Encircling an exceptional point

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We calculate analytically the geometric phases that the eigenvectors of a parametric dissipative two-state system described by a complex symmetric Hamiltonian pick up when an exceptional point (EP) is encircled. An EP is a parameter setting where the two eigenvalues and the corresponding eigenvectors of the Hamiltonian coalesce. We show that it can be encircled on a path along which the eigenvectors remain approximately real and discuss a microwave cavity experiment, where such an encircling of an EP was realized. Since the wave functions remain approximately real, they could be reconstructed from the nodal lines of the recorded spatial intensity distributions of the electric fields inside the resonator. We measured the geometric phases that occur when an EP is encircled four times and thus confirmed that for our system an EP is a branch point of fourth order.

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I. INTRODUCTION

Since Berry's pioneering work [1] geometric phases, i.e., contributions to a quantum system's phase which depend only on the geometry of the path traversed by the system in its parameter space, have been the focus of intense theoretical and experimental research. The majority of the theoretical works discuss various generalizations of Berry's original paper, see, e.g., Refs. [2–5] and for a very early work Ref. [6], and investigate the appearance of geometric phases in systems with complex eigenfunctions, such as, e.g., open or dissipative systems [7,8]. Most experimental works observe geometric phases by tracing the pattern of nodal lines of a wave function during adiabatic and cyclic processes [2,9], a technique suggested originally by Berry and Wilkinson [10].

The dissipative nature of a system is commonly suppressed or neglected in experiments (see, e.g., Ref. [11]). This was not the case in the recently reported observation of a so-called exceptional point (EP) in a microwave cavity experiment [12]. Such an EP, i.e., the coalescence of two levels of a quantum system, occurs only in dissipative systems, where it is associated with crossings and avoided crossings of the eigenvalues [13–15]. One of the key features of an EP is the appearance of a geometric phase [12,14–16] when it is encircled in parameter space. If the EP is isolated, in its vicinity the dynamics is predominantly determined by the two states corresponding to the resonances, which coalesce at the EP. There, our system may be modeled by a two-dimensional non-Hermitian, symmetric matrix. Such systems have been analyzed in Refs. [3,15], where one can find a complete and very detailed treatise on the essential features of the eigenvalues and eigenvectors of parameter-dependent two-dimensional matrices associated with the singularities.

For two-state systems described by a complex symmetric Hamiltonian, the geometric phase associated with an EP [12,14] differs from that associated with a diabolic point (DP) [10], a simple degeneracy between two levels. The only way to determine a geometric phase with our experimental setup [12] is by recording the change of the pattern of nodal lines when encircling an EP in parameter space. In our microwave cavity experiments we can only measure the intensity distribution of the electric field, that is, the absolute value of the complex eigenfunctions. However, a premise for the mere existence of a pattern of nodal lines is that the eigenfunctions remain real throughout the cyclic process. Accordingly, the question arises, whether Berry's reconstruction technique can be applied to the complex eigenfunctions of the dissipative microwave resonator discussed in Refs. [12,17].

In the present work we first focus on a parametric two-state model adequate for the simulation of our experiment with the dissipative microwave cavity. In doing so, we restrict ourselves to the analysis of those properties of the eigenvalues and eigenvectors, which are observable using our experimental setup. For more detailed information one might consult Ref. [3]. Accordingly, we calculate the geometric phase that occurs when an EP is encircled. Moreover, we show that for this model a path around the EP exists along which the eigenvectors are approximately real, that is, have an imaginary part negligible compared to its real part. Indeed, as is explained in more detail below, from the mere fact that we observe nodal lines rather than nodal points in our experiment we may already conclude that along the path chosen in our experiment the eigenfunctions have exactly this property. This then allows us to employ Berry's reconstruction technique in a microwave cavity experiment, where the development of the nodal line patterns with the parameters is studied.

The paper is organized as follows. Using a two-state model adequate for the simulation of our experiment we analytically calculate in Sec. II the geometric phases that occur when an EP is encircled. By this calculation a path around the EP is defined, along which the eigenvectors of the system remain approximately real. In Sec. IV the actual microwave cavity experiment is discussed. Using the same experimental techniques and a setup similar to the one discussed in Ref. [12], we here present data from the encircling of a different

EP. From the transformation properties of the eigenfunctions and from the existence of a nodal line pattern for every setting of the cavity's parameters we conclude that the eigenfunctions remain approximately real in the experiment while the EP is encircled. A conclusion is given in Sec. V.

