which fluctuates on a characteristic time scale that may vary over a large range. In particular, the metastable states with fluctuating barriers are common to chemical and biological models [3–5], and to a wide range of physical problems, such as nonequilibrium transport models, molecular dissociation in strongly coupled chemical systems [6], ratchet models for the action of molecular motors [7], noise in microstructures and generation process of carrier traps in semiconductors [8–10]. The escape from metastable state with fluctuating or randomly switching barrier was studied in the past mainly by well known mean first passage time (MFPT) technique. In Refs. [11,12] exact results for the MFPT of escape process over fluctuating barrier that switches between two configurations have been obtained. However, the MFPT method requires the implication of absorbing boundary in the system, and it does not take into account the inverse probability current through this boundary. The nonlinear relaxation time (NLRT) method is devoid of this disadvantage [13]. Nevertheless the theory for the NLRT is not well developed and the equations for the NLRT are unknown for the case of time varying potential.

In the present paper we derive general equations for the NLRT for potentials randomly switching between two arbitrary configurations with a sink. We find the exact solution of these equations for a piece-wise linear potential flipping between unstable and metastable configurations, for arbitrary white noise intensity and fluctuation rate of the potential. Analyzing this exact result we focus on the noise enhanced stability (NES) effect, which implies that the system remains in the metastable state for a longer time than in the absence of additive white noise, and the lifetime of the metastable state has a maximum at some noise intensity. This effect, which cannot be described by Kramers-like behavior, was observed and investigated theoretically and experimentally in various physical systems and mainly concerning MFPT in periodically or randomly driven metastable states [3,9,14–24]. In these papers the nonmonotonic behavior of the average escape time was observed: (i) In physical systems, like tunnel diode [14], and Josephson junction [18], where the influence of thermal fluctuations on the superconductive state lifetime and the turn-on delay time for a single Josephson element with high damping was investigated; (ii) in chemical systems, like the one-dimensional return map of the Belousov-Zhabotinsky reaction, by investigating the behavior of the length of the laminar region as a function of the noise intensity [22], and (iii) in biologically motivated models, such that investigated in Ref. [3], where the overdamped motion of a Brownian particle moving in an asymmetric fluctuating potential shows noise induced stability.

Here we study the NES phenomenon for the NLRT in randomly switching metastable state, and we obtain analytically the region of system parameters, where this effect takes place. We find also resonant activation phenomenon by investigating the mean lifetime as a function of switchings mean rate. Moreover, we find that the NLRT exhibits a maximum as a function of barrier switching rate. This new resonant-like phenomenon is related to the NES effect [9,14].

The paper is organized as follows. In the second section we derive the general equations for the nonlinear relaxation time of Brownian diffusion in randomly switching potential with a metastable state. In the third section we analytically derive the mean lifetime for piece-wise linear potential. In the next section we obtain the condition to observe the NES phenomenon and investigate the behavior of the mean lifetime as a function of switchings mean rate. In the final section we draw the conclusions.

II. GENERAL EQUATIONS

We consider the one-dimensional overdamped Brownian motion in switching potential profile

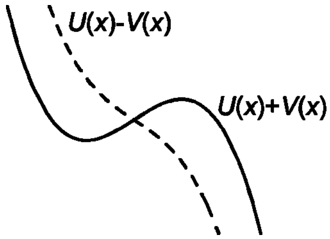

FIG. 1. Switching potential with metastable state.

$$\frac{dx}{dt} = -\frac{\partial \Phi(x,t)}{\partial x} + \xi(t),$$

$$\Phi(x,t) = U(x) + V(x)\eta(t). \tag{1}$$

Here $x(t)$ is the Brownian particle displacement, $\xi(t)$ is the white Gaussian noise with zero mean and correlation function $\langle \xi(t)\xi(t+\tau)\rangle = 2D\delta(\tau)$. The potential $\Phi(x,t)$ is the sum of two terms: The fixed potential $U(x)$ and the randomly switching term $V(x)\eta(t)$. The variable $\eta(t)$ is the Markovian dichotomous noise, which takes the values ± 1 with the mean flipping rate ν. If we invoke the following expression for probability density in terms of the average

$$W(x,t) = \langle \delta(x - x(t))\rangle \tag{2}$$

and introduce auxiliary function $Q(x,t)$

$$Q(x,t) = \langle \eta(t)\delta(x - x(t))\rangle, \tag{3}$$

we obtain the next closed set of equations (see [25,26], and the Appendix)

