Design and robustness of delayed feedback controllers for discrete systems

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We study a matrix form of time-delay feedback control in the context of discrete time maps of high dimension. In almost all cases where standard proportional feedback control methods can achieve control, time-delay feedback controllers containing only static elements can be designed to achieve identical linear stability properties. Analysis of an example involving a ring of coupled maps that can be controlled at only two sites demonstrates that the time-delay controller equivalent to a standard optimal controller can be equally robust in the presence of noise, except at special points in parameter space where the uncontrolled system has a mode with Floquet multiplier exactly equal to 1. Numerical simulations confirm the results of the analysis.

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I. INTRODUCTION

Over the past decade, it has become increasingly clear that robust and reliable methods of controlling chaotic dynamical systems would have important applications in a variety of engineering and scientific contexts. An important element of the control problem is the question of whether periodic orbits that would be unstable in the absence of control can be stabilized using noninvasive methods (in which the feedback signal vanishes on the desired orbit). Standard proportional control (SPC) methods have been thoroughly investigated in the classical control theory literature and well-developed techniques exist for determining whether an orbit is controllable and for constructing an optimal (linear) controller. In SPC, one compares the current state of the system to some external representation of the desired orbit and applies appropriate feedback when a difference is detected. The theory is particularly well understood for the case where the discrete time map is available describing the dynamics of the system on a Poincaré section of its phase space. Optimal control theory provides constructive methods for designing a noninvasive controller that will render stable an intrinsically unstable fixed point of such a map, which corresponds to an unstable periodic orbit of the continuous dynamical system.

An alternative to SPC is the use of a time-delay element in the control loop that allows comparison of the current state of the system to its state one or more periods in the past, rather than to an externally produced reference signal. Time-delay feedback has been shown to be a plausible technique in a variety of theoretical and experimental contexts [1–6]. It is particularly advantageous in fast systems where the reference state required for SPC cannot be readily produced [7–9]. Analytical understanding of time-delay controllers, however, lags far behind that of SPC. The design of optimal time-delay controllers is usually accomplished by experimentally or numerically scanning the space of parameters associated with a given control scheme.

In this paper we consider a particular form of time-delay feedback for discrete time maps. We point out that the optimal control theory for SPC can be translated directly into a constructive method for designing a time-delay controller whose linear stability properties will be identical to the optimal SPC for the same system. We then investigate the robustness of the time-delay controller in the presence of noise, adapting a method of analysis previously applied to SPC [10]. Our results indicate that the time-delay controller is no less robust than the standard one. This suggests that the advantages of time-delay feedback can be realized without an accompanying loss in performance.

The control scheme we investigate is a generalization of a method called "extended time-delay autosynchronization" (ETDAS) [9]. In ETDAS, the controller has a recursive structure that effectively stores information from previous periods, with the nth iterate in the past weighted by a factor of R. It is known that such a scheme can dramatically enlarge the domain of control over the $R=0$ case, but also that it fails in some cases of interest [11]. The generalization involves promoting both R and the feedback gain to matrices that act on all the system variables available for measurement rather than just one. It was first suggested by Nakajima [12] and we will refer to it as "generalized ETDAS" or GETDAS.

As noted by Nakajima, GETDAS controllers form a subset of the dynamic delayed feedback controllers introduced by Yamamoto *et al.* [13]. This subset has a property that is particularly important for high-speed applications: all of the elements in the controller are passive, performing the same linear transformations on their inputs at all times. The dynamic controllers, on the other hand, include elements in the feedback loop that must be adjusted at each iteration in response to the system's behavior, which requires construction of a dynamical system operating at the same high frequencies as the system being controlled.

In Sec. II we introduce the generalized form of ETDAS, discuss its ability to stabilize unstable modes, and show how to convert a standard control matrix into an equivalent GETDAS form. In Sec. III we illustrate the method with the example of a ring of diffusively coupled logistic maps. In Sec. IV we derive a formula for the size of deviations from the fixed point in the long time statistically steady state for a linear system with a given level of intrinsic noise and compare the results for the ring of logistic maps controlled by SPC or GETDAS. In Sec. V we discuss the effects of nonlinearities, comparing numerical results to heuristic formula for the maximum tolerable noise level in a GETDAS controller. Section VI summarizes our results and frames some open questions.

II. STANDARD CONTROL AND TIME-DELAY CONTROL

The standard approach to stabilizing a fixed point of a dynamical system governed by a discrete time map is to apply a feedback signal proportional to the difference between the state of the system and the desired fixed point. For a system with L dynamical variables, we write

$$\mathbf{y}_{n+1} = \mathbf{F}(\mathbf{y}_n) + \hat{\mathbf{B}}\mathbf{u}_n, \tag{1}$$

$$\mathbf{u}_n = -\hat{\mathbf{K}}(\mathbf{y}_n - \mathbf{y}^*). \tag{2}$$

Here $\mathbf{y}_n$ is an L-dimensional vector, with n indexing discrete time steps; $\mathbf{u}_n$ is the control; $\mathbf{F}$ specifies the dynamics of the system; $\hat{\mathbf{B}}$ is an $L\times L$ matrix specifying which system variables can be adjusted externally and how variations in $\mathbf{u}$ affect them; and $\hat{\mathbf{K}}$, a matrix we are free to choose, transforms the measured variables into one or more control signals. It is assumed that $\mathbf{F}$ has a fixed point $\mathbf{y}^*$ that is unstable in the absence of control; i.e., for $\hat{\mathbf{K}}=0$.

