Small numerators cancelling small denominators of the high-temperature scaling variables: A systematic explanation in arbitrary dimensions

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We describe a method to express the susceptibility and higher derivatives of the free energy in terms of the scaling variables (Wegner's nonlinear scaling fields) associated with the high-temperature (HT) fixed point of the Dyson hierarchical model in arbitrary dimensions. We give a closed form solution of the linearized problem. We check that up to order 7 in the HT expansion, all the poles ("small denominators") that would naively appear in some positive dimension are canceled by zeros ("small numerators"). The requirement of continuity in the dimension can be used to lift ambiguities which appear in calculations at fixed dimension. We show that the existence of a HT phase in the infinite volume limit for a continuous set of values of the dimension, requires that this mechanism works to all orders. On the other hand, most poles at negative values of the dimensional parameter [where the free energy density is not well-defined, but renormalization group (RG) flows can be studied] persist and reflect the fact that for special negative values of the dimension, finite-size corrections become leading terms. We show that the inverse problem is also free of small denominator problems and that the initial values of the scaling variables can be expressed in terms of the infinite volume limit of the susceptibility and higher derivatives of the free energy. We discuss the existence of an infinite number of conserved quantities (RG invariants) and their relevance for the calculation of universal ratios of critical amplitudes.

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I. INTRODUCTION

In many problems, one faces the challenge of deriving the macroscopic consequences of a microscopic theory. As we look at the problem at increasingly large scales, a sequence of effective theories appear and under some appropriate conditions, an infinite volume limit can be taken. A general method that allows us to construct these flows in the space of theories is the renormalization group (RG) method [1]. The study of some RG fixed points and of the linearized flows close to these fixed points has produced a successful picture of the universal behavior in second order phase transitions. On the other hand, controlling the RG flows beyond the linearized approximation and calculating the related nonuniversal behavior are more difficult issues. This is unfortunately necessary to calculate the critical amplitudes.

As a first step, one can deal with the nonlinear RG flows for simplified models where the RG transformation can be implemented without major technical difficulty. One possibility is to use approximate versions of the exact RG equations [2,3] such as the local potential approximation [4]. Another possibility to address nonlinear questions [5–7] is to use Dyson's hierarchical model [8,9]. In the following, we use this lattice model for which the block-spin method can be easily implemented. This model is briefly reviewed in Sec. II. Other approaches of nonlinear aspects of the RG flows can be found, for instance, in Refs. [10–13].

In the context of ordinary differential equations, a standard method [14] to go beyond the linearized approximation in the vicinity of a fixed point, consists in constructing a new system of coordinates where the equations become linear. However, this type of procedure is often plagued with the "small denominator problem" initially encountered by Poincaré in his study of perturbed integrable Hamiltonians. In the context of the RG method, these new coordinates are called the scaling variables (or the nonlinear scaling fields) and were first introduced by Wegner [15]. Recently, we have proposed an *ab initio* calculation of the critical amplitudes in the high-temperature (HT) phase of this model [7]. In this calculation, the critical amplitudes are RG invariant made out of the nonlinear scaling variables associated with Wilson's nontrivial IR fixed point *and* the nonlinear scaling variables associated with the HT fixed point. In this approach, the two fixed points are in some approximate sense dual [16] to each others. The scaling variables associated with Wilson's fixed point have been extensively discussed, but much less is known about those associated with the HT fixed point. We emphasize that being able to use both kind of variables is quite convenient for the study of the RG flows in the intermediate region between the two fixed points.

At first sight, the construction of the scaling variables associated with the HT fixed point is impossible for $D=3$ and more generally for rational values of D, because some of the denominators are exactly zero. However, a numerical study in $D=3$ showed [17] that in all of the 36 zero denominators considered, a zero numerator miraculously appears. This strongly indicates the existence of a general mechanism enabling us to overcome the small denominator problem.

In this paper, we show that such a mechanism exists and is closely related to the existence of the infinite volume limit of the susceptibility and higher derivatives of the free energy. In addition, we address the issue that whenever zero numera-

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tor and denominator appear at the same time, the coefficients of the nonlinear expansion appear to be undetermined $\left(\frac{0}{0}\right)$. We show that this indeterminacy can be lifted by a procedure similar to the dimensional regularization [18] used for the evaluation of Feynman diagrams. It should however be emphasized that it is not used here to take care of a UV problem, since we will be working with a lattice model. We will consider the construction of the HT scaling field in arbitrary dimensions. In our construction, the zero denominators appear as poles at particular dimensions and one can study the mechanism of cancellation close to a pole but not exactly at the pole.