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x}[U'(x)W + V'(x)Q] + D\frac{\partial^2 W}{\partial x^2},$$

$$\frac{\partial Q}{\partial t} = -2\nu Q + \frac{\partial}{\partial x}[U'(x)Q + V'(x)W] + D\frac{\partial^2 Q}{\partial x^2}. \tag{4}$$

Let $x = x_0$ be the initial position of Brownian particles. Then

$$W(x,0) = \delta(x - x_0), \tag{5}$$

and $W(x,t)$ becomes the conditional probability density $W(x,t|x_0,0)$. Since $\eta(0)$ is a deterministic value, the initial condition for the function $Q(x,t)$ is [see Eqs. (3) and (5)]

$$Q(x,0) = W(x,0)\eta(0) = \pm\,\delta(x - x_0). \tag{6}$$

Let us consider the potential profiles $U(x) \pm V(x)$ with a wall at $x \to -\infty$ and a sink at $x \to +\infty$ (see Fig. 1). The potential profile $U(x)+V(x)$ corresponds to a metastable state, and $U(x)-V(x)$ corresponds to an unstable one.

Thus, we investigate the system with randomly switching metastable state.

The nonlinear relaxation time (NLRT) for the state located in the interval (L_1, L_2) is defined as follows [13]

$$\tau(x_0) = \int_0^\infty dt \int_{L_1}^{L_2} W(x,t|x_0,0)\,dx, \tag{7}$$

where $x_0 \in (L_1, L_2)$. The NLRT is also interpreted as mean lifetime of Brownian particles in the interval (L_1, L_2) or average residence time, because, in accordance with Eqs. (2) and (7) can be rewritten as conditional time average

$$\tau(x_0) = \left\langle \int_0^\infty \theta(x(t) - L_1)\,\theta(L_2 - x(t))\,dt \Big| x(0) = x_0 \right\rangle,$$

where $\theta(x)$ is the step function.

Let us rewrite the definition (7) in the form

$$\tau(x_0) = \int_{L_1}^{L_2} Y(x,x_0,0)\,dx, \tag{8}$$

where $Y(x,x_0,s)$ is the Laplace transform of conditional probability density $W(x,t|x_0,0)$. After Laplace transforming Eqs. (4), with initial conditions (5) and (6), we obtain the following closed set of ordinary differential equations

$$DY'' + [U'(x)Y + V'(x)R]' - sY = -\delta(x - x_0),$$

$$DR'' + [U'(x)R + V'(x)Y]' - (s + 2\nu)R = \mp\,\delta(x - x_0), \tag{9}$$

where $R(x,x_0,s)$ is the Laplace transform of auxiliary function $Q(x,t)$, defined by Eq. (6). Using the method proposed in Ref. [27], we expand the functions $sY(x,x_0,s)$ and $sR(x,x_0,s)$ in power series in s

$$sY(x,x_0,s) = Z_0(x,x_0) + sZ_1(x,x_0) + \cdots$$

$$sR(x,x_0,s) = R_0(x,x_0) + sR_1(x,x_0) + \cdots. \tag{10}$$

Since all Brownian particles move to the sink located at the point $x = +\infty$ (see Fig. 1) we have zero stationary probability distribution, i.e.,

$$\lim_{t\to\infty} W(x,t|x_0,0) = \lim_{s\to 0} sY(x,x_0,s) = 0.$$

As a consequence, in expansions (10) $Z_0(x,x_0) = 0$, $R_0(x,x_0) = 0$, and the definition (8) becomes

$$\tau(x_0) = \int_{L_1}^{L_2} Z_1(x,x_0)\,dx. \tag{11}$$

Substituting the expansions (10) in Eqs. (9) and equating the terms without s, we obtain the following set of equations for the functions $Z_1(x,x_0)$ and $R_1(x,x_0)$

$$DZ_1'' + [U'(x)Z_1 + V'(x)R_1]' = -\delta(x - x_0),$$

$$DR_1'' + [U'R_1 + V'Z_1]' - 2\nu R_1 = \mp\,\delta(x - x_0). \tag{12}$$

Because of the reflecting boundary at $x = -\infty$, the probability flow equals zero at this point, and from Eqs. (4) we have

$$\left[D\frac{\partial W}{\partial x} + U'(x)W + V'(x)Q \right]_{x=-\infty} = 0,$$