II. ANALYTIC TREATMENT OF ENCIRCLING AN EP

In this section, we analyze the behavior of the eigenvectors of a two-state quantum system when encircling an EP. As outlined in Ref. [12], the two-state Hamiltonian appropriate for the simulation of our system in the vicinity of an isolated EP is written in terms of a two-dimensional complex symmetric matrix. Accordingly, in the sequel we restrict ourselves to such two-state systems. In particular, we show that a closed path exists along which the eigenvectors are approximately real and that in accordance with Refs. [3,14] one needs four turns around the EP in order to restore the original situation. We use the same notation as in Refs. [17,18] and write the Hamiltonian of the two-state system in the form

$$H = \begin{pmatrix} E_1 - i\gamma_1 & H_{12} \\ H_{12} & E_2 - i\gamma_2 \end{pmatrix}. \tag{1}$$

Here, the parameters $E_{1,2}$ and $\gamma_{1,2}$ are real and H_{12} may be complex. Expression (1) is a complex symmetric Hamiltonian. By defining

$$\mathcal{E} \equiv \frac{E_1 + E_2 - i(\gamma_1 + \gamma_2)}{2} \tag{2}$$

and

$$e \equiv (E_1 - E_2)/2,$$

$$\gamma \equiv (\gamma_1 - \gamma_2)/2, \tag{3}$$

the eigenvalues $E_{1,2}$ of H can be written as

$$E_{1,2} = \mathcal{E} \pm \sqrt{(e - i\gamma)^2 + H_{12}^2}. \tag{4}$$

Two-state systems described by such a Hamiltonian have been studied both analytically and numerically in Ref. [15]. Moreover, subtracting $\mathcal{E}$ from the diagonal elements of H given in Ref. [3] and performing a transformation of the type defined in Eq. (3.6) of Ref. [3], the Hamiltonian (1) can be brought to the form given in Eq. (6.2) of Ref. [3] with $G=0$, whose eigenvalues and eigenvectors provide the refractive indices and the associated polarization vectors of dichroic, nonchiral crystals. Reference [3] provides a very detailed description of such crystals at and around three types of singularities (called singular axes, C points of circular polarization, which in the absence of chirality coincide with the singular axes, and L lines of linear polarization) that may occur dependent on the choice of the three parameters e, γ, and H_{12}. In the following we will rederive those properties of the eigenvalues and eigenvectors of the Hamiltonian (1), which are observable using our experimental setup.

The complex eigenvalues coincide if the square root vanishes. Hence, at the two EPs of H, one has the relation

$$H_{12} = \pm i(e - i\gamma) \tag{5}$$

between the parameters of the Hamiltonian. Furthermore, the parameter H_{12} is nonzero at an EP. Else, if H_{12} vanishes, the space of eigenvectors is two dimensional [see Eq. (1)], and a degeneracy rather than an EP occurs.

Since we are interested in the behavior of the eigenvectors of H in the vicinity of an EP, we furthermore define the complex parameter

$$B \equiv \frac{e - i\gamma}{H_{12}}, \tag{6}$$

which becomes

$$B^{\mathrm{EP}} = \pm i \tag{7}$$

at an EP, i.e., when Eq. (5) is fulfilled. The eigenvectors $|r_1\rangle$ and $|r_2\rangle$ of H can then be written as functions of B. Normalizing the left-hand eigenvectors $\langle l_k|$ and the right-hand eigenvectors $|r_k\rangle$ in the biorthogonal sense, they can be defined as

$$\langle l_1| = (\cos\,\theta, \sin\,\theta), \quad |r_1\rangle = \begin{pmatrix} \cos\,\theta \\ \sin\,\theta \end{pmatrix},$$

$$\langle l_2| = (-\sin\,\theta, \cos\,\theta), \quad |r_2\rangle = \begin{pmatrix} -\sin\,\theta \\ \cos\,\theta \end{pmatrix}, \tag{8}$$

where θ is defined by

$$\tan\,\theta \equiv -B + \sqrt{B^2+1} = -B + \sqrt{(B+i)}\sqrt{(B-i)}. \tag{9}$$

This choice of normalization of course defines the left and right eigenvectors only up to an additional phase, which cancels out when evaluating the absolute value of the eigenvectors. Hence, this additional phase may not be observed with our experimental setup (see Sec. III).