For Dyson's hierarchical model, the dimension D appears in a continuous parameter $c=2^{1-2/D}$ introduced explicitly in Sec. II. The infinite volume limit is well defined for $0<c<2$, or in other words $D>0$. The linear variables associated with the HT fixed point are introduced in Sec. III. A closed form expression for the linear transformation which diagonalizes the linear RG transformation is given in arbitrary dimensions. The restriction to the first l_{max} of these variables can be interpreted as a HT expansion. In Sec. IV, we expand the linear variables in terms of the scaling variables. We show that up to order 7 in the HT expansion, the poles corresponding to zero denominators in positive dimensions $(0<c<2)$ are exactly canceled by a zero at the numerator. The coefficients of the expansion are then unambiguously defined rational functions of c with no poles for $0<c<2$. Their poles appear only at negative values of the dimension where the statistical mechanics model does not have a well defined infinite volume limit.

The linear variables are linear combinations of the average values of the total field $\Sigma_x \phi_x$. In Sec. V, we use this fact to reexpress the connected parts of the average values of the total field divided by the volume, or in other words, the susceptibility and the higher derivatives of the free energy density, in terms of the scaling variables. We show that up to order 7, the linear contribution is the *only* leading term in the infinite volume limit. In Sec. VI, we explain why this should happen to all orders. In Sec. VII we explain why it guarantees the cancellations discussed in Sec. IV to all orders.

Having showed that it is possible to construct a solution of the RG flows in the HT phase, we then need to calculate the initial values of the scaling variables in terms of the local measure (for instance, a Ising measure or a Landau-Ginzburg measure) used to specify the statistical mechanics model. This amounts to inverting the previous expansions. In Sec. VIII, we construct the scaling variables in terms of the linear variables and show that the coefficients are free of poles for $0<c<2$. We also show that, up to numerical constants, the initial values of the scaling variables are the infinite volume limit of the susceptibility and higher order derivatives of the free energy density. This concludes our construction of a complete solution of the RG flows in the HT phase. To be precise, we have shown that various expansions can be constructed order by order without encountering any small denominator problems and that it is possible to study empirically the convergence of these series. In Sec. IX, we show with an example how everything we have done can be used to calculate the HT expansion at finite volume. We also check explicitly that it yields results in agreement with calculations performed using independent methods [19]. In Sec. X, we discuss the existence of an infinite number of conserved quantities and their relevance for the calculation of universal ratios of critical amplitudes.

II. DYSON'S HIERARCHICAL MODEL

In this section, we remind some basic facts about Dyson's hierarchical model that will be needed in the following. For more details, the reader may consult Refs. [20,21]. We consider fields located at $2^{n_{max}}$ sites labeled with n_{max} indices $x_{n_{max}},\ldots,x_1$, each being 0 or 1. We divide the $2^{n_{max}}$ sites into two blocks, each containing $2^{n_{max}-1}$ sites. If $x_{n_{max}}=0$, the site is in the first box, if $x_{n_{max}}=1$, the site is in the second box and so on. The nonlocal part of the energy reads

$$H_{nl}=-\frac{1}{2}\sum_{n=1}^{n_{max}}\left(\frac{c}{4}\right)^n\sum_{x_{n_{max}},\ldots,x_{n+1}}\left(\sum_{x_n,\ldots,x_1}\phi_{(x_{n_{max}},\ldots x_1)}\right)^2. \quad (1)$$

The partition function for a constant source J (or external magnetic field) reads

$$Z(J)=\prod_x\int d\phi_x W(\phi_x)\exp\left(-\beta H_{nl}+J\sum_y\phi_y\right). \quad (2)$$

We call $W(\phi_x)d\phi_x$ the local measure. The most common examples are the Ising measure $W(\phi)=\delta(\phi^2-1)$ or the Landau-Ginzburg measure $W(\phi)=\exp(-A\phi^2-B\phi^4)$. The RG transformation consists in integrating over the fields keeping their sum constant in increasingly large boxes. After each integration the fields are rescaled by a factor $\sqrt{c/4}$ in order to keep the form of H_{nl} identical, and the RG transformation generates a flow in the space of local measures.