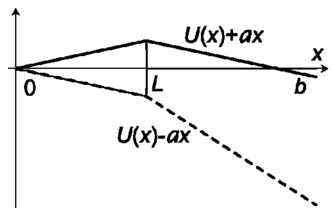

FIG. 2. Switching piece-wise linear potential.

$$\left[D\frac{\partial Q}{\partial x}+U'(x)Q+V'(x)W\right]_{x=-\infty}=0. \quad (13)$$

Making the Laplace transform of Eqs. (13) and substituting the expansions (10), we obtain the following conditions for the functions $Z_1(x,x_0)$ and $R_1(x,x_0)$

$$[DZ_1'+U'(x)Z_1+V'(x)R_1]_{x=-\infty}=0,$$

$$[DR_1'+U'(x)R_1+V'(x)Z_1]_{x=-\infty}=0. \quad (14)$$

By integrating the system (12) from $-\infty$ to x, with boundary conditions (14), we obtain the following closed set of integro-differential equations for the functions $Z_1(x,x_0)$ and $R_1(x,x_0)$

$$DZ_1'+U'(x)Z_1+V'(x)R_1=-\theta(x-x_0),$$

$$DR_1'+U'(x)R_1+V'(x)Z_1=2\nu\int_{-\infty}^{x}R_1dy\mp\theta(x-x_0). \quad (15)$$

These general equations allow to calculate the NLRT for potential profiles above-defined. We may consider two mean lifetimes $\tau_+(x_0)$ and $\tau_-(x_0)$, depending on the initial configuration of the randomly switching potential profile $\Phi(x,0)$: $U(x)+V(x)$ or $U(x)-V(x)$. The NLRT (11) is equal to $\tau_+(x_0)$, when we take the sign "−" in the second equation of system (15), and vice versa for $\tau_-(x_0)$.

III. LIFETIMES FOR PIECE-WISE LINEAR POTENTIAL

Let us consider a piece-wise linear potential profile (see Fig. 2) with $V(x)=ax$ $(x>0,\ 0<a<k)$ and

$$U(x)=\begin{cases}+\infty, & x<0\\ 0, & 0\leqslant x\leqslant L\\ k(L-x), & x>L\end{cases}. \quad (16)$$

Hereafter we shall analyze the average residence time $\tau_-(0)$ from the interval $(L_1=0,L_2=b)$ with $b>L$, which is finite in deterministic case. We consider the initial position of all Brownian particles at the origin, i.e., $x_0=0$. The potential profile $U(x)+ax$ corresponds to metastable state and $U(x)-ax$ corresponds to unstable one, as indicated in Fig. 2. After substituting the potential (16) and $V(x)=ax$ in Eqs. (15) and choosing the sign "+" we arrive at

$$DZ_1'-k\theta(x-L)Z_1+aR_1=-1,$$

$$DR_1'-k\theta(x-L)R_1+aZ_1=1+2\nu\int_0^x R_1dy. \quad (17)$$

We solve the set of differential equations (17) in the regions $0<x<L$ and $x>L$ separately, and then use the continuity conditions at the point $x=L$

$$Z_{1|L-0}=Z_{1|L+0},\quad R_{1|L-0}=R_{1|L+0}. \quad (18)$$

For $0<x<L$, the solutions of Eqs. (17) read

$$Z_1(x)=c_1\left(\cosh\,\gamma x+\frac{2\nu D}{a^2}\right)+c_2\sinh\,\gamma x+\frac{1}{a}-\frac{2\nu x}{\gamma^2D^2},$$

$$R_1(x)=-\frac{\gamma D}{a}(c_1\sinh\,\gamma x+c_2\cosh\,\gamma x)-\frac{a}{\gamma^2D^2}, \quad (19)$$

where $Z_1(x)\equiv Z_1(x,0)$, $R_1(x)\equiv R_1(x,0)$ and

$$\gamma=\sqrt{\frac{a^2}{D^2}+\frac{2\nu}{D}}. \quad (20)$$

The finite solutions of Eqs. (17) in the interval $(L,+\infty)$ are

$$Z_1(x)=c_3e^{\mu(x-L)}+\frac{1}{k},$$

$$R_1(x)=\frac{c_3(k-\mu D)}{a}e^{\mu(x-L)}, \quad (21)$$

where

$$\mu=\frac{2k}{3D}\left[1+\sqrt{1+3\frac{\gamma^2D^2}{k^2}}\cos\left(\frac{\theta+2\pi}{3}\right)\right],$$