Defining $\hat{\mathbf{A}}$ as the Jacobean of $\mathbf{F}$ evaluated at $\mathbf{y}^*$ and $\mathbf{x}_n$ as the deviation $\mathbf{y}_n-\mathbf{y}^*$, the linearized system in the vicinity of the fixed point is

$$\mathbf{x}_{n+1} = \hat{\mathbf{A}}\mathbf{x}_n + \hat{\mathbf{B}}\mathbf{u}_n, \tag{3}$$

$$\mathbf{u}_n = -\hat{\mathbf{K}}\mathbf{x}_n. \tag{4}$$

We define $\hat{\mathbf{M}}$,

$$\hat{\mathbf{M}} \equiv \hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}}. \tag{5}$$

The control problem is to find a matrix $\hat{\mathbf{K}}$ such that all eigenvalues of $\hat{\mathbf{M}}$ have a magnitude smaller than 1, thereby making the controlled system linearly stable. Using quadratic optimal control theory [14] an appropriate matrix $\hat{\mathbf{K}}$ can be constructed whenever $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ satisfy the *controllability* condition Rank$[\hat{\mathbf{C}}]=L$, where

$$\hat{\mathbf{C}} \equiv [\hat{\mathbf{B}},\hat{\mathbf{A}}\hat{\mathbf{B}},\hat{\mathbf{A}}^2\hat{\mathbf{B}},\ldots,\hat{\mathbf{A}}^{L-1}\hat{\mathbf{B}}]. \tag{6}$$

If one of the eigenvalues of $\hat{\mathbf{A}}$ lies on the unit circle, the uncontrolled system has a marginal eigenvector. In the special case where the marginal eigenvalue is exactly 1, rather than any other complex number of unit magnitude, we say that the uncontrolled system has a *stationary* mode or eigenvector. A stationary eigenvector is completely invariant under the action of $\hat{\mathbf{A}}$.

The form of a GETDAS control signal may be written in three equivalent ways:

$$\mathbf{u}_n = \hat{\mathbf{G}}\cdot(\mathbf{x}_n - \mathbf{x}_{n-1}) + \hat{\mathbf{R}}\cdot\mathbf{u}_{n-1} \tag{7}$$

$$= \hat{\mathbf{G}}\cdot\sum_{l=0}^{\infty}\hat{\mathbf{R}}^l\cdot(\mathbf{x}_{n-l} - \mathbf{x}_{n-l-1}) \tag{8}$$

$$= \hat{\mathbf{G}}\cdot\left[\mathbf{x}_n + (\hat{\mathbf{R}} - \hat{\mathbf{I}})\cdot\sum_{l=0}^{\infty}\hat{\mathbf{R}}^l\cdot\mathbf{x}_{n-l-1}\right]. \tag{9}$$

Note that expanding $\mathbf{u}_{n-1}$ in terms of $\mathbf{x}_{n-l}$ gives an infinite sum over previous states of the system, as in the scalar case discussed by Socolar and Gauthier [15]. When there is only one variable available for monitoring and one adjustable system parameter, or when both the matrices $\hat{\mathbf{G}}$ and $\hat{\mathbf{R}}$ are just multiples of the identity matrix, the situation reduces to the scalar case that has been studied previously [15,16].

Using Eqs. (3) and (7), the GETDAS scheme can be written as follows:

$$\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix}_{n+1} = \hat{\mathbf{Q}}\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix}_{n} \tag{10}$$

with

$$\hat{\mathbf{Q}} \equiv \begin{bmatrix}\hat{\mathbf{A}} & \hat{\mathbf{B}}\\ \hat{\mathbf{G}}[\hat{\mathbf{A}} - \hat{\mathbf{I}}] & \hat{\mathbf{G}}\hat{\mathbf{B}} + \hat{\mathbf{R}}\end{bmatrix}. \tag{11}$$

The stability of the controlled system is determined by the eigenvalues of $\hat{\mathbf{Q}}$.

Much attention has been given to the fact that ETDAS with scalar parameters G and R is incapable of suppressing instabilities in cases where $\hat{\mathbf{A}}$ has an odd number of real eigenvalues larger than unity [17]. Yamamoto *et al.* suggested a dynamical delayed feedback control scheme that avoids this weakness [13]. In the dynamical delayed feedback control (DDFC) scheme, the feedback signal is given by

$$\mathbf{u}_n = \hat{\mathbf{a}}(\mathbf{x}_n - \mathbf{x}_{n-1}) + \hat{\mathbf{b}}\mathbf{w}_n \tag{12}$$

with

$$\mathbf{w}_{n+1} = \hat{\mathbf{c}}(\mathbf{x}_n - \mathbf{x}_{n-1}) + \hat{\mathbf{d}}\mathbf{w}_n. \tag{13}$$

The control signal $\mathbf{u}$ is composed of two parts, the first one is (the matrix version of) the traditional TDAS term (the $R=0$ case of ETDAS). The second term involves an introduced dynamical variable $\mathbf{w}$.

