Radiation reaction and relativistic hydrodynamics

V. I. Berezhiani
Institute of Physics, The Georgian Academy of Science, 6 Tamarashvili Street, 380077 Tbilisi, Georgia

R. D. Hazeltine and S. M. Mahajan
Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712, USA
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By invoking the radiation reaction force, first perturbatively derived by Landau and Lifschitz, and later shown by Rohrlich to be exact for a single particle, we construct a set of fluid equations obeyed by a relativistic plasma interacting with the radiation field. After showing that this approach reproduces the known results for a locally Maxwellian plasma, we derive and display the basic dynamical equations for a general magnetized plasma in which the radiation reaction force augments the direct Lorentz force.

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I. INTRODUCTION

In recent years relativistic plasmas have attracted considerable attention, primarily in connection with their possible role in a variety of astrophysical phenomena. Observations strongly suggest the existence of relativistic plasmas in the early universe, in active galactic nuclei, relativistic jets, black hole magnetospheres, etc. [1–3]. In addition to these astrophysical settings, the advances in short petawatt pulse laser technology have been harnessed to create laboratory relativistic plasmas by irradiation on gases [4].

A kinetic description of relativistic plasmas is difficult and sometimes not even possible. The relatively less detailed and simpler hydrodynamic description, however, may be an adequate framework for modeling the most significant and complex phenomena which take place in relativistic plasmas. Different kinds of extant hydrodynamic models [5–7] are based on the integration of relativistic kinetic equation to derive moment equations with a subsequent recipe for closing the moment chain. Ideology and practice of the procedure is relatively simple in the collisional limit while in the collisionless case, it becomes less straightforward. The so-called Maxwellian or thermodynamic closure, in which the chain of moment equations is truncated by assuming a Maxwellian distribution function (with varying parameters, such as density, temperature, and fluid element velocity) is the most popular choice. Recently, however, a different closure approach has been developed for magnetized plasmas in which important physics associated with pressure anisotropy and parallel heat flow has been included [8–10]. The latter is not accessible in a Maxwellian closure.

The closure problem in a relativistic plasma is further exacerbated by the fact that such a plasma is strongly radiative, and in the bulk of astrophysical situations it is embedded in the intense incoherent radiation field of other astrophysical objects [1,2]. In most publications devoted to relativistic plasmas, the radiation reaction force is neglected; it is ordered small compared to the Lorentz force and is not expected to be a major determinant of plasma dynamics. This assumption may not be correct in astrophysical conditions where spatiotemporal scales of plasma motion are sufficiently large. The radiation pressure could also be important; in fact, the acceleration of a plasma by radiation pressure has been considered as a possible mechanism for producing relativistic outflows (jets) from very luminous radiation sources, such as the active galactic nuclei or compact galactic objects [11–13].

Through Compton scattering of external photons, individual particles in a plasma lose energy and at the same time there is momentum transfer to the plasma. The bulk flow can be either accelerated or decelerated (i.e., radiative drag). The radiative drag force is derived by resorting to a phenomenological, test-particle approach. In this approach the energy-momentum conservation (in the Thompson or the Klein-Nishina regime) is invoked to treat the particle-photon interaction with subsequent integration of the obtained force over the distribution function (see, for instance, Ref. [14]).

It is interesting to remark that Landau and Lifshitz [15] demonstrated that the radiation drag force acting on an electron which scatters photons can be derived (in the Thompson regime) not only through energy-momentum considerations but also by averaging the radiation reaction force. Similarly Gun and Ostriker [16] found that in the field of electromagnetic (EM) radiation, the radiative losses will ultimately lead to an increase in particle energy. The drag appears due to the appropriate phase lag between velocity and the field. The situation turns out to be similar to what is known in the collisional case, i.e., despite the fact that collisions are dissipative, the particles acquire energy due to inverse bremsstrahlung.

Thus the fluid equations, derived from the relativistic kinetic equation in which the radiation reaction force is included, provide not only a proper description of hot plasma dynamics on the long scale, but also contain self-consistent expression of the drag force acting on a plasma embedded in the radiation field of other hot objects. Our aim here is to construct the relativistic hydrodynamics taking into account the radiation reaction in magnetized plasmas. This theory is the natural generalization of the recently developed relativistic theory of magnetized plasmas [8–10] and will considerably extend its domain of applicability. Naturally the notation and definitions used in these references will be closely followed here.

Before deriving the equations for the magnetized case, we study the role played by the radiation reaction force in two elementary but important physical problems: (1) in the dynamics of a single charged particle, and (2) the hydrodynamics of an *e-p* plasma in the Maxwellian unmagnetized case (derived in Appendix A). We will compare our results with the old results obtained by the standard test-particle approach.