$$\cos\,\theta=-\frac{1+9(\nu D-a^2)/k^2}{[1+3\gamma^2D^2/k^2]^{3/2}}, \quad (22)$$

is the negative root of the following cubic equation

$$\lambda\left(\lambda-\frac{k}{D}\right)^2-\gamma^2\lambda+\frac{2\nu k}{D^2}=0. \quad (23)$$

Substituting the solutions (19) and (21) in the continuity conditions (18) and in the second equation (17), we obtain, after rearrangements, the following compact system of algebraic equations for unknown constants c_1,c_2,c_3

$$c_1\cosh\,\gamma L+c_2\sinh\,\gamma L+c_3\left(\frac{2\nu k}{\mu\Gamma^2}-1\right)=0,$$

$$c_1\sinh\,\gamma L+c_2\cosh\,\gamma L+c_3\frac{k-\mu D}{\Gamma}=-\frac{a^2}{\Gamma^3},$$

$$c_1-c_3\frac{ka^2}{\mu\Gamma^2D}=\frac{ah}{2\nu D}, \quad (24)$$

where

$$\Gamma=\gamma D,$$

$$h = \frac{a}{k} + \frac{2\nu a L}{\Gamma^2} - 1. \tag{25}$$

The solutions of Eqs. (24) are

$$c_1 = \frac{ah}{2\nu D} + \frac{a^3 k(2\nu Da \sinh \gamma L - h\Gamma^3)}{2\nu\Gamma^3 D[a^2 k + D(2\nu k - \mu\Gamma^2) \cosh \gamma L + \mu\Gamma D(\mu D - k) \sinh \gamma L]},$$

$$c_2 = \frac{a\{(\mu\Gamma^2 - 2\nu k)(2\nu Da + h\Gamma^3 \sinh \gamma L) + [h\mu\Gamma^4(k - \mu D) - 2\nu a^3 k]\cosh \gamma L\}}{2\nu\Gamma^3[a^2 k + D(2\nu k - \mu\Gamma^2)\cosh \gamma L + \mu\Gamma D(\mu D - k)\sinh \gamma L]},$$

$$c_3 = \frac{\mu a(2\nu Da \sinh \gamma L - h\Gamma^3)}{2\nu\Gamma[a^2 k + D(2\nu k - \mu\Gamma^2)\cosh \gamma L + \mu\Gamma D(\mu D - k)\sinh \gamma L]}. \tag{26}$$

Substituting the expressions (19) and (21) of the function $Z_1(x)$ in Eq. (11), and using Eqs. (26), we obtain finally the following result for mean lifetime

$$\tau_-(0) = \frac{b}{k} + \frac{\nu L^2}{\Gamma^2} + \frac{a}{2\nu\Gamma^4}\left(\frac{D\Gamma\{2\nu a[\Gamma^2(e^{\mu(b-L)} - 1) + 2\nu kL] + h\Gamma^2(2\nu k - \mu\Gamma^2)\}\sinh \gamma L - 2\nu Da(a^2 k + \mu\Gamma^2 D - 2\nu kD)}{a^2 k + D(2\nu k - \mu\Gamma^2)\cosh \gamma L + \mu\Gamma D(\mu D - k)\sinh \gamma L}\right.$$
$$\left. + \frac{D[h\mu\Gamma^4(\mu D - k) + 2\nu a(a^2 k + \mu\Gamma^2 D - 2\nu kD)]\cosh \gamma L - h\Gamma^4[\Gamma^2(e^{\mu(b-L)} - 1) + 2\nu kL + \mu D(\mu D - k)]}{a^2 k + D(2\nu k - \mu\Gamma^2)\cosh \gamma L + \mu\Gamma D(\mu D - k)\sinh \gamma L}\right). \tag{27}$$

Equation (27) is exact, and was derived without any assumptions on the white noise intensity D and on the mean rate of flippings ν.