When varying B continuously along a closed curve around one of the EPs, i.e., around $B^{\mathrm{EP}}=+i$ or $B^{\mathrm{EP}}=-i$, the phase of $B-B^{\mathrm{EP}}$ will change by 2π. Accordingly, *one* of the square-root functions in the second line of Eq. (9), namely, that corresponding to $\sqrt{B-B^{\mathrm{EP}}}$, changes its sign, whereas $\sqrt{B+B^{\mathrm{EP}}}$ will return to its original value as long as exactly one EP is encircled. Hence, encircling an EP implies a change from $\tan\,\theta$ to $\tan\,\theta_1$, where

$$\tan\,\theta_1 \equiv -B - \sqrt{B^2+1} = -\cot\,\theta, \tag{10}$$

that is,

$$\theta_1 = \theta \pm \frac{\pi}{2}. \tag{11}$$

To compare the eigenvectors of H before and after encircling an EP in the B plane, we use the abbreviation

$$|1,2\rangle \equiv |r_{1,2}\rangle_{B_0}, \tag{12}$$

where B_0 denotes some value of B. Starting from this initial value B_0 we track the development of the eigenvectors of H when an EP is encircled. Comparing the eigenvectors before and after encircling an EP, i.e., inserting θ and $\theta_1=\theta+\pi/2$ into Eq. (8), then yields the transformation scheme

$$\begin{Bmatrix}|1\rangle\\|2\rangle\end{Bmatrix}\circlearrowleft\begin{Bmatrix}|2\rangle\\-|1\rangle\end{Bmatrix}\circlearrowleft\begin{Bmatrix}-|1\rangle\\-|2\rangle\end{Bmatrix}\circlearrowleft\begin{Bmatrix}-|2\rangle\\|1\rangle\end{Bmatrix}\circlearrowleft\begin{Bmatrix}|1\rangle\\|2\rangle\end{Bmatrix}, \qquad (13)$$

while inserting θ and $\theta_1=\theta-\pi/2$ leads to

$$\begin{Bmatrix}|1\rangle\\|2\rangle\end{Bmatrix}\circlearrowright\begin{Bmatrix}-|2\rangle\\|1\rangle\end{Bmatrix}\circlearrowright\begin{Bmatrix}-|1\rangle\\-|2\rangle\end{Bmatrix}\circlearrowright\begin{Bmatrix}|2\rangle\\-|1\rangle\end{Bmatrix}\circlearrowright\begin{Bmatrix}|1\rangle\\|2\rangle\end{Bmatrix}. \qquad (14)$$

Both schemes have been observed experimentally [12,17] in a particularly shaped microwave billiard (see Sec. IV) by changing the orientation of the closed loop around the EP, as indicated by the symbols $\circlearrowleft$ and $\circlearrowright$. Changing the orientation of the loop is equivalent to following a given scheme backwards. In both cases, *four* turns around the EP are needed in order to restore the original situation (see also Refs. [3,12,14]). At this point we note that if B changes continuously along a closed curve, which encircles *both* EPs, then the sign of *both* square-root functions in the second line of Eq. (9) will change, that is, each of the eigenvectors $|1\rangle$ and $|2\rangle$ transforms into itself. If, however, both EPs are encircled with opposite orientation by tracing out an "8," then both eigenvectors will acquire an extra phase π, that is, a phase which coincides with Berry's phase for the encircling of a diabolical point (see Sec. III).

The resulting transformation schemes [Eqs. (13) and (14)] are not surprising since the eigenvectors (8) have a branch point of a fourth root at the EP. This can be easily seen by noting that (i) the tangent in Eq. (9) depends on $\sqrt{B\pm B^{\mathrm{EP}}}$ and (ii) an additional square root is needed for the computation of the sine and cosine functions in Eq. (8), from the tangent, viz.,

$$\cos\,\theta=\frac{1}{\sqrt{1+\tan^2\,\theta}},$$

$$\sin\,\theta=\frac{1}{\sqrt{1+\cot^2\,\theta}}. \qquad (15)$$

If B is real, the components of $|1\rangle$ and $|2\rangle$ are real. Hence, if B is sufficiently far from the EPs, i.e.,

$$|B|\gg 1, \qquad (16)$$

the eigenvectors are approximately real. Keeping only terms up to the first order in B^{-1} we obtain

$$|r_1\rangle\approx\begin{pmatrix}1\\(2B)^{-1}\end{pmatrix},$$

$$|r_2\rangle\approx\begin{pmatrix}(2B)^{-1}\\-1\end{pmatrix}. \qquad (17)$$

Therefore the eigenfunctions are approximately real along the path sketched in Fig. 1, which either follows the real axis of the complex B plane or fulfills Eq. (16). We note here that along the lower part of the path sketched in Fig. 1, i.e., far away from $B^{\mathrm{EP}}=-i$, the components of the eigenvectors depend only linearly on B^{-1}. We therefore expect that the experimentally measured eigenvectors vary only slightly along this part of the path. If the nodal line patterns are tracked along this path in an experiment, they can be used to reconstruct the eigenstates according to Berry and Wilkinson [10].