II. RADIATION REACTION FORCE AND KINETIC EQUATIONS

There have been a few attempts in the past to construct the kinetic theory of plasmas that encompasses effects related to radiation reaction [17]. The construction of a kinetic theory of classical charged particles is based on the equation of motion for a single particle. Although, the methodology for deriving the kinetic equation obeyed by the one-particle distribution function when Lorentz force dominates the charged particle dynamics is standard, the inclusion of the radiation reaction force in the system, in general, is problematic. Several questions about the radiation reaction force—its physically correct derivation, realm of validity, its "defects," etc., have been a "permanent" topic of active discussion for more than a century. Interested readers can find it in Ref. [18]. For all their differences, every investigation begins with the following equation of motion for a single charged particle (speed of light, $c=1$):

$$m\frac{du^{\mu}}{ds} = eF^{\mu\nu}u_{\nu} + s^{\mu}. \tag{1}$$

Here u^{α} is the αth component of the contravariant reduced four-momentum $u^{\mu}=[\gamma,\gamma\boldsymbol{u}]$, $\gamma=(1-\boldsymbol{u}^2)^{-1/2}$, where $\boldsymbol{u}$ is the particle velocity, s is the proper time, $ds=dt/\gamma$, $F^{\mu\nu}$ is the electromagnetic tensor $[u^{\mu}u_{\mu}=-1$ consistent with the time-like metric $\eta^{\mu\nu}=\mathrm{diag}(-1,1,1,1)]$ used throughout this study, and s^{μ} is the contravariant radiation reaction force:

$$s^{\mu} = \frac{2e^2}{3}\left(\frac{d^2u^{\mu}}{ds^2} + \frac{d^2u_{\nu}}{ds^2}u^{\mu}u^{\nu}\right). \tag{2}$$

Equation (1), with the radiation reaction force in the form Eq. (2), is known as the Lorentz-Abraham-Dirac (LAD) equation. The LAD equation is of third order implying that the initial position and velocity do not uniquely determine the particle dynamics; initial acceleration is also needed. It would appear that relativistic kinetic equation for the distribution function would have to be constructed in 12- rather than eight-dimensional phase space [17]. Such a theory, however, cannot be free from the "defects" which already are contained in the LAD equation. A particularly damaging defect is the existence of the so called runaway solutions [15,18]—the exponential growth of velocity in the absence of an external Lorentz force. According to Rohrlich [19], the most basic defect in LAD equation is that the radiation reaction force does not vanish when the external force (the cause of acceleration and of corresponding radiation) goes to zero.

This serious defect seems to disappear in the form of Eq. (1) suggested by Landau and Lifshitz [15] in which the radiation reaction force reads as follows:

$$s^{\mu} \equiv \frac{2}{3}\frac{e^3}{m}\left[u_{\kappa}u^{\lambda}\partial_{\lambda}F^{\mu\kappa} + \frac{e}{m}(F^{\mu\kappa}F_{\kappa\lambda}u^{\lambda} + u^{\mu}u_{\kappa}u_{\lambda}F^{\kappa\sigma}F_{\sigma\lambda})\right]. \tag{3}$$

The main assumption in deriving Eq. (3) (let us call it the LL equation) is the smallness (as compared to the Lorentz force) of the radiation reaction force in the particle's rest frame. Note that under certain conditions the radiation reaction force (even when it is small in the particles rest frame) can become larger than the Lorentz force in the Lab frame [15].

This "problem-free" expression for s^{μ} prompted Rohrlich (encouraged by the results obtained by Spohn [20]) into making a very strong statement. He claims that the LL equation, despite the fact that it was derived as an approximation by Landau and Lifshitz, is exact for a point particle. Though this claim needs careful examination, Rohrlich has made a sufficiently convincing case that the LL equation could be used with little risk to describe the radiation reaction in most problems of interest. The Landau-Lifshitz-Rohrlich prescription not only corrects the defects of the LAD equation, it also yields a dynamic equation that is a conventional second order differential equation for the particle position. From our current perspective this feature of the LL equation is, perhaps, the most attractive of all; it allows the construction of a relativistic kinetic theory in the conventional phase space.

Before going to the kinetic formulation, we would like to explicitly demonstrate the role of the radiation reaction force in the scattering processes. In the Thompson regime, the equation of motion of the particle exposed to a photon source has been suggested by Phinney [12], and by Sikora and Wilson [13]. In the current notation, the equation reads

$$\frac{dp^{\alpha}}{ds} = -\sigma(\overline{T}^{\alpha\beta}u_{\beta} + \overline{T}^{\beta\gamma}u^{\alpha}u_{\beta}u_{\gamma}), \tag{4}$$

where $p^{\alpha}=mu^{\alpha}$ is the particle four-momentum, $\sigma=8\pi e^4/3m^2$ is the Thompson cross section, and $\overline{T}^{\alpha\beta}$ is the energy-momentum tensor of the averaged (in high frequency) radiation field. According to Phinney this equation provides the most elegant derivation of the Compton energy losses (cooling) (see Blumenthal and Gould in Ref. [14]); in particular, for an isotropic radiation field with energy density $\overline{T}^{00}=U$, we have $dp^0/dt=-(4/3)\sigma\gamma^2(\boldsymbol{u})^2U$. At this stage, we would like to remark that Eq. (4) was constructed even earlier by Landau and Lifshitz [15] for the particle energy loss in the field of an isotropic distribution of EM waves (photons). However, in that treatment, the relation between the derived expression for force, and the radiation reaction force was not worked out explicitly. It is straightforward to show that the radiation reaction force of the LL equation, Eq. (3), can be rewritten as

$$s^{\alpha} = \frac{2e^3}{3m}\frac{\partial F^{\alpha\beta}}{\partial x^{\gamma}}u_{\beta}u^{\gamma} - \sigma(\overline{T}^{\alpha\beta}u_{\beta} + \overline{T}^{\beta\gamma}u^{\alpha}u_{\beta}u_{\gamma}), \tag{5}$$

where

$$\overline{T}^{\alpha\beta} = \frac{1}{4\pi}\left[- F^{\alpha\gamma}F^{\beta}_{\gamma} + \tfrac{1}{4}g^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}\right] \tag{6}$$

is the EM field energy-momentum tensor; Eqs. (5) and (6) are entirely equivalent to Eq. (4) for the problem solved by Landau and Lifshitz.