IV. CONDITION TO OBSERVE NOISE ENHANCED STABILITY (NES)

Because of the complicated expression (27) of the mean lifetime, we analyze the limiting cases of very large and very small noise intensities. Using the approximate estimations for small parameters γ and μ in the limit $D \to \infty$

$$\gamma \simeq \sqrt{\frac{2\nu}{D}}\left(1 + \frac{a^2}{4\nu D}\right), \quad \mu \simeq -\sqrt{\frac{2\nu}{D}}\left(1 - \frac{k}{2\sqrt{2\nu D}}\right),$$

obtained from Eqs. (20) and (22), we find from Eq. (27)

$$\tau_-(0) = \frac{b}{k} + \frac{L^2}{2D}\left[1 - \frac{bq(1-q)}{\omega L}\right] + o\left(\frac{1}{D}\right). \tag{28}$$

Here $\omega = \nu L/k$ and $q = a/k$ are dimensionless parameters. The parameter q quantifies the degree of potential flatness after the point L (see Fig. 2). Under very large noise intensity D, Brownian particles "do not see" the fine structure of potential profile and move as in the fixed potential $-kx$. Therefore, the NLRT decreases with noise intensity, tending to the value b/k as follows from Eq. (28).

The NES phenomenon should be searched in opposite limiting case of very slow diffusion ($D \to 0$) [9,14,24]. The approximate expressions, obtained in this limit, for parameters γ, Γ and μ are

$$\gamma \simeq \frac{a}{D}\left(1 + \frac{\nu D}{a^2}\right), \quad \Gamma \simeq a\left(1 + \frac{\nu D}{a^2}\right),$$

$$\mu \simeq -\frac{2\omega}{L(1-q^2)}\left[1 - \frac{2\omega D(1+q^2)}{kL(1-q^2)^2}\right]. \tag{29}$$

Substituting Eqs. (29) into Eq. (27), and retaining the terms up to first order in D, we obtain the following expression for NLRT at small noise intensity

$$\tau_-(0) = \tau_0 + \frac{D}{a^2} f(q,\omega,s) + o(D). \tag{30}$$

Here

$$f(q,\omega,s) = \frac{3q^2 + 4q - 5}{2(1-q^2)} + 2\omega\frac{3q^2 + q - 3}{q(1-q^2)} - \frac{2\omega^2}{q^2} + se^{-s}\frac{q^3(1+q^2)}{(1+q)(1-q^2)} + (1-e^{-s})\frac{q(1-q^2-2q^3)}{2(1-q^2)} \tag{31}$$

and

$$\tau_0 = \frac{2L}{a} + \frac{\nu L^2}{a^2} + \frac{b-L}{k} - \frac{q(1-q)}{2\nu}(1-e^{-s}), \tag{32}$$

is the mean lifetime in the absence of white Gaussian noise ($D=0$). In Eqs. (31) and (32) new dimensionless parameter

$$s = \frac{2\omega(b/L - 1)}{1 - q^2},$$

is introduced.

For very slow switchings $\nu \to 0$, we find from Eq. (32)

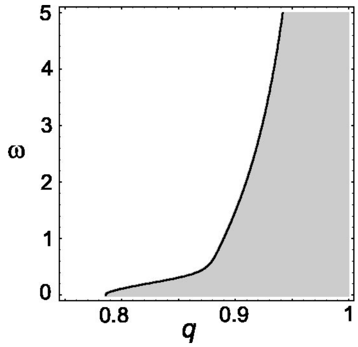

FIG. 3. Shaded area is the parameter region on the plane (q,ω) where NES effect can be observed. Here $\omega=(\nu L)/k$, $q=a/k$, and $b=2L$.

$$\tau_0(\nu\to 0)=\frac{2L}{a}+\frac{b-L}{k+a}, \tag{33}$$

which is different from the deterministic time

$$\tau_d=\tau_0(\nu=0)=\frac{L}{a}+\frac{b-L}{k+a}. \tag{34}$$

Difference between the results (33) and (34) is due to a nonzero probability of one switching within the deterministic time interval $(0,\tau_d)$ in the case $\nu\to 0$.

The condition to observe the NES effect can be expressed by the inequality

$$f(q,\omega,s)>0. \tag{35}$$

Let us analyze the structure of NES region on the plane (q,ω) from Eqs. (31) and (35). At very slow and fast flippings we obtain

$$q>\frac{\sqrt{19}-2}{3}\simeq 0{,}7863, \quad \omega\to 0$$

$$\omega<\frac{q(3q^2+q-3)}{1-q^2}, \quad \omega\to\infty. \tag{36}$$

In Fig. 3 we show the NES region (shaded area) on the plane (q,ω) for $b/L=2$, obtained from inequality (35).