Yamamoto *et al.* showed that for almost any matrix $\hat{\mathbf{K}}$, the DDFC parameters can be chosen to give $\mathbf{u}_n=-\hat{\mathbf{K}}\cdot\mathbf{x}_n$, which means that the behavior of $\mathbf{x}_n$ and $\mathbf{u}_n$ in the DDFC system will be identical to that in a SPC system with the same $\hat{\mathbf{K}}$. The choices leading to this equivalence are

$$\hat{\mathbf{a}} = -\hat{\mathbf{K}}\cdot\hat{\mathbf{A}}\cdot[\hat{\mathbf{A}} - \hat{\mathbf{I}}]^{-1}, \tag{14}$$

$$\hat{\mathbf{b}} \cdot \hat{\mathbf{c}} = -[\hat{\mathbf{K}} + \hat{\mathbf{a}}] \cdot \hat{\mathbf{B}} \cdot \hat{\mathbf{a}}, \tag{15}$$

$$\hat{\mathbf{d}} = \hat{\mathbf{c}} \cdot \hat{\mathbf{a}}^{-1} \cdot \hat{\mathbf{b}}. \tag{16}$$

The condition for constructing the equivalent DDFC controller is that $\hat{\mathbf{A}} - \hat{\mathbf{I}}$ be invertible, which is true as long as the uncontrolled system does not have a stationary mode.

Yamamoto's result shows that DDFC, which requires no comparison of a system variable to its fixed point value, can be almost as versatile as SPC, failing only in cases where the uncontrolled system has a stationary mode. Some examples of such modes are the Goldstone modes associated with a continuous symmetry and therefore cannot be avoided. In other cases, such as the coupled map lattice discussed below, stationary modes occur at special isolated points in parameter space. We will consider below the effect of isolated stationary modes on the robustness of a time-delay controller, but first we consider an important drawback of the DDFC scheme.

In order to implement DDFC, it is in general necessary to construct a dynamical system that produces the appropriate behavior of the variable $\mathbf{w}$ in Eq. (13). Unlike the feedback signal $\mathbf{u}$ of GETDAS, $\mathbf{w}$ cannot be written as a simple sum over past measurements of $\mathbf{x}$. There must be some element in the feedback system that generates an independent dynamical process. In many contexts, this defeats the purpose of time-delay control, whose primary advantage is the avoidance of the need for an externally generated dynamical signal that helps determine the feedback. In contrast, the GETDAS scheme can be implemented using only passive devices that repeatedly perform the same linear transformations on signals generated by the system of interest. Moreover, the inclusion of the sum over all past iterates can be accomplished in a simple way by implementing a scheme suggested by Eq. (7). A schematic diagram of a GETDAS controller is shown in Fig. 1. For implementation of the discrete controllers shown here, the output of $\hat{\mathbf{B}}$ is sampled once every period and the sampled value determines the feedback signal.

Nakajima, in introducing GETDAS, has pointed out that it is a special case of DDFC [12]. For the case $\hat{\mathbf{d}} = \hat{\mathbf{c}}\hat{\mathbf{a}}^{-1}\hat{\mathbf{b}}$, Eqs. (12) and (13) give $\mathbf{w}_n = \hat{\mathbf{c}} \cdot \hat{\mathbf{a}}^{-1} \cdot \mathbf{u}_{n-1}$, so $\mathbf{w}$ can be eliminated from the equations, leaving them in a form identical to GETDAS with

$$\hat{\mathbf{G}} = \hat{\mathbf{a}},$$

$$\hat{\mathbf{R}} = \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{a}}^{-1}. \tag{17}$$

Here we show that the GETDAS subset of DDFC controllers is big enough to contain an equivalent controller to any SPC scheme (with the usual exception for stationary modes). *Theorem*. Given a discrete linear system $\mathbf{x}_{n+1} = \hat{\mathbf{A}} \cdot \mathbf{x}_n$ that has no stationary eigenvectors (i.e., for which $\hat{\mathbf{A}}$ has no eigenvalues exactly equal to 1), if the system can be stabilized using SPC, then it can also be stabilized using GETDAS.

Proof: In analogy with Eq. (10), the SPC scheme can be represented in the following form:

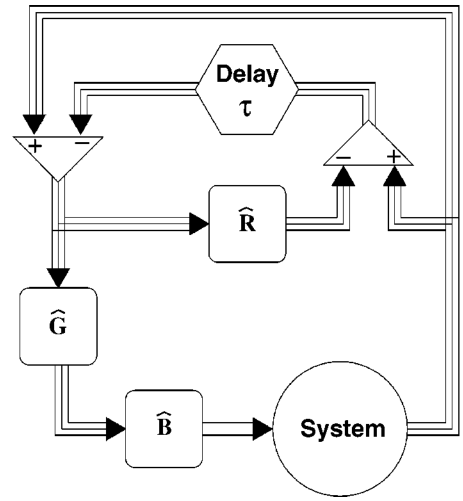

FIG. 1. Schematic circuit diagram for implementing GETDAS to control a dynamical system. Triple lines indicate multiple signals that are fed into each element in the loop. The hexagon represents an element whose only effect is a time delay of all incoming signals. Triangles represent devices that form the difference of each pair of incoming signals. Each labeled square represents a device that performs a linear transformation on its inputs. The square labeled $\hat{\mathbf{B}}$ may be considered as a part of the system that cannot be changed and may have fewer outputs than the full number of system variables. It contains the information about which system variables can be monitored, which system parameters can be adjusted through feedback, and how those adjustments affect all of the different system variables.