After providing a perspective for the LL equation, we turn to the kinetic description of a radiating plasma. Incorporating the LL equation, the generalization to the hierarchy of Bogolyubov equations for a radiating plasma has been carried out by Kuzmenkov [21]. Neglecting correlations, the kinetic equation for a one-particle distribution turns out to be the appropriately modified Vlasov-Boltzmann equation which will be systematically treated in the following section on magnetized plasmas. Realizing, however, that there are problems of interest in which the plasmas may not satisfy the “conditions for magnetization,” we derive in Appendix A the radiation-modified equations pertinent to a plasma with Maxwellian closure.

III. MAGNETIZED PLASMA

A. General formalism

In this section we wish to introduce the effects of radiation missing in the fluid description of magnetized relativistic plasmas presented in Refs. [8–10]. For notational as well as other details the reader is requested to consult these references. For an optically thick plasma, radiation enters the fluid dynamics in two ways. First, the equilibrium photon bath contributes to the total energy-momentum tensor (radiation pressure). This effect is relatively easy to include, essentially by adding the photon pressure to that of the plasma, but for clarity and simplicity it is omitted here. We focus attention on the radiation reaction, whose inclusion is not trivial even in the optically thick case.

We start with the kinetic equation

$$\frac{p^{\mu}}{m}\partial_{\mu}f + \frac{\partial (fF^{\mu})}{\partial p^{\mu}} = C, \tag{7}$$

where C is a collision operator, and F^{μ} is the total force,

$$F^{\mu} = eF^{\mu\nu}p_{\nu} + s^{\mu},$$

consisting of the Lorentz force $eF^{\mu\nu}p_{\nu}$ to which the radiative reaction force of Landau, Lifshitz, and Rohrlich,

$$s^{\mu} \equiv \frac{2}{3}\frac{e^3}{m}\left[u_{\kappa}u^{\lambda}\partial_{\lambda}F^{\mu\kappa} + \frac{e}{m}(F^{\mu\kappa}F_{\kappa\lambda}u^{\lambda} + u^{\mu}u_{\kappa}u_{\lambda}F^{\kappa\sigma}F_{\sigma\lambda})\right],$$

has been added. Using the definition $F^{\mu\kappa}F_{\kappa\lambda} = -We^{\kappa}_{\lambda}$, where $W = B^2 - E^2$ is the first relativistic invariant of the electromagnetic field, the radiative reaction force is expressed as

$$s^{\mu} = \frac{2}{3}\left(\frac{e}{m}\right)^3\left[p_{\kappa}p^{\lambda}\partial_{\lambda}F^{\mu\kappa} - eWe^{\kappa}_{\lambda}p^{\lambda}\left(\eta^{\mu}_{\kappa} + \frac{p_{\kappa}p^{\mu}}{m^2}\right)\right]. \tag{8}$$

To find a fluid description of the radiative plasma that includes this radiative effect, we need to find the moments of the force.

We define the general moment

$$S^{\alpha\beta\cdots\gamma} \equiv \int \frac{d^3p}{E}p^{\alpha}p^{\beta}\cdots p^{\gamma}\frac{\partial s^{\nu}f}{\partial p^{\nu}}.$$

Observe that

$$p_{\mu}\left(\eta^{\mu}_{\kappa} + \frac{p_{\kappa}p^{\mu}}{m^2}\right) = 0.$$

It follows (in view of the antisymmetry of the Faraday tensor) that

$$p_{\nu}s^{\nu} = 0.$$

Therefore, one can rearrange the derivative as if $E \equiv p^0(p)$ were constant. In particular, we may write

$$S^{\alpha\beta\cdots\gamma} = -\int \frac{d^3p}{E}s^{\nu}f\frac{\partial}{\partial p^{\nu}}(p^{\alpha}p^{\beta}\cdots p^{\gamma})$$

or

$$S^{\alpha\beta\cdots\gamma} = -\int \frac{d^3p}{E}fs^{\{\alpha}p^{\beta}\cdots p^{\gamma\}},$$

where the curly brackets indicate indicial symmetrization as usual. It is convenient to introduce

$$\overline{S}^{\alpha\beta\cdots\gamma} \equiv -\int \frac{d^3p}{E}fs^{\alpha}p^{\beta}\cdots p^{\gamma},$$

whence

$$S^{\alpha\beta\cdots\gamma} = \overline{S}^{\{\alpha\beta\cdots\gamma\}}. \tag{9}$$