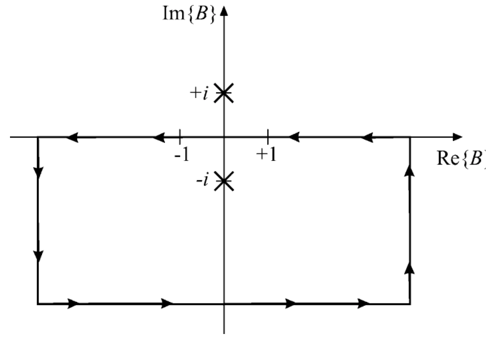

FIG. 1. A path in the complex B plane [cf. Eq. (6) in the main text] surrounding an EP situated at $B^{\mathrm{EP}}=-i$. Along this path the eigenfunctions of H remain approximately real.

III. PHASE OF THE EIGENVECTORS

In the following section we will discuss why *only* the transformation schemes (13) and (14) are observed in experiments (see Refs. [12] and Sec. IV). One can redefine the eigenvectors such that

$$\langle l_k|\to\langle\tilde{l}_k|=e^{-i\phi}\langle l_k|,\quad |r_k\rangle\to|\tilde{r}_k\rangle=e^{i\phi}|r_k\rangle. \qquad (18)$$

This conserves the biorthogonal normalization. If the phase ϕ is, e.g., constructed [25] so that after two loops around an EP it equals π then one obtains $|\tilde{r}_k\rangle\to|\tilde{r}_k\rangle$ after two loops. This seems to disagree with the schemes (13) and (14) claiming that $|\tilde{r}_k\rangle\to-|\tilde{r}_k\rangle$ after two loops. In the present section we show that the experimental result (13) for the phase change of the eigenvectors (8) remains unchanged under the replacement (18). In the first section the essence of the argument is presented in a rather general and abstract way. In the second section the loops around an EP are described in a more physical way. In the third section, we consider eigenstates with an additional phase, as defined in Eq. (18), and show that the results of the second section remain unchanged. In the last section we show that the present arguments yield the well-known geometrical phase occurring when a DP is encircled.

A. Smoothest interpolation between the experimental pictures

In the experiment described in Sec. IV and in Refs. [12,17] the development of the eigenvectors (8) is tracked. Their coefficients are analytical functions of B everywhere except at the EPs. Analytical functions are arbitrarily often differentiable. In this sense they are the smoothest possible functions.

Except for a parameter-independent phase, the vectors (8) are the only analytical representation of the eigenvectors, because a system of biorthogonal eigenvectors is well defined up to an arbitrary phase $\phi=\phi(B)$ as introduced in Eq. (18) with a complex B. The phase is real in all the domain where one is allowed to choose the path C. Except for the constant there is no analytical function which is real on some area in the complex plane. Hence, multiplying Eq. (8) with a phase factor depending on B yields nonanalytical eigenvectors.

Instead of claiming that the experiment follows Eq. (8), one can therefore state that Eq. (8) is the smoothest interpolation between the experimental pictures of the eigenfunctions. In this sense it is the simplest mathematical interpretation of the sequence of wave functions presented in Fig. 3. According to the argument of Ockham's razor [19] one cannot hope for anything else.

In the following section, a physical process is discussed that allows us to explicitly follow a given eigenvector on a path encircling an EP. It yields in Sec. III C the result announced above.

B. Parameter-dependent state

The physical process is the one introduced by Berry in Ref. [1]—modified to the treatment of complex symmetric (instead of Hermitian) H and to loops around an EP (instead of a DP).

Let $H=H(\vec{R})$ depend on a set $\vec{R}$ of parameters. The eigenvectors $\langle l_k(\vec{R})|$, $|r_k(\vec{R})\rangle$, and eigenvalues $E(\vec{R})$ depend on $\vec{R}$.