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}_{n+1} = \begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ -\hat{\mathbf{K}}\hat{\mathbf{A}} & -\hat{\mathbf{K}}\hat{\mathbf{B}} \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}_n. \tag{18}$$

From the general form of GETDAS given in Eqs. (10) and (11), one sees immediately that the two schemes are equivalent if and only if the matrices $\hat{\mathbf{G}}$ and $\hat{\mathbf{R}}$ are chosen as follows:

$$\hat{\mathbf{G}} = -\hat{\mathbf{K}}[\hat{\mathbf{A}} - \hat{\mathbf{I}}]^{-1}\hat{\mathbf{A}}, \tag{19}$$

$$\hat{\mathbf{R}} = \hat{\mathbf{K}}[\hat{\mathbf{A}} - \hat{\mathbf{I}}]^{-1}\hat{\mathbf{B}}. \tag{20}$$

(Note that $\hat{\mathbf{A}}$ commutes with $[\hat{\mathbf{A}} - \hat{\mathbf{I}}]^{-1}$.) The correspondence fails if and only if $\hat{\mathbf{A}} - \hat{\mathbf{I}}$ is not invertible; i.e., if none of its eigenvalues vanishes. Let $\{\lambda_i\}$ be the eigenvalues of $\hat{\mathbf{A}}$. Since the eigenvalues of $\hat{\mathbf{A}} - \hat{\mathbf{I}}$ are equal to $\lambda_i - 1$, the condition for invertibility of $\hat{\mathbf{A}} - \hat{\mathbf{I}}$ is $\lambda_i \neq 1$ for all i. Q.E.D

The reason the GETDAS cannot stabilize a system with a stationary mode is conceptually clear. For a long-lived perturbation in the stationary direction we have $\hat{\mathbf{A}} \cdot \mathbf{s} - \mathbf{s} = 0$, hence $\mathbf{x}_{n+1} - \mathbf{x}_n = 0$ and no feedback signal is generated.

The matrices in Eqs. (18) and (11) are $2L \times 2L$ matrices. In the former case, the obvious degeneracy between the top L and bottom L rows causes L of the eigenvalues to be zero, as expected given that the stability of the SPC scheme is determined completely by the matrix $\hat{\mathbf{M}}$. In contrast, $\hat{\mathbf{Q}}$ has $2L$ nontrivial eigenvalues. It is only for special choices of the pair $\hat{\mathbf{G}}$ and $\hat{\mathbf{R}}$ that L of these vanish. Investigation of the most general conditions on $\hat{\mathbf{G}}$ and $\hat{\mathbf{R}}$ for stability is beyond the scope of this work.

We have shown that any $\hat{\mathbf{K}}$ that stabilizes the system via SPC can be converted into a GETDAS control scheme. The converse, however, is not true. The system may be stabilizable via GETDAS using some other $\hat{\mathbf{G}}$ and $\hat{\mathbf{R}}$ which may or may not correspond to a SPC matrix $\hat{\mathbf{K}}$. The reverse conversion is possible if and only if the GETDAS matrices are related by

$$\hat{\mathbf{R}} = -\hat{\mathbf{G}}\hat{\mathbf{A}}^{-1}\hat{\mathbf{B}}. \tag{21}$$

Thus set of GETDAS controllers is larger than the set of SPC controllers.

III. A COUPLED MAP LATTICE

Given the possibility of constructing a time-delay controller that has the same linear stability properties as any given standard one, it is natural to ask whether one must pay a price for using the time delay. One issue that might be relevant is the sensitivity of the controller to noise. One might worry that the time-delay controller will be less robust than its SPC counterpart. As a preliminary investigation of this issue, we consider a nontrivial example of a high-dimensional system: a coupled map lattice (CML) in which all variables can be measured but only two adjacent sites are accessible for control.

We consider a ring of L diffusively coupled, identical logistic maps. The general equation describing the uncontrolled system is

$$y_l^{(n+1)} = (1-2\epsilon)f(y_l^{(n)}) + \epsilon[(1-a)f(y_{l-1}^{(n)}) + (1+a)f(y_{l+1}^{(n)})]. \tag{22}$$

The subscript l indicating spatial position runs from 0 to $L-1$, and all indices are taken modulo L. The superscripts in parentheses indicate temporal iterates. $f(y)$ is a function describing a single site evolution in one time step; the constant $\epsilon \in (0,1)$ indicates the coupling strength between neighboring sites; and $a \in (-1,1)$ is an asymmetry parameter in the coupling strengths to left and right neighbors. For $a=0$ the dynamics are governed by symmetric diffusion and for $a=\pm 1$ we have one-way coupling.