Let $M^{\alpha\beta\cdots\gamma}_{(n)}$ denote the nth moment of the distribution (a tensor of rank n). Then we have shown that the nth moment of radiative reaction is given by Eq. (9) with

$$\overline{S}^{\alpha\beta\cdots\gamma} = -\frac{2}{3}\left(\frac{e}{m}\right)^3[(\partial_{\lambda}F^{\alpha\kappa})M^{\lambda\beta\cdots\gamma}_{(n+1)\kappa} - eW(e^{\alpha}_{\lambda}M^{\lambda\beta\cdots\gamma}_{(n)} + e^{\kappa}_{\lambda}m^{-2}M^{\alpha\lambda\beta\cdots\gamma}_{(n+2)\kappa})]. \tag{10}$$

There are two moments of primary interest: the energy-momentum conservation law

$$\partial_{\nu}T^{\mu\nu} - eF^{\mu\nu}\Gamma_{\nu} = C^{\mu} - S^{\mu} \tag{11}$$

requires the first moment,

$$S^{\mu} = -\frac{2}{3}\left(\frac{e}{m}\right)^3[(\partial_{\lambda}F^{\mu}_{\kappa})T^{\kappa\lambda} - eWe_{\kappa\lambda}(\eta^{\kappa\mu}\Gamma^{\lambda} + m^{-2}M^{\mu\kappa\lambda})], \tag{12}$$

while the evolution of $M^{\alpha\beta\gamma} \equiv M^{\alpha\beta\gamma}_{(3)}$ depends upon the second moment,

$$S^{\alpha\beta} = \overline{S}^{\alpha\beta} + \overline{S}^{\beta\alpha}$$

with

$$\overline{S}^{\mu\nu} = -\frac{2}{3}\left(\frac{e}{m}\right)^3[(\partial_\lambda F^{\mu}_{\kappa})M^{\nu\kappa\lambda} + (\partial_\lambda F^{\nu}_{\kappa})M^{\mu\kappa\lambda} - eWe_{\kappa\lambda}(T^{\nu\lambda}\eta^{\kappa\mu} + T^{\mu\lambda}\eta^{\kappa\nu} + 2m^{-2}M^{\mu\nu\kappa\lambda})]. \tag{13}$$

These results are fully general. However, fluid closure might require knowledge of higher moments of s. We next specialize to a situation in which the above two moments are sufficient for a closed description.

B. Small gyroradius limit

The evolution of a magnetized plasma will be affected by radiation reaction force as well as other radiative processes. However, magnetization of the plasma requires that the dominant electromagnetic field vary slowly on the gyroscale, and in that case the *forms* of the various moments are not changed by radiation. Thus a description in terms of the familiar variables,

$$n_R, V_{\parallel}, p_{\parallel}, p_{\perp}, q_{\parallel},$$

remains possible; only the evolution equations are changed. As remarked previously, the parallel and perpendicular pressures appearing here will, in general, include radiation pressure (which is by itself isotropic) along with plasma pressure.

Using the known, magnetized forms of the tensors Γ^{α}, $T^{\alpha\beta}$, and $M^{\alpha\beta\gamma}$ we find that

$$e^{\kappa\lambda}\Gamma_{\lambda} = 0, \tag{14}$$

$$e^{\mu}_{\kappa}T^{\mu\nu} = e^{\mu\nu}p_{\perp}, \tag{15}$$

$$e^{\kappa\lambda}M_{\kappa\lambda\mu} = 2(m_1 U_{\mu} + m_3 k_{\mu}), \tag{16}$$

where k_{μ} is a four-vector orthogonal to U_{μ} [8]. The radiation reaction moments, then, reduce to

$$S^{\mu} = -\frac{2}{3}\left(\frac{e}{m}\right)^3\left[(\partial_\lambda F^{\mu}_{\kappa})T^{\kappa\lambda} - \frac{2eW}{m^2}(m_1 U^{\mu} + m_3 k^{\mu})\right] \tag{17}$$

and

$$S^{\mu\nu} = -\frac{2}{3}\left(\frac{e}{m}\right)^3[(\partial_\lambda F^{\mu}_{\kappa})M^{\nu\kappa\lambda} + (\partial_\lambda F^{\nu}_{\kappa})M^{\mu\kappa\lambda} - 2eW(e^{\mu\nu}p_{\perp} + m^{-2}e_{\kappa\lambda}M^{\mu\nu\kappa\lambda})]. \tag{18}$$

A stronger simplification is possible, although perhaps questionable. We have already assumed that the Faraday tensor is dominated by its slowly varying part. If the rapidly varying part, corresponding to radiation, is neglected on the right-hand sides of Eqs. (17) and (18), then the terms involving gradients of the Faraday tensor become small, of order δ compared to the remaining terms. In that case the lowest δ-order moments are simply

$$S^{\mu} = \frac{4}{3}\frac{e^4}{m^5}W(m_1 U^{\mu} + m_3 k^{\mu}) \tag{19}$$

and

$$S^{\mu\nu} = \frac{4}{3}\frac{e^4}{m^3}W(e^{\mu\nu}p_{\perp} + m^{-2}e_{\kappa\lambda}M^{\mu\nu\kappa\lambda}). \tag{20}$$

The remaining unknowns, the fourth-rank tensor contributions to Eq. (20), are computed in Appendix B.