Note that for a constant configuration where all the fields take the same value $\overline{\phi}$, the nonlocal part of the energy takes the value

$$H_{nl}(\overline{\phi})=-2^{n_{max}}(\overline{\phi})^2\frac{1}{2}\sum_{n=1}^{n_{max}}\left(\frac{c}{2}\right)^n. \quad (3)$$

In the infinite volume limit $(n_{max}\to\infty)$, the sum converges only for $|c|<2$. Proofs of the existence of the thermodynamical limit for a Ising measure [22,23] require that the energy does not scale faster than the number of sites. This means $|c|<2$ for the model considered here.

In the following, we will make a change of variables in order to get rid of β in front of H_{nl} in Eq. (2) and reabsorb it in the local measure. Our main object of study will be the generating function (obtained by Fourier transforming the local measure)

$$R_n(k)=1+a_{n,1}k^2+a_{n,2}k^4+\cdots, \quad (4)$$

with

$$a_{n,l} = (-\beta)^l \frac{1}{2l!}\left(\frac{c}{4}\right)^{ln}\left\langle\left(\sum_{2^n sites} \phi_x\right)^{2l}\right\rangle. \qquad (5)$$

The RG transformation can be summarized in terms of the recursion formula

$$R_{n+1}(k) = C_{n+1}\exp\left[-\frac{1}{2}\frac{\partial^2}{\partial k^2}\right]\left[R_n\left(\frac{\sqrt{c}k}{2}\right)\right]^2. \qquad (6)$$

We fix the normalization constant C_n so that $R_n(0)=1$. Note that compared to Eq. (2.5) of Ref. [21], there is no factor β in the argument of the exponential because β has been reabsorbed in k and the $a_{n,l}$ according to Eq. (5).

It is important to remember that in the notation $a_{n,l}$, the first index refers to the number of RG steps and the second to the powers of the total field. Sometimes, the number of RG steps n will be omitted, sometimes the vector index l will be replaced by boldface notations. We use the parametrization $c=2^{1-2/D}$ such that a free massless field scales in the same way as in a usual D-dimensional theory. For reference, Dyson's parametrization [8] was $c=2^{2-\alpha}$. The logarithm of R generates the connected zero-momentum Green's functions at finite volume. We emphasize that in the following, the temperature dependence has been absorbed in the initial $R_0(k)$. For instance, in the case of an Ising measure, $R_0(k) = \cos(\sqrt{\beta}k)$.

In the HT phase, polynomial truncations of order l_{max} in k^2 provide rapidly converging approximations [6,20]. The RG flows can be expressed in terms of a quadratic map in a l_{max} dimensional space

$$a_{n+1,l} = \frac{u_{n,l}}{u_{n,0}}, \qquad (7)$$

with

$$u_{n,\sigma} = \Gamma_\sigma^{\mu\nu} a_{n,\mu} a_{n,\nu}, \qquad (8)$$

and

$$\Gamma_\sigma^{\mu\nu} = (c/4)^{\mu+\nu}\frac{(-1/2)^{\mu+\nu-\sigma}[2(\mu+\nu)]!}{(\mu+\nu-\sigma)!\,(2\sigma)!} \qquad (9)$$

for $\mu+\nu \geqslant \sigma$ and zero otherwise. We use "relativistic" notations. The greek indices μ and ν go from 0 to l_{max}, while latin indices i, j go from 1 to l_{max}. Repeated indices mean summation unless specified differently. With the normalization of Eq. (7), $a_n, 0=1$ for any n and is not a dynamical variable. Note that a truncation to order l_{max} is always implicit in the following. However, for reasons that will be explained in the following section, there is no explicit dependence in l_{max}.

III. THE LINEAR RG TRANSFORMATION

In this section we discuss the linearized RG transformation near the HT fixed point $a_i=0$ for all $i \geqslant 1$. For small departure from the HT fixed point $\delta a_{n,i}$ the linear RG transformation reads

$$\delta a_{n+1,i} \simeq \mathcal{M}_i^j \delta a_{n,j}, \qquad (10)$$

with

$$\mathcal{M}_i^j = 2\Gamma_i^{j0} = 2\left(\frac{c}{4}\right)^j\left(-\frac{1}{2}\right)^{j-i}\frac{(2j)!}{(2i)!\,(j-i)!}, \qquad (11)$$

for $i \leqslant j$ and zero otherwise.