In a first step, we consider $\vec{R}=\vec{R}(t)$ as a function of time. The state $|\psi(t)\rangle$ of the system is the solution of the time-dependent Schrödinger equation

$$H(\vec{R}(t))|\psi(t)\rangle = i\frac{\partial}{\partial t}|\psi(t)\rangle. \tag{19}$$

At $t=0$ the system shall be in the eigenstate $|r_n\rangle$, i.e.,

$$|\psi(0)\rangle = |r_n(\vec{R}(0))\rangle. \tag{20}$$

Expanding $|\psi(t)\rangle$ into the instantaneous eigenstates $|r_n(\vec{R}(t))\rangle$ at time t and assuming that the parameter $\vec{R}(t)$ is changed so slowly that the adiabatic approximation is applicable [24], we obtain

$$|\psi(t)\rangle = \exp\left(-i\int_0^t dt' E_n(\vec{R}(t'))\right)\exp[i\phi(\vec{R}(t))]|r_n(\vec{R}(t))\rangle. \tag{21}$$

Hence, the adiabatic approximation implies that at each instant t the state $|\psi(t)\rangle$ is proportional to the instantaneous eigenstate $|r_n(\vec{R}(t))\rangle$, if it is proportional to $|r_n(\vec{R}(0))\rangle$ at time $t=0$. The dynamical phase

$$-i\int_0^t dt' E_n(\vec{R}(t'))$$

is well known. Of special interest is the additional phase ϕ which is due to the motion in parameter space. With the ansatz (21), Schrödinger equation (19) yields the equation

$$\dot{\phi}|r_n\rangle = i\frac{\partial}{\partial t}|r_n\rangle \tag{22}$$

for the phase ϕ. Using biorthogonality this gives

$$\dot{\phi} = i\left\langle l_n \middle| \frac{\partial}{\partial t} r_n\right\rangle = i\langle l_n|\vec{\nabla}_R r_n\rangle \dot{\vec{R}}. \tag{23}$$

The solution is

$$\phi = i\int_0^t dt' \langle l_n|\vec{\nabla}_R r_n\rangle\dot{\vec{R}} = i\int_C d\vec{R}\langle l_n|\vec{\nabla}_R r_n\rangle. \tag{24}$$

The last integral is a path integral in $\vec{R}$ space. The parameters move along the path C between time zero and time t. This means that the time has only served to parametrize the path. The last integral is independent of time and should therefore be valid not only for an adiabatic process but also in the present experimental context, where a continuous variation of the parameters is considered. Let

$$\vec{R} = \begin{pmatrix}\mathrm{Re}B\\ \mathrm{Im}B\end{pmatrix} \tag{25}$$

be the real and imaginary parts of B, and let $\langle l_n|,|r_n\rangle$ be defined as in Eq. (8). Then Eq. (24) is a path integral in the complex plane of B, viz.,

$$\phi = i\int_C dB\left\langle l_n \middle| \frac{d}{dB} r_n\right\rangle. \tag{26}$$

The integrand is analytic everywhere except at the EPs. The function $\langle l_n|(d/dB)r_n\rangle$ is multivalued—it is defined on a Riemannian surface rather than the complex plane. But on that surface, it is analytic everywhere. Especially it is continuous on every path C that avoids the EPs. Along such a path one has

$$\begin{aligned}\left\langle l_n \middle| \frac{d}{dB} r_n\right\rangle &= \frac{1}{2}\left(\left\langle \frac{d}{dB} l_n \middle| r_n\right\rangle + \left\langle l_n \middle| \frac{d}{dB} r_n\right\rangle\right)\\ &= \frac{1}{2}\frac{d}{dB}\langle l_n|r_n\rangle = 0.\end{aligned} \tag{27}$$

The first line of this equation holds because $\langle l_n|$ is just the transpose of $|r_n\rangle$. The result is due to the biorthogonal normalization.

Hence, the choice (8) for the eigenfunctions leads to

$$\phi(C) = 0 \tag{28}$$

if the path C does not cross an EP. In other words, our choice of the normalization implies that the total phase acquired by a state $|\psi(t)\rangle$ when encircling an EP in parameter space is obtained from the change of the eigenstates, that is, from the transformation schemes (13) and (14). As will be shown in the following section, this result is independent of the phase convention chosen for the eigenstates.