IV. MODIFIED FLUID EQUATIONS

The new radiation reaction terms enter the fluid closure in simple combination with moments of the collision operator $\mathcal{C}$ as seen in Eq. (11). Hence the necessary radiation reaction moments are the same as the needed moments of $\mathcal{C}$:

$$S_{\parallel} = \boldsymbol{b}\cdot\boldsymbol{S}, \tag{21}$$

$$S_E = S^0, \tag{22}$$

$$S_{\parallel E} = -\tfrac{1}{2}(U_{\alpha}k_{\beta} + k_{\alpha}U_{\beta})S^{\alpha\beta}, \tag{23}$$

$$S_{\perp} = e_{\alpha\beta}S^{\alpha\beta}. \tag{24}$$

Each of these quantities enter the closed set of fluid equations by means of the replacement

$$\mathcal{C} \rightarrow \mathcal{C} - \mathcal{S}. \tag{25}$$

The two first-order moments are found from Eq. (19):

$$S_{\parallel} = \frac{4}{3}\frac{e^4}{m^5}W\left(m_1\gamma V_{\parallel} + m_3\gamma\sqrt{\frac{W}{B^2}}\right), \tag{26}$$

$$S_E = \frac{4}{3}\frac{e^4}{m^5}W\left(m_1 + m_3\gamma\sqrt{\frac{B^2}{W}}V_{\parallel}\right). \tag{27}$$

To compute the second-order moments, we first note the identities

$$e_{\mu\nu}\eta^{\mu\nu} = e_{\mu\nu}e^{\mu\nu} = 2,$$

$$e_{\mu\nu}U^{\mu}U^{\nu} = 0,$$

$$e_{\mu\nu}(k^{\mu}U^{\nu} + U^{\mu}k^{\nu}) = \eta_{\mu\nu}(k^{\mu}U^{\nu} + U^{\mu}k^{\nu}) = 0,$$

$$U_{\mu}U_{\nu}b(k^{\mu}U^{\nu} + U^{\mu}k^{\nu}) = 0,$$

$$(k^{\mu}U^{\nu} + U^{\mu}k^{\nu})(k_{\mu}U_{\nu} + U_{\mu}k_{\nu}) = -2.$$

Then Eq. (B6) (see Appendix B) can be seen to yield

$$S_{\parallel E} = \frac{4}{3}\frac{e^4}{m^3}WQ_3, \tag{28}$$

$$S_{\perp} = \frac{8}{3}\frac{e^4}{m^3}W(Q_0 + Q_1 + p_{\perp}). \tag{29}$$

Finally we substitute these expressions for the radiative reaction into the appropriate set of closed moment equations. For a magnetized plasma, such a closed system is available [8]; we need only add the radiative reaction terms computed here. As we have noted, the new terms do not introduce new moments and therefore do not require additional equations. The covariant form of the closed fluid system for each plasma species has a compact expression

$$\mathcal{F}_{\mu\alpha}\left(\frac{\partial T^{\alpha\kappa}}{\partial x^{\kappa}} + S^{\alpha} - C^{\alpha}\right) = eE_{\parallel}B\Gamma_{\mu}, \tag{30}$$

$$e_{\alpha\beta}\left(\frac{\partial M^{\kappa\alpha\beta}}{\partial x^{\kappa}} + S^{\alpha\beta} - C^{\alpha\beta}\right) = 0, \tag{31}$$

$$(U_{\alpha}k_{\beta} + U_{\beta}k_{\alpha})\left(\frac{\partial M^{\kappa\alpha\beta}}{\partial x^{\kappa}} + S^{\alpha\beta} - C^{\alpha\beta}\right) = -2eE_{\parallel}h. \tag{32}$$

The set of Eqs. (30)–(32), in tensor notation, or their equivalent three-vector expression, Eqs. (C1)–(C4) of Appendix C, constitutes the main result of this paper. Future work will deal with laboratory and astrophysical applications of this basic system, in which the radiation reaction force is a codeterminant (with the direct Lorentz force) of the dynamics of a magnetized plasma.

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APPENDIX A: MAXWELLIAN CLOSURE

In this appendix we derive the radiation-modified hydrodynamic equations for systems for which the local Maxwellian (thermodynamic) closure can be invoked. These equations, unlike the equations derived in Secs. III and IV, are valid for arbitrary magnetic fields. In addition, Maxwellian closure allows us to compare our results with what is already known—much of which pertains to precisely this case.

We shall concentrate on a collisionless plasma, i.e., when particles are correlated with themselves due to the delayed interaction, while correlations with other particles (collisions) are neglected. Usually by collisions we mean the correlations between particles where interactions over long times are replaced by effective short-time interactions. For a relativistic plasma this procedure leads to the collisional integral (or generalized Landau integral) of Beliaev and Budker [22].