The fixed point of interest is the homogeneous one, $y_l = y^*$ for all l. The linearized equations for $x_l = y_l - y^*$ are written as

$$\mathbf{x}^{(n+1)} = \hat{\mathbf{A}}\mathbf{x}^{(n)}, \tag{23}$$

where $\mathbf{x}$ is the vector $(x_1, x_2, \ldots, x_L)$ and $\hat{\mathbf{A}}$ has the following form:

$$\hat{\mathbf{A}} = \nu(1-2\epsilon)\hat{\mathbf{I}} + \nu\epsilon((1-a)\hat{\mathbf{J}} + (1+a)\hat{\mathbf{J}}^T), \tag{24}$$

where $\nu \equiv \partial f/\partial y|_{y=y^*}$: $\hat{\mathbf{I}}$ is the identity matrix; and

$$\hat{\mathbf{J}} = \begin{bmatrix} 0 & 0 & \ldots & & 0 & 1 \\ 1 & 0 & 0 & \ldots & & 0 \\ 0 & 1 & 0 & 0 & \ldots & \\ & & & \ddots & & \\ 0 & \ldots & 0 & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 \end{bmatrix}. \tag{25}$$

It has been shown previously [18,19] that for any L the symmetric system is controllable using SPC if all sites are measurable and just two neighboring sites can be directly affected by the feedback signal.

Because stationary modes make GETDAS control impossible, it is important to know the parameter values where such modes exist. A straightforward analysis reveals that for fixed ν, L, and a, the present CML does indeed have stationary modes, but only for special, isolated values of ϵ, which we denote ϵ_s. Writing out the components of Eq. (23) gives

$$x_l^{(n+1)} = \nu(1-2\epsilon)x_l^{(n)} + \nu\epsilon_s[x_{l-1}^{(n)} + x_{l+1}^{(n)} + a(x_{l+1}^{(n)} - x_{l-1}^{(n)})] \tag{26}$$

for every l and n. We write a solution as $x_l^{(n)} = \xi(n)\exp[ilk]$, with k restricted by the periodic boundary conditions to take on the values $2s\pi/L$ for integer s. For a stationary mode, $\xi(n)$ is constant. This occurs only if

$$\frac{\nu-1}{4\nu} = \epsilon \sin\left(\frac{s\pi}{L}\right)\left[\sin\left(\frac{s\pi}{L}\right) - ia\cos\left(\frac{s\pi}{L}\right)\right]. \tag{27}$$

For values of ϵ satisfying this equation, the uncontrolled system will have a stationary mode.

For L even, Eq. (27) is satisfied for $s=L/2$, independent of a, by

$$\epsilon_s = \frac{\nu-1}{4\nu}. \tag{28}$$

For $a=0$ which corresponds to the symmetric diffusive CML, additional solutions exist with

$$\epsilon_s = \frac{\nu-1}{4\nu}\frac{1}{\sin^2(s\pi/L)} \tag{29}$$

for all integers s. For $a \neq 0$, there are no stationary modes other than the $s=L/2$ case.

As shown by Grigoriev *et al.*, control of a symmetric CML can be achieved with just two controllers placed at adjacent sites [18]. In the present notation, this corresponds to an $L \times L$ matrix $\hat{\mathbf{B}}$ with elements $B_{11} = B_{22} = 1$ and all other elements equal to 0. An appropriate $L \times L$ control matrix $\hat{\mathbf{K}}$ may be determined by iterative solution of the Riccati equation of optimal control theory [14]. Due to the structure of $\hat{\mathbf{B}}$ only the first two rows of $\hat{\mathbf{K}}$ are relevant; all other elements vanish identically. Converting any SPC matrix $\hat{\mathbf{K}}$ into the GETDAS form results in a matrix $\hat{\mathbf{G}}$ with all elements outside the first two rows equal to 0, and a matrix $\hat{\mathbf{R}}$ with all elements vanishing except for the upper left 2×2 block.

IV. NOISE AMPLIFICATION IN THE LINEAR VICINITY OF THE FIXED POINT

In the presence of bounded noise, the controlled system may remain confined to a region near the fixed point or may exhibit deviations from the fixed point limited only by nonlinear saturation effects. To determine which occurs for a given level of intrinsic noise, the first step is to calculate the linear amplification of the noise due to non-normality of the eigenvectors of the controlled system [10]. This amounts to a computation of the size of the cloud of points that will occur around the fixed point when the intrinsic noise level is low enough that the system never leaves the linear regime. We discuss the effects of nonlinearity in Sec. V below. Here we study the effect of a GETDAS controller on the noise level, assuming the noise is so small that even with amplification it remains well within the linear regime. The goal is to see whether the GETDAS controller amplifies noise significantly more strong than its SPC counterpart.