C. Redefining the phase of the eigenstates

Let us use in Sec. III B the eigenstates $|\tilde{r}_k\rangle,\langle\tilde{l}_k|$ of Eq. (18) instead of $|r_k\rangle,\langle l_k|$. The phase $\phi=\phi(\vec{R})$ shall be a function of the parameters. Then, in analogy to Eq. (21) we may write

$$|\psi(t)\rangle = \exp\left(-i\int_0^t dt' E_n(\vec{R}(t'))\right)\exp[i\beta(\vec{R}(t))]|\tilde{r}_n(\vec{R}(t))\rangle$$
$$= \exp\left(-i\int_0^t dt' E_n(\vec{R}(t'))\right)\exp[i\beta(\vec{R}(t)) + i\phi(\vec{R}(t))]$$
$$\times|r_n(\vec{R}(t))\rangle, \tag{29}$$

where in the second line we used the definition of $|\tilde{r}_n\rangle$ [see Eq. (18)]. Hence, the total change of the phase of $|\psi(t)\rangle$ is given as $[\beta(C)+\phi(C)]$ plus the change of $|r_n\rangle$. But, proceeding as in Sec. III A one shows that $[\beta(C)+\phi(C)]=0$, as long as the path does not cross an EP. As a result, our choice of the normalization of the eigenstates has the specific property that all phase changes acquired by a state $|\psi(t)\rangle$ when encircling an EP are solely obtained from the transformation schemes (13) and (14) independent of the phase conventions chosen for the eigenstates—in agreement with the argument given in Sec. III A. For comparison we briefly discuss—in the following section—the well-known phase that occurs when a diabolic point is encircled.

D. Encircling a diabolic point

A diabolic point is a degeneracy of the eigenvalues such that there are two linearly independent eigenvectors. It is most easily obtained as a degeneracy in a system described by a real, symmetric H. Let us set in Eq. (1)

$$\gamma_1 = 0, \quad \gamma_2 = 0, \tag{30}$$

and

$$H_{12} = \omega = \text{real}. \tag{31}$$

Then the diabolic point occurs when $e=0$ and $H_{12}=0$. Encircling corresponds to moving the vector

$$\begin{pmatrix} e \\ H_{12} \end{pmatrix}$$

around the origin, e.g., with

$$e = \varrho \cos\xi, \quad H_{12} = \varrho \sin\xi, \quad 0 \leqslant \xi < 2\pi. \tag{32}$$

This yields the eigenvectors

$$|r_1\rangle = \begin{pmatrix} \cos(\xi/2) \\ \sin(\xi/2) \end{pmatrix} \tag{33}$$

and

$$|r_2\rangle = \begin{pmatrix} \sin(\xi/2) \\ -\cos(\xi/2) \end{pmatrix}. \tag{34}$$

One sees that encircling a DP changes the sign of the eigenvectors—this is Berry's phase [1].

Strictly speaking, the path (32) cannot be represented as a closed curve in the complex plane of B. According to Eq. (32), B is real for all ξ except $\xi=0,\pi$, where it goes to infinity. In order to circumvent this difficulty one can add a small imaginary part to H_{12} and replace Eq. (31) by

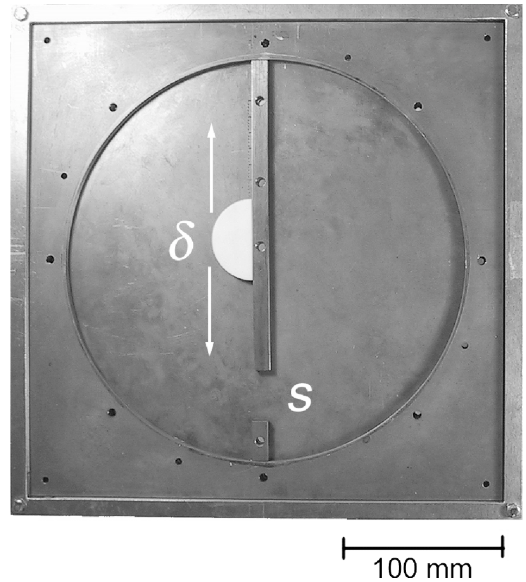

FIG. 2. A photograph of the opened microwave billiard employed for the observation of EPs. A circular copper cavity is divided into two semicircular parts. The two parts are variably coupled by a slit of width s. One of the semicircular cavities can be perturbed by adjusting the position δ of a Teflon stub inside the resonator.

$$H_{12} = \omega + i\epsilon. \tag{35}$$

By virtue of Eq. (32), the parameter

$$B = \frac{e}{\omega + i\epsilon} \tag{36}$$

then moves on a path that has the shape of a figure-eight and encircles the EPs at i and $-i$ in opposite directions. We have convinced ourselves that such a closed path changes the phase of the eigenvectors by π—in agreement with Berry's phase.

The parametrization in terms of B is similar to the one discussed in Ref. [15]. However, the authors of Ref. [15] did not exactly specify the path chosen for encircling a DP. Note that a closed path which encircles both EPs in the same sense does not change the phase of the eigenvectors.