The procedure is absolutely similar to but simpler than the one carried out in Sec. III. In addition to the standard moments defined in the main text, it is convenient to introduce the entropy four-flow vector

$$S^{\alpha} = -\int \frac{d^3\boldsymbol{p}}{p^0} p^{\alpha} f\{\ln[(2\pi\hbar)^3 f] - 1\}, \tag{A1}$$

which will be found to obey

$$\frac{\partial S^{\alpha}}{\partial x^{\alpha}} = -m\int \frac{d^3\boldsymbol{p}}{p^0} s^{\alpha}\frac{\partial f}{\partial p^{\alpha}}. \tag{A2}$$

The local Maxwellian distribution function [5],

$$f^{\mathrm{eq}}(p) = \frac{1}{(2\pi\hbar)}\exp\left(\frac{\mu - p^{\alpha}U_{\alpha}}{T}\right), \tag{A3}$$

has the following local parameters: μ, the chemical potential; T, the temperature; and U^{α}, the hydrodynamic four velocity: $U^{\alpha}=(\gamma,\gamma\boldsymbol{U})$, $\gamma=(1-\boldsymbol{U}^2)^{-1/2}$ $(U^{\alpha}U_{\alpha}=-1)$. This distribution yields the flux four-vector: $\Gamma^{\alpha}=nU^{\alpha}$, where n_R is the rest frame density

$$n_R = \frac{4\pi Tm^2}{(2\pi\hbar)^3}K_2\left(\frac{m}{T}\right)\exp\left(\frac{\mu}{T}\right) \tag{A4}$$

and K_n is the modified Bassel function of order n.

The energy-momentum tensor for the Maxwellian is

$$T^{\alpha\beta} = hU^{\alpha}U^{\beta} + \eta^{\alpha\beta}P, \tag{A5}$$

where $h=mc^2n[K_3(z)/K_2(z)]$ $(z=m/T)$ is the total enthalpy per unit volume, and the pressure $P=nT$.

The third moment (stress flow tensor) for this distribution may be expressed as

$$M^{\alpha\beta\gamma} = A_1U^{\alpha}U^{\beta}U^{\gamma} - A_2\eta^{\{\alpha\beta,U^{\gamma}\}}, \tag{A6}$$

where $A_1=m^2n[1+6K_3(z)/zK_2(z)]$ and $A_2=-m^2nK_3 \times(z)/zK_2(z)$. Constructing and using $\overline{T}_{\alpha\beta}M^{\alpha\beta\gamma} =A_1\overline{T}^{\alpha\beta}U_{\alpha}U_{\beta}U^{\gamma}-2A_2\overline{T}^{\alpha\gamma}U_{\alpha}$, the equation of motion (the second moment of the kinetic equation), Eq. (12), becomes

$$\frac{\partial T^{\alpha\beta}}{\partial x^{\beta}} - eF^{\alpha\beta}nU_{\beta} = F^{\alpha}_{\mathrm{rad}}, \tag{A7}$$

where

$$F^{\alpha}_{\mathrm{rad}} = \frac{2e^3}{3m^2}\frac{\partial F^{\alpha\beta}}{\partial x^{\gamma}}T^{\gamma}_{\beta} - \sigma n\{[1 + 2G(z)z^{-1}]\overline{T}^{\alpha\beta}U_{\beta} + [1 + 6G(z)z^{-1}]\overline{T}^{\beta\gamma}U_{\beta}U_{\gamma}U^{\alpha}\} \tag{A8}$$

with $G(z)=K_3(z)/K_2(z)$.

The entropy four-flow may be written as $S^{\alpha}=nSU^{\alpha}$, where $S=\ln[K_2\exp(zG)/Pz^2]+c_1$ is the entropy per particle. The equation for S can be obtained directly from Eq. (16), or by contracting Eq. (21) with U^{α}:

$$U^{\alpha}\frac{\partial S}{\partial x^{\alpha}} = \frac{z}{nm}U_{\alpha}F^{\alpha}_{\mathrm{rad}}. \tag{A9}$$

One can see that without the radiation reaction force the plasma dynamics is isentropic with a corresponding adiabatic equation of state.

In the cold plasma limit $(z\to\infty, G\to 1)$, Eq. (21) reduces to

$$m\frac{dU}{ds}^{\alpha} = eF^{\alpha\beta}U_{\beta} + s^{\alpha}, \tag{A10}$$

where s^{α} has the same form as the single particle, Eq. (1), but with u^{α} replaced by U^{α}, and the "time" derivative replaced by the convective derivative $d/ds=\gamma(\partial_t+\boldsymbol{U}\cdot\boldsymbol{\nabla})$. Thus, as expected, the cold plasma fluid equation has a form similar to the one for particle motion. However, for high temperatures, the fluid equations turn out to be considerably more complex. In the ultrarelativistic limit $(z \ll 1, G \approx 4/z)$, for example, the force balance reads

$$U^{\beta}\frac{\partial}{\partial x^{\beta}}(4TU^{\alpha}) + \frac{1}{n}\frac{\partial P}{\partial x_{\alpha}} = eF^{\alpha\beta}U_{\beta} - \frac{\sigma T}{\pi e}\frac{\partial F^{\alpha\gamma}}{\partial x^{\beta}}U^{\beta}U_{\gamma} - \frac{\sigma T}{e}J^{\alpha} + 8\sigma\frac{T^2}{m^2}[\overline{T}^{\alpha\beta}U_{\beta} + 3\overline{T}^{\beta\gamma}U_{\beta}U_{\gamma}U^{\alpha}]. \tag{A11}$$

Here J^{α}, the four-current $(J^{\alpha} = \Sigma e\Gamma^{\alpha}$, summed over particle spices), satisfies the Maxwell equation: $\partial F^{\alpha\beta}/\partial x^{\beta} = -(4\pi/c)J^{\alpha}$. It is interesting to note that the last term in the right-hand side of the equation is proportional to T^2, implying that, for the ultrarelativistic case, it could dominate the flow dynamics.