The NES effect occurs at $q\simeq 1$, i.e., at very small steepness $k-a=k(1-q)$ of the reverse potential barrier for the metastable state. For this potential profile, a small noise intensity can return particles into potential well, after they crossed the point L. Then Brownian particles stay for long time in the metastable state. This means that, for a fixed mean flipping rate, the NES effect increases when $q\to 1$. For fixed parameter q the effect increases when $\omega\to 0$, because Brownian particles have enough time to move back into potential well.

In Fig. 4 we show the plots of the normalized mean lifetime $\tau_-(0)/\tau_0$, Eq. (27), as a function of the noise intensity D for three values of the dimensionless mean flipping rate $\omega=\nu L/k$: 0.01, 0.05, 0.1.

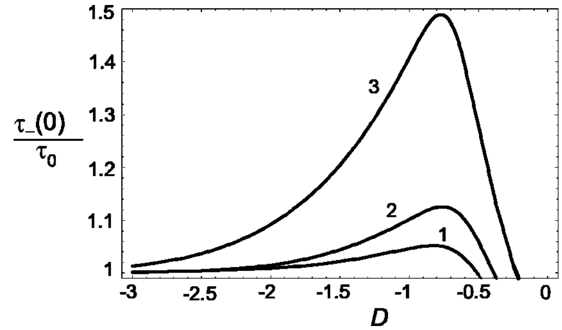

FIG. 4. Semilogarithmic plot of the normalized mean lifetime $\tau_-(0)/\tau_0$ vs the white noise intensity D for three values of the dimensionless mean flipping rate $\omega=\nu L/k$: 0.1 (curve 1), 0.05 (curve 2), 0.01 (curve 3). Parameters are $L=1$, $k=1$, $b=2$, and $a=0.995$.

The maximum value of the NLRT and the range of noise intensity values, where NES effect occurs, increases when ω decreases.

By using exact Eq. (27) we have also investigated the behavior of the mean lifetime $\tau_-(0)$ as a function of switchings mean rate ν. In Fig. 5 we plot this behavior for seven values of noise intensity.

At very slow flippings $(\nu\to 0)$ we obtain

$$\tau_-(0)\simeq\tau_d-\frac{D(1-e^{-aL/D})}{a^2(1+q)}, \tag{37}$$

i.e., the NLRT of the fixed unstable potential $U(x)-ax$. While for very fast switchings $(\nu\to\infty)$ we obtain

$$\tau_-(0)\simeq\frac{b}{k}+\frac{L^2}{2D}, \tag{38}$$

i.e., the mean lifetime for average potential $U(x)$. All limiting values of the NLRT expressed by Eqs. (37) and (38) are shown in Fig. 5. At intermediate rates the escape from the metastable state exhibits a minimum at $\omega=0.1$, which is the signature of resonant activation (RA) phenomenon [2,4,12,28].

Moreover, in Fig. 5 we observe a new resonant-like behavior for the NLRT as a function of mean fluctuation rate of potential. The NLRT exhibits a *maximum* between the slow

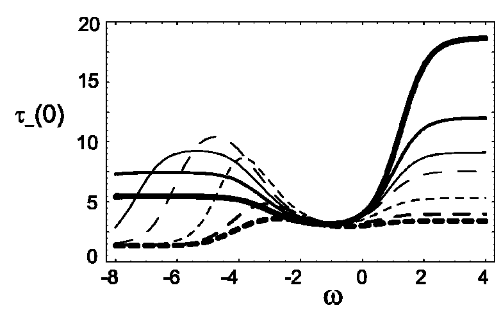

FIG. 5. Semilogarithmic plot of the mean lifetime $\tau_-(0)$ vs the dimensionless mean flipping rate $\omega=\nu L/k$ for seven noise intensity values. Specifically from top to bottom on the right side of the figure: $D=0.03$, 0.05, 0.07, 0.09, 0.15, 0.25, 0.35. The other parameters are the same as in Fig. 4.

limit of potential fluctuations (static limit) and the RA minimum. This maximum occurs for a value of the barrier fluctuation rate on the order of the inverse of the time $\tau_{\mathrm{up}}(D)$ required to escape from the metastable fixed configuration

$$\tau_{\mathrm{up}}(D) = \frac{b-L}{k-a} - \frac{L}{a} + \frac{D(e^{aL/D}-1)}{a^2(1-q)}. \tag{39}$$

This suggests that, the enhancement of stability of metastable state is strongly correlated with the potential fluctuations, when the Brownian particle "sees" the barrier of the metastable state [9,14,24].