IV. EXPERIMENT

The geometric phases which occur when an EP is encircled have been observed for the first time in the microwave cavity experiment described in Ref. [12]. Flat microwave resonators as the one used in Ref. [12] are commonly known as microwave billiards and form one cornerstone for the experimental investigation of quantum chaotic phenomena (for an overview see, e.g., Refs. [20,21]). They compose an analog computer that solves the Schrödinger equation for quantum billiards. The circular resonator employed in the experiment [12] was manufactured of copper and divided by a conducting wall into two approximate half-circles. Figure 2 shows a photograph of the cavity without its lid. An opening of length s in the wall couples the two semicircular parts of the cavity. A second adjustable parameter, called δ, is given by the position of a semicircular Teflon stub in one part of the cavity.

In the vicinity of an EP [12], we model the microwave billiard through a two-state system as described by H of Eq. (1). The connection between the parameters of H and the observables of the microwave cavity experiment can be illustrated directly for the uncoupled case which implies $s=0$ mm, that is, $H_{12}=0$, in Eq. (1). The resonance frequency f_1 of a mode in one semicircular cavity corresponds to E_1 in Eq. (1) while the total width of it, Γ_1, corresponds to γ_1. The resonance frequency and total width of a mode in the adjacent semicircular cavity gives then E_2 and γ_2. The situation is slightly more involved for $s>0$ mm, i.e., $H_{12}\neq 0$. However, the diagonal elements of H are the same as in the uncoupled case. Moreover, the coupling mechanism via the slit implies that the off-diagonal elements of H coincide, that is, H is complex symmetric. The resonance frequencies and the widths measured in the experiment correspond to the real and the imaginary part of the eigenvalues of H.

The nodal line patterns of the electric-field distributions could be mapped by using a perturbation body method [12,22,23]. Applying Berry's procedure [1] to these patterns we were able to reconstruct the eigenfunctions of the cavity. Extending Ref. [12] we show here a pair of modes, which for small couplings are localized in the adjacent semicircular halves of the cavity (see Fig. 3). Figure 3 exhibits two reconstructed eigenfunctions of the resonator for various parameter settings (s,δ). The two shadings in Fig. 3 can be associated with the two different orientations of the electric field inside the microwave billiard [21]. The wave functions of these modes can be mapped out separately even at a frequency crossing if the resonator is excited via different antennas [17]. Following the theoretical analysis, cf. Eq. (12), we chose the wave functions for $s=10$ mm and $\delta=42$ mm as basis states, i.e.,

$$|1,2\rangle \equiv |r_{1,2}\rangle_{(s=10\ \mathrm{mm},\delta=42\ \mathrm{mm})}. \tag{37}$$

The basis states are labeled as $|1\rangle$ in Fig. 3(a) and $|2\rangle$ in Fig. 3(b), respectively. They are chosen far away from the EP, beforehand identified by studying the behavior of the eigenvalues [12], so that Eq. (16) is fulfilled. This implies that the basis states are approximately real [cf. Eq. (17)]. At all other parameter settings, the eigenstates $|r_1\rangle$ and $|r_2\rangle$ are linear combinations of $|1\rangle$ and $|2\rangle$, cf. Ref. [17]. Let α, β be the expansion coefficients of $|r_1\rangle$ so that

$$|r_1\rangle = \alpha|1\rangle + \beta|2\rangle. \tag{38}$$

The eigenstates are orthonormal in the biorthogonal sense, which requires

$$|r_2\rangle = \beta|1\rangle - \alpha|2\rangle. \tag{39}$$

and

$$\alpha^2 + \beta^2 = 1. \tag{40}$$

The eigenfunctions remain approximately real for *all* steps displayed in Fig. 3, since a superposition of $|1\rangle,|2\rangle$ with complex expansion coefficients would have removed the nodal lines while a superposition of the basis states with real expansion coefficients simply shifts the nodal lines. The reason for this is that the absolute value of the wave function of the complex superposition is zero only where the coefficients of both $|1\rangle$ and $|2\rangle$ vanish.