We assume that noise enters the system in the form of an independent random addition q to x at every site and on every time step, with q drawn from a bounded distribution with variance σ^2. (For the numerical simulations discussed below we assume a uniform distribution with $q \in [-\sqrt{3}\sigma, \sqrt{3}\sigma]$.) The equation describing the evolution of the controlled system is

$$\mathbf{x}^{(n+1)} = \hat{\mathbf{A}}\mathbf{x}^{(n)} + \hat{\mathbf{B}}\mathbf{u}^{(n)} + \mathbf{q}^{(n)}, \tag{30}$$

with

$$\langle q_j^{(n)} \rangle_a = 0 \quad \forall\, n,j, \tag{31}$$

$$\langle q_j^{(n)} q_i^{(m)} \rangle_a = \sigma^2 \delta_{ij}\delta_{nm}. \tag{32}$$

Here the notation $\langle\rangle_a$ represents an ensemble average. The SPC case has been analyzed previously [10,18].

Using $\hat{\mathbf{M}}$ as defined in Eq. (5) and the GETDAS scheme equivalent to SPC with the given $\hat{\mathbf{K}}$, Eq. (30) becomes

$$\begin{aligned}\mathbf{x}^{(n+1)} &= \hat{\mathbf{M}}\mathbf{x}^{(n)} + \mathbf{q}^{(n)} - \hat{\mathbf{R}}\mathbf{q}^{(n-1)} \\ &= \hat{\mathbf{M}}^n\mathbf{x}^{(1)} + \sum_{l=0}^{n-1} \hat{\mathbf{M}}^l (\mathbf{q}^{(n-l)} - \hat{\mathbf{R}}\mathbf{q}^{(n-l-1)}).\end{aligned} \tag{33}$$

Below we use a notation familiar from quantum mechanics: $\langle \mathbf{a} | \mathbf{b} \rangle \equiv \mathbf{a}^* \cdot \mathbf{b}$ (with no ensemble average implied). We let λ_j be the eigenvalues of $\hat{\mathbf{M}}$ and $\{|\mathbf{e}^j\rangle\}$ be the corresponding eigenvectors. The vectors $\{|\mathbf{v}^i\rangle\}$ are defined by the relation $\langle \mathbf{e}^i | \mathbf{v}^j \rangle = \delta_{ij}$. We may then write $\mathbf{q} = \Sigma \langle \mathbf{v}_i | \mathbf{q} \rangle | \mathbf{e}_i \rangle$.

Defining a noise amplification factor γ as the ratio of the standard deviation of the distribution of states around the fixed point in the controlled system to the standard deviation of the intrinsic noise,

$$\gamma \equiv \frac{\sqrt{\langle \mathbf{x}^{(n+1)} \cdot \mathbf{x}^{(n+1)} \rangle_a}}{L\sigma}, \tag{34}$$

an ensemble average of Eq. (33) yields

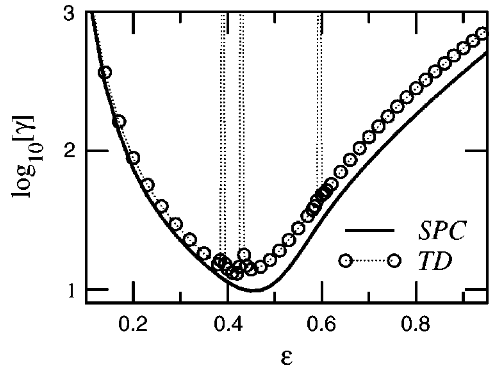

FIG. 2. Noise amplification in a symmetrically coupled ring of linear maps with periodic boundary conditions as a function of the coupling constant (ϵ). The size of the ring is $L=10$ and the map multiplier is $\nu=-1.8$. The solid line shows γ for an optimal SPC stabilization scheme and the circles connected by the dotted line for a GETDAS scheme. Note the divergence of γ for $\epsilon=\epsilon_s$ in the latter case.

$$\gamma = \sqrt{\frac{t_1 + t_2 + t_3}{L}} \tag{35}$$

with

$$t_1 = \sum_{i,j=1}^{L} \frac{\langle \mathbf{e}^i | \mathbf{e}^j \rangle \langle \mathbf{v}^j | \mathbf{v}^i \rangle}{1 - \lambda_i^* \lambda_j}, \tag{36}$$

$$t_2 = -2\Re \left[\sum_{i,j,r=1}^{L} \frac{\langle \mathbf{v}^j | \mathbf{v}^i \rangle \langle \mathbf{e}^i | \mathbf{e}^r \rangle \langle \mathbf{v}^r | \hat{\mathbf{R}} | \mathbf{e}^j \rangle \lambda_i^*}{1 - \lambda_i^* \lambda_r} \right], \tag{37}$$

$$t_3 = \sum_{i,j,r,s=1}^{L} \frac{\langle \mathbf{v}^j | \mathbf{v}^i \rangle \langle \mathbf{e}^i | \hat{\mathbf{R}}^T | \mathbf{v}^r \rangle \langle \mathbf{e}^r | \mathbf{e}^s \rangle \langle \mathbf{v}^s | \hat{\mathbf{R}} | \mathbf{e}^j \rangle}{1 - \lambda_r^* \lambda_s}, \tag{38}$$

where we have used the fact that $\hat{\mathbf{R}}$ is real and $\hat{\mathbf{M}}^n\mathbf{x}^{(1)}$ goes to zero for large n, the latter being true by the assumption that $\hat{\mathbf{K}}$ has been chosen such that SPC renders the noiseless system asymptotically stable. As previously shown by Egolf and Socolar [10], the result for the SPC scheme is given by the t_1 term alone. The contributions t_2 and t_3 describe the effect produced in the GETDAS scheme by the repeated passage of noise signals through the time-delay feedback loop.