V. CONCLUSIONS

We have investigated the nonlinear relaxation time for one-dimensional system with additive white Gaussian noise, and potential profile switching between two configurations, due to a Markovian dichotomous noise. From the general equations (15), we provide exact expression of the mean lifetime for piece-wise linear potential, for arbitrary noise intensity, and arbitrary fluctuation rate of the potential. We find the noise enhanced stability and the resonant activation phenomena in the system investigated. We obtained analytically the region on the (q,ω) plane, where the NES effect can be observed. Moreover, when we fix white noise intensity D, flatness q, and vary switchings mean rate ν, we can observe new resonant-like behavior of the mean lifetime, which is related to the NES phenomenon. The NLRT shows a maximum as a function of the mean flipping rate of potential, with the NES effect strongly correlated with the potential fluctuations. The general equations derived in this paper enable us to perform the analysis of the NES effect conditions in physical systems with more complex potential profiles.

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APPENDIX: EQUATIONS FOR PROBABILITY DENSITY

Upon differentiation of Eq. (2) on t, we obtain

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}\langle \dot{x}(t)\,\delta(x-x(t))\rangle. \tag{A1}$$

Substituting $\dot{x}(t)$ from Eq. (1), and using the definition (3) of auxiliary function, we can rewrite Eq. (A1) as

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x}[U'(x)W] + \frac{\partial}{\partial x}[V'(x)Q] - \frac{\partial}{\partial x}\langle \xi(t)\,\delta(x-x(t))\rangle. \tag{A2}$$

To obtain the evolution of function $Q(x,t)$, we use the Shapiro-Loginov's formula for Markovian dichotomous noise [29]

$$\frac{d}{dt}\langle \eta(t)R_t[\eta]\rangle = -2\nu\langle \eta(t)R_t[\eta]\rangle + \langle \eta(t)\dot{R}_t[\eta]\rangle, \tag{A3}$$

where $R_t[\eta]$ is an arbitrary functional depending on the history of random process $\eta(\tau)$, $0\leqslant\tau\leqslant t$. Replacing $R_t[\eta]$ with $\delta(x-x(t))$ in Eq. (A3), using Eq. (1) and taking into account that $\eta^2(t)=1$, we arrive at

$$\frac{\partial Q}{\partial t} = -2\nu Q + \frac{\partial}{\partial x}[U'(x)Q] + \frac{\partial}{\partial x}[V'(x)W] - \frac{\partial}{\partial x}\langle \xi(t)\,\eta(t)\,\delta(x - x(t))\rangle. \tag{A4}$$

To split the functional averages in Eqs. (A2) and (A4), we use the Furutsu-Novikov's formula for white Gaussian noise $\xi(t)$ [30]

$$\langle \xi(t)F_t[\xi]\rangle = \int_0^t \langle \xi(t)\xi(\tau)\rangle \left\langle \frac{\delta F_t[\xi]}{\delta\xi(\tau)}\right\rangle d\tau = D\left\langle \frac{\delta F_t[\xi]}{\delta\xi(t)}\right\rangle, \tag{A5}$$

where $F_t[\xi]$ is an arbitrary functional of $\xi(t)$. Replacing sequentially $F_t[\xi]$ with $\delta(x-x(t))$ and with $\eta(t)\delta(x-x(t))$ in Eq. (A5), and taking into account that, in accordance with Eq. (1), $\delta x(t)/\delta\xi(t)=1$, we find

$$\langle \xi(t)\,\delta(x-x(t))\rangle = -D\frac{\partial W}{\partial x},$$

$$\langle \xi(t)\,\eta(t)\,\delta(x-x(t))\rangle = -D\frac{\partial Q}{\partial x}. \tag{A6}$$

Substituting the expressions (A6) in Eqs. (A2) and (A4), we obtain the desired closed set of Eqs. (4).

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