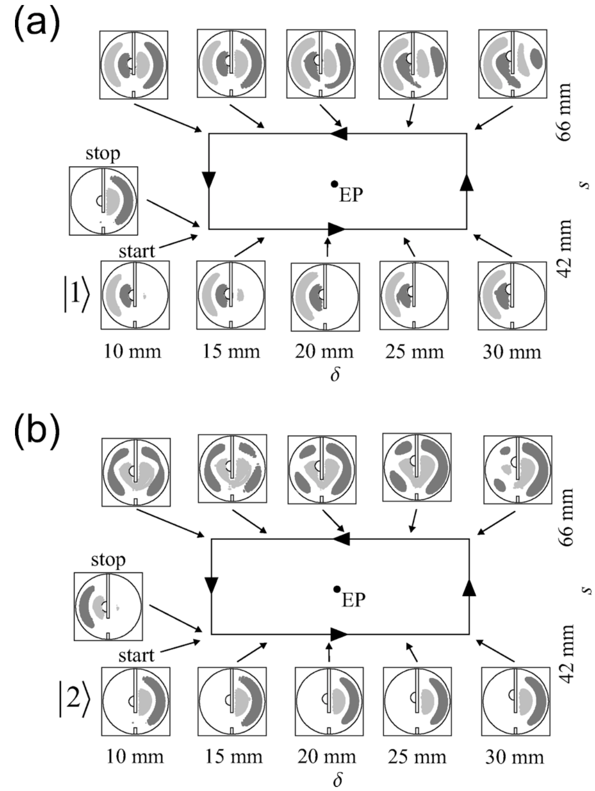

FIG. 3. Development of the reconstructed electrical field distributions of two modes of the resonator shown in Fig. 2 while an EP is encircled. The initial states, i.e., the "start" configurations, are labeled as $|1\rangle$ and $|2\rangle$ in agreement with the definition (37). Their field distributions can be reconstructed from the recorded nodal line patterns for all settings (s,δ).

By varying (s,δ) in small steps, one EP has been encircled in the (s,δ) plane. Both eigenfunctions were tracked continuously during the sequence of 11 steps that form the closed loop around the EP. The reconstructed wave functions clearly show that the basis state $|1\rangle$ transforms to $|2\rangle$ [see Fig. 3(a)], and that $|2\rangle$ transforms to $-|1\rangle$, which implies a geometric phase of π. The data presented in Fig. 3 therefore confirm in Berry's sense [1,11] the appearance of a geometric phase which is picked up by *one* eigenvector when an EP is encircled:

$$\left\{\begin{matrix}|1\rangle\\|2\rangle\end{matrix}\right\} \circlearrowleft \left\{\begin{matrix}|2\rangle\\-|1\rangle\end{matrix}\right\}. \tag{41}$$

A recently suggested additional geometric phase [25], which also leads to mathematical inconsistencies [26], does not appear.

The transformation scheme for four consecutive turns around a single EP has been measured by repeatedly tracking the nodal line patterns along the path shown in Fig. 3. The resulting geometric phases for the basis states $|1\rangle$ and $|2\rangle$ can

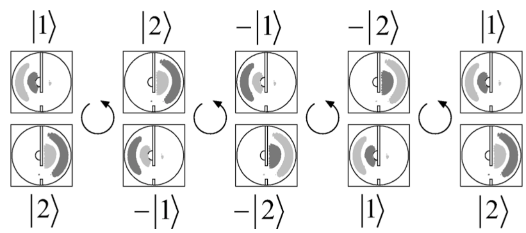

FIG. 4. Development of the basis states $|1\rangle$ and $|2\rangle$ during four consecutive turns around an EP. For each loop, symbolized by ↻, the development from the initial to the final states has been tracked according to Fig. 3. Four turns are needed in order to restore the original situation.

be derived from the reconstructed wave functions shown in Fig. 4. Our experimental results are in accordance with the analytical result, Eq. (13); see Fig. 4. The transformation scheme (13) implies that the eigenfunctions of the resonator have a branch point of fourth root at the EP.

V. CONCLUSION

We have shown analytically that the eigenfunctions of the two-state system described by the complex symmetric Hamiltonian modeling our microwave cavity experiment transform according to Eq. (13) and (14) when the parameters of the Hamiltonian are taken around one EP, i.e., a fourth-order branch point. The appearing geometric phases are a consequence of the normalization of the eigenfunctions (8). The eigenfunctions are approximately real on a path encircling the EP, a property which is essential for their experimental reconstruction according to Berry [1] from distributions of nodal lines mapped in microwave cavity experiments.

We verified these results by performing an experiment with a normal conducting microwave billiard consisting of two variably coupled semicircular resonators. The two modes we report on here are for small couplings localized in the adjacent semicircular halves of the resonator. This allowed us to completely map their nodal line patterns when the EP is encircled. The reconstructed wave functions confirm the transformation schemes derived analytically. The experimental results presented here show that the geometric phases occurring when an EP is encircled agree with those observed in earlier experiments [12] and with analytical and numerical calculations [3,14,15]. There is no experimental evidence for any additional geometric phase factors [25].

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