A comparison of the noise amplification in SPC and the equivalent GETDAS schemes is shown in Fig. 2 for a ring of coupled maps with $a=0$ (the symmetric diffusion case). It is assumed that all of the sites can be monitored, but only two adjacent sites can be affected by the feedback. The specific case shown has $\nu=-1.8$. Results for other values of ν are qualitatively similar. For almost all values of the coupling strength, there is a price paid in amplification for using GETDAS rather than SPC, but it is rather small—at most a factor of 2 for very strong coupling. There are special points, however, where GETDAS generates extremely large amplifications. These are points very close to the coupling strengths where stationary modes are present in the uncontrolled system.

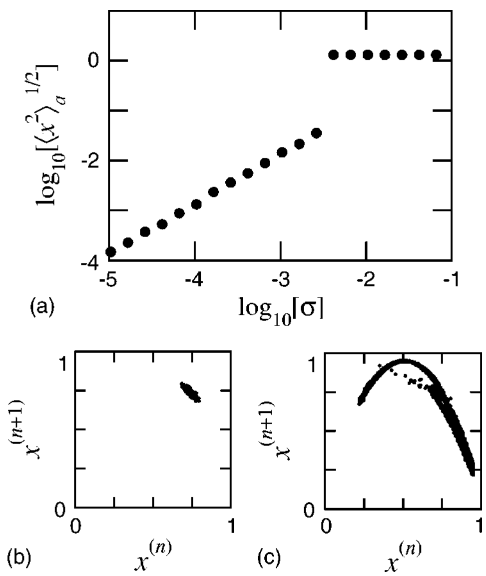

FIG. 3. (a) The radius of the cloud of points scattered around the fixed point, as a function of the intrinsic noise level σ, for a ring of symmetrically coupled logistic maps controlled via GETDAS as described in the text at the end of Sec. III. The size of the CML is $L=10$ and the map multiplier at the fixed point is $\nu=-1.8$ and the coupling constant is $\epsilon=0.40$. (b) The return map for the site farthest from the controller for $\sigma=10^{-2.4}$. The cloud of points is confined to the vicinity of the fixed point. (c) The return map for the site farthest from the controller for $\sigma=10^{-2.3}$, just beyond the point where control is lost.

In a purely linear system, noise never destroys the stability of a fixed point. Though the cloud of points the system visits may have a large variance due to non-normal effects that can make γ large, the ratio of this variance to the variance of the intrinsic noise is independent of the intrinsic noise level.

V. STRONG NOISE AND THE TOLERANCE LIMIT

In a nonlinear system, the situation is different. As the intrinsic noise level σ is increased, the standard deviation of the cloud around the fixed point grows in proportion for small noise levels, but then increases rapidly as σ crosses some threshold. Figure 3 illustrates this behavior in the case of coupled logistic maps controlled via GETDAS. For this study, the logistic map $f(x)=\mu x(1-x)$ was modified so as to avoid the trivial divergence associated with points that leave the domain $[0,1]$. On each iteration, points that fall outside the unit interval are reflected about 0 or 1, as needed. The jump in the plot in Fig. 3(a), which we interpret as signaling a loss of control, occurs when deviations about the fixed point are still much less than 1. The jump is caused by the nonlinearity in the map, not by the crossing of an attractor basin boundary. The value of σ at the threshold is denoted σ_{max}, as it is the maximum noise strength that can be tolerated by the given controller. Figures 3(b) and 3(c) show the return maps just below and above the transition, respectively, for the site in the ring that is farthest from the two controllers.

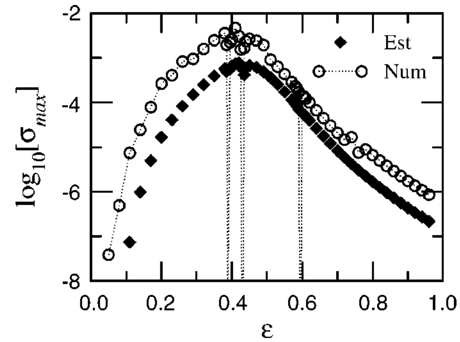

FIG. 4. The maximum noise level controlled using GETDAS as a function of coupling constant in a ring of symmetrically coupled logistic maps controlled as described in the text at the end of Sec. III. The size of the CML is $L=10$ and the map multiplier at the fixed point is $\nu=-1.8$.