We now apply the formalism to an electron-positron fluid. For a one fluid description of the e-p plasma, we assume that the temperature $T_{\pm} = T_0$, the density $n_{\pm} = n_0/2$ and the velocity $U^{\alpha}_{\pm} = U^{\alpha}_0$, and by implication $|J^{\alpha}| \ll en_0$, an assumption valid for flows with large spatiotemporal scales [23]. The equation describing the dynamics of the electron-positron fluid can now be written as (omitting the subscript 0 for simplicity)

$$\frac{\partial}{\partial x^{\beta}}(T^{\alpha\beta} + \overline{T}^{\alpha\beta}) = F^{\alpha}, \tag{A12}$$

where

$$F^{\alpha} = -\sigma n\{[1 + 2G(z)z^{-1}]\overline{T}^{\alpha\beta}U_{\beta} + [1 + 6G(z)z^{-1}]\overline{T}^{\beta\gamma}U_{\beta}U_{\gamma}U^{\alpha}\}, \tag{A13}$$

and $\overline{T}^{\alpha\beta}$ and $T^{\alpha\beta}$ have already been defined.

To compare our results with Phinney, we evaluate the forces in the rest frame of the fluid element: $F^0_{\text{rest}} = -4\sigma nGz^{-1}\overline{T}^{00}_{\text{rest}}$, and $F^i_{\text{rest}} = \sigma n(1 + 2Gz^{-1})\overline{T}^{i0}_{\text{rest}}$ $(i = 1, 2, 3)$. We also notice that $\langle\gamma'^2 - 1\rangle = \langle\gamma'^2\beta'^2\rangle = 3Gz^{-1}$, where $\langle\cdots\rangle$ denotes averaging over the distribution function, and γ'^2 and $\beta' c$ are, respectively, the particles' relativistic factor and velocity. Using the notation $(J'_0 = \overline{T}^{00}_{\text{rest}}, J'_1 = \overline{T}^{30}_{\text{rest}})$, we arrive at the equations

$$F^0_{\text{rest}} = -\sigma n\tfrac{4}{3}\ J'_0\langle\gamma'^2\beta'^2\rangle, \tag{A14}$$

$$F^z_{\text{rest}} = \sigma nJ'_1\left(1 + \tfrac{2}{3}\langle\gamma'^2\beta'^2\rangle\right), \tag{A15}$$

in agreement with Phinney [12] (see also Ref. [14]). Note also that in Phinney's equations, the term $\overline{T}^{\alpha\beta}$ present in the current force equation is absent.

We find that the formalism developed here reproduces, in the appropriate limit, the results obtained in past publications. This increases our confidence in the framework we are employing to describe the general dynamics of a magnetized plasma evolving under the combined influence of the Lorentz and radiation-reaction forces.

APPENDIX B: THE FOURTH MOMENT

The tensor needed in Eq. (20) is

$$Q^{\alpha\beta} \equiv m^{-2}e_{\kappa\lambda}M^{\alpha\beta\kappa\lambda}.$$

In a magnetized plasma, any second-rank tensor involving moments will have the form

$$Q^{\alpha\beta} = Q_0\eta^{\alpha\beta} + Q_1e^{\alpha\beta} + Q_2U^{\alpha}U^{\beta} + Q_3(k^{\alpha}U^{\beta} + k^{\beta}U^{\alpha}). \tag{B1}$$

Here the Q_i are scalars. By considering Eq. (B1) in the rest frame (subscript R), we see that

$$Q_0 = Q^{33}_R,$$

$$Q_1 = Q^{11}_R - Q^{33}_R,$$

$$Q_2 = Q^{00}_R + Q_0,$$

$$Q_3 = Q^{03}_R.$$

Because of the simple form of the perpendicular quasiprojector in the rest frame, it is also easy to express Q_R in terms of M_R; for example,

$$m^2Q^{33}_R = M^{1133}_R + M^{2233}_R = 2M^{1133}_R,$$

in view of the obvious symmetry. We also use the symmetry,

$$M^{1111}_R = 3M^{1122}_R,$$

to conclude that

$$m^2Q_0 = 2M^{1133}_R,$$

$$m^2Q_1 = 4M^{1122}_R - 2M^{1133}_R,$$

$$m^2Q_2 = 2(M^{0011}_R + M^{1133}_R),$$

$$m^2Q_3 = 2M^{1103}_R.$$

Next we use the known (model) distribution function to compute directly the four necessary moments:

$$M^{1133}_R = \frac{n_RT^3}{K_2}\left[\zeta K_3 + \Delta\left(4K_4 - \frac{2K_3^2}{K_2}\right)\right], \tag{B2}$$

$$M^{1122}_R = \frac{n_RT^3}{K_2}\left[\zeta K_3 + \Delta\left(6K_4 - \frac{2K_3^2}{K_2}\right)\right], \tag{B3}$$

$$M^{0011}_R = \frac{n_RT^3}{K_2}\left[\zeta^2K_2 + 5\zeta K_3 + \Delta\left(2\zeta K_3 + 28K_4 + \frac{10K_3^2}{K_2}\right)\right], \tag{B4}$$

$$M^{1103}_R = -\frac{m^2q_{\|}}{1 + \zeta\mathcal{K}}\left(1 + \frac{K_4}{K_2} + \frac{7K_4}{\zeta K_3}\right). \tag{B5}$$