A simple argument gives a useful estimate of σ_{max} [10]. From the point of view of the linear control theory, nonlinearities introduce errors that can be treated as an additional source of noise for the linear analysis. Assuming that the errors associated with the nonlinearity are uncorrelated between time steps and uncorrelated with the intrinsic noise, we can write the total standard deviation from the fixed point Δ as

$$\Delta = \gamma\sqrt{\sigma^2 + \sigma_{nl}^2}, \tag{39}$$

where σ_{nl} is the standard deviation of the errors induced by nonlinearity. Assuming that the quadratic nonlinearity does not vanish, a rough estimate of σ_{nl} for relatively small x is $f''(\mathbf{x}^*)\Delta^2/2$. The maximum noise level σ_{max} is then determined by a self-consistency condition on Δ:

$$\Delta = \gamma\sqrt{\sigma^2 + \tfrac{1}{2}f''(\mathbf{x}^*)\Delta^4}, \tag{40}$$

which has a real solution for all $\sigma < \sigma_{max}$ if and only if

$$\sigma_{max} = \frac{1}{\gamma^2|f''(\mathbf{x}^*)|}. \tag{41}$$

For a symmetrically coupled ring of identical logistic maps, we have measured σ_{max} via simulation and compared the results to the estimate of Eq. (41). Figure 4 shows that the analytic estimate of σ_{max} for all values of the coupling constant ϵ is lower than the numerically determined σ_{max}. (It appears that the correlations neglected in the analytic estimate tend to help rather than hurt the controller.) Figure 5 shows that even though the amplification in the GETDAS controlled CML is greater than the SPC amplification, which implies a lower σ_{max} for GETDAS, the noise level tolerated by the GETDAS controller is actually *higher* than that for SPC over a wide range of ϵ. It appears that, at least in this one case, the use of information about past iterates can improve the robustness of the linear controller against noise.

Figure 6 shows the noise tolerance results for both SPC and GETDAS, this time as a function of γ, the linear noise amplification factor. The two branches visible for both schemes correspond to two regimes of coupling: the lower and upper branches in each case are data for $\epsilon \in (0, 0.5)$ and

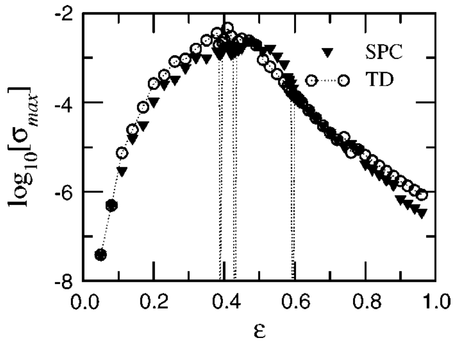

FIG. 5. Comparison of the maximum noise level controlled using SPC or the equivalent GETDAS scheme. The system studied is the same as that of Fig. 4.

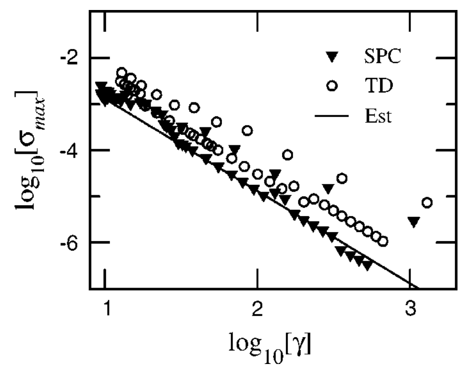

FIG. 6. Maximum controllable noise levels in a CML control using SPC or GETDAS as a function of the linear amplification factor. The upper branch of each data set is for $\epsilon < 0.5$ and the lower branch is for $\epsilon > 0.5$. The system studied is the same as that of Fig. 4. The solid line is the estimated maximum tolerable noise level as computed from Eq. (41).

$\epsilon \in (0.5, 1)$, respectively. Here one observes that for comparable γ, the GETDAS scheme consistently tolerates more noise than the SPC scheme.

VI. CONCLUSION

We have shown a simple way to implement a time-delay feedback control scheme for discrete-time systems that matches the performance of an optimal controller designed by standard methods. The technique is a generalization of the ETDAS method that has previously been shown to be useful for stabilizing high-speed systems, and suggests the possibility of controlling more complex high-speed systems than have previously been studied. The one drawback of the GETDAS scheme is that it cannot be applied to systems that have a stationary eigenmode for specific parameter values, but the effects of such modes appear to be manifested only when the parameters are extremely close to those values. A simple theoretical argument for the maximum noise level that can be tolerated by the linear controller has been shown to give a reasonable estimate for when control will be lost in the case of a ring of coupled logistic maps. Detailed comparison of the GETDAS and SPC noise tolerance properties suggests that GETDAS may actually be marginally better over substantial ranges of parameter space.

Several questions now arise. First and foremost, it will be important to study the behavior of the GETDAS controller when applied to a continuous time dynamical system. The discrete time analysis of maps has proven to be a useful guide to intuition regarding simpler versions of ETDAS, but the connection is not a direct one. Within the context of discrete maps, there is also an interesting optimal control problem to be solved. Since there is a wider parameter space for the choice of $\hat{\mathbf{G}}$ and $\hat{\mathbf{R}}$ than there is for the choice of $\hat{\mathbf{K}}$, it is possible that GETDAS schemes having no SPC counterpart could be the method of choice for some systems, even if high-speed operation is not an issue.

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