We combine these results with Eq. (20) and conclude that the second radiation reaction moment is given by

$$S^{\mu\nu} = \frac{4}{3}\frac{e^4 W}{m^3}[\eta^{\mu\nu}Q_0 + e^{\mu\nu}(p_\perp + Q_1) + U^\mu U^\nu Q_2 + (k^\mu U^\nu + U^\mu k^\nu)Q_3], \tag{B6}$$

where

$$Q_0 = \frac{2p_\parallel}{\zeta^2 K_2}\left[\zeta K_3 + 2\Delta\left(2K_4 - \frac{K_3^2}{K_2}\right)\right], \tag{B7}$$

$$Q_1 = \frac{2p_\parallel}{\zeta^2 K_2}\left[\zeta K_3 + 2\Delta\left(4K_4 - \frac{K_3^2}{K_2}\right)\right], \tag{B8}$$

$$Q_2 = \frac{2p_\parallel}{\zeta^2 K_2}\left[\zeta^2 K_2 + 6\zeta K_3 + 2\Delta\left(16K_4 + \zeta K_3 - 6\frac{K_3^2}{K_2}\right)\right], \tag{B9}$$

$$Q_3 = \frac{-2q_\parallel}{1+\zeta\mathcal{K}}\left(1 + \frac{K_4}{K_2} + \frac{7K_4}{\zeta K_3}\right). \tag{B10}$$

APPENDIX C: THE SYSTEM IN THREE-VECTOR FORM

To write the fluid equations in terms of conventional three-vectors, we substitute our results for the various components of the radiative reaction into known equations for the magnetized, relativistic plasma fluid [9]. Here we neglect collisions for simplicity. The corresponding equations of motion are given by a parallel acceleration law,

$$\sqrt{W}\boldsymbol{\nabla}_\parallel\left(\frac{p_\parallel}{\sqrt{W}}\right) + \frac{p_\perp}{2}\boldsymbol{\nabla}_\parallel \ln W + \gamma n_R \boldsymbol{b}\cdot\frac{d}{dt}\left(\frac{h\gamma\boldsymbol{V} + \boldsymbol{q}}{n_R}\right) + q_\parallel \boldsymbol{b}\cdot\frac{d\gamma\boldsymbol{V}}{ds} + \gamma V_\parallel\left(\frac{\partial q^0}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{q}\right) = \frac{4}{3}\frac{e^4}{m^5}W\left(m_1\gamma V_\parallel + m_3\gamma\sqrt{\frac{W}{B^2}}\right) + e\gamma n_R E_\parallel; \tag{C1}$$

a parallel-energy evolution law,

$$\sqrt{W}\frac{d}{dt}\frac{p_\parallel}{\sqrt{W}} + \frac{p_\perp}{2}\frac{d\ln W}{dt} - n_R\frac{d}{dt}\frac{h}{n_R} - \gamma\boldsymbol{q}\cdot\frac{d\boldsymbol{V}}{dt} - \frac{1}{\gamma}\times\left(\frac{\partial \boldsymbol{q}^0}{\partial t} + \nabla\cdot q\right) = \frac{4}{3}\frac{e^4}{m^5}W\left(m_1 + m_3\gamma\sqrt{\frac{B^2}{W}}V_\parallel\right); \tag{C2}$$

a perpendicular-energy evolution law,

$$\sqrt{W}\frac{\partial}{\partial x^\nu}\left(\frac{m_3 k^\nu + m_1 U^\nu}{\sqrt{W}}\right) = \frac{4}{3}\frac{e^4}{m^3}W(Q_0 + Q_1 + p_\perp); \tag{C3}$$

and a law for the evolution of heat flow,

$$5m_3\gamma\frac{d\ln(m_3 n_R^{-6/5})}{dt} + (m^2 n_R + 5Th - 2m_2)\gamma^2\boldsymbol{k}\cdot\frac{d\boldsymbol{V}}{dt} + \frac{dTh}{ds} - m_2\frac{d\ln\sqrt{W}}{ds} + 7m_3\gamma\boldsymbol{k}\cdot\frac{d\boldsymbol{V}}{ds} = \frac{4}{3}\frac{e^4}{m^3}WQ_3 + ehE_\parallel. \tag{C4}$$

Here the Q_i are given by Eqs. (B7)–(B10) and we use abbreviations introduced previously [8]:

$$m_1 = \frac{m}{K_2}\left[K_3 p_\parallel + (p_\parallel - p_\perp)\left(K_3 - 2\frac{K_4}{K_3}K_2\right)\right], \tag{C5}$$

$$m_2 = m(p_\parallel - p_\perp)\frac{K_4}{K_3}, \tag{C6}$$

$$m_3 = q_\parallel\frac{m\mathcal{K}}{1+\zeta\mathcal{K}}. \tag{C7}$$

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