The diagonalization of $\mathcal{M}$ is not too difficult because of its upper triangular form. The spectrum is given by the diagonal elements:

$$\lambda_{(r)} = 2(c/4)^r, \qquad (12)$$

in agreement with Ref. [5]. We need to construct $\mathcal{R}$, a matrix of right eigenvectors, such that

$$\mathcal{M}_l^i \mathcal{R}_i^r = \lambda_{(r)} \mathcal{R}_l^r \qquad (13)$$

(with no summation over r). For convenience, the columns of $\mathcal{R}$ are ordered as the eigenvalues, $0<c<4$ being assumed. We will then introduce the linear coordinates $h_{n,l}$ defined by

$$a_{n,l} = \mathcal{R}_l^r h_{n,r}, \qquad (14)$$

and which transform as

$$h_{n+1,r} \simeq \lambda_{(r)} h_{n,r} \qquad (15)$$

in the linear approximation. The matrix $\mathcal{R}_i^r$ and its inverse are also upper triangular. This implies that $h_{n,l}$ is of order β^l, just as $a_{n,l}$ is. We will fix the normalization of the right eigenvectors in $\mathcal{R}$ in such way that all the diagonal elements are 1. This guarantees that $h_{n,l}=a_{n,l}+\mathcal{O}(\beta^{l+1})$.

Before entering into the technical details of the construction of $\mathcal{R}$, an important consequence of the upper triangular form of $\mathcal{M}$ should be noticed. The eigenvectors and eigenvalues of $\mathcal{M}$ are independent of a possible truncation. In other words, the fact that $\mathcal{R}$ is upper triangular means that the polynomial truncations of $\mathcal{R}$ to order $k^{2l_{max}}$mentioned in Sec. II are indeed a projection in the subspace spun by the first l_{max} eigenvectors of $\mathcal{M}$.

We now construct $\mathcal{R}$. We first notice that for $j>i$,

$$\mathcal{R}_i^j = \left(\frac{c}{4-c}\right)^{j-i}\mathcal{P}_i^j, \qquad (16)$$

with P_i^j c-independent For indices no larger than 7, the entries of P are

$$\begin{pmatrix} 1 & 6 & 45 & 420 & 4725 & 62370 & 945945 \\ 0 & 1 & 15 & 210 & 3150 & 51975 & 945945 \\ 0 & 0 & 1 & 28 & 630 & 13860 & 315315 \\ 0 & 0 & 0 & 1 & 45 & 1485 & 45045 \\ 0 & 0 & 0 & 0 & 1 & 66 & 3003 \\ 0 & 0 & 0 & 0 & 0 & 1 & 91 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix $\mathcal{P}$ has remarkable properties

$$\mathcal{P}_i^{i+1}\mathcal{P}_{i+1}^{i+2} = 2\mathcal{P}_i^{i+2} \qquad (17a)$$

$$\mathcal{P}_i^{i+1}\mathcal{P}_{i+1}^{i+3}+\mathcal{P}_i^{i+2}\mathcal{P}_{i+2}^{i+3}=\mathcal{P}_i^{i+1}\mathcal{P}_{i+1}^{i+2}\mathcal{P}_{i+2}^{i+3} \tag{17b}$$

and higher order ones that as we will see are related to the very simple form of the inverse matrix. We can condense these relations into the more compact recursion

$$\mathcal{P}_i^{i+m}\mathcal{P}_{i+m}^{i+m+q}=\binom{m+q}{q}\mathcal{P}_i^{i+m+q}. \tag{18}$$

This implies the closed form expression

$$\mathcal{P}_i^{j}=\left(-\frac{1}{2}\right)^{j-i}\frac{(2j)!}{(2i)!\,(j-i)!}. \tag{19}$$

Using Eqs. (16) and (19), it is easy to check that, as a consequence of the binomial formula, we have provided an exact solution of Eq. (13) with the required normalization (ones on the diagonal). Similarly, one can show that the inverse has the very simple form

$$(\mathcal{R}^{-1})_i^j=(-1)^{j-i}\mathcal{R}_i^j. \tag{20}$$

This equation does not hold for an arbitrary upper triangular matrix. It implies the identities (17) and many others.

IV. EXPRESSION OF THE LINEAR VARIABLES IN TERMS OF THE SCALING VARIABLES

In this section, we express the linear variables h_l in terms of the (nonlinear) scaling variables y_l for which the approximate multiplicative transformation of Eq. (15) becomes exact. If we use $ln(y_l)$ as our new coordinates, the RG flows become parallel straight lines. All the dynamics is then contained in the mapping that we now proceed to construct.

We first rewrite the RG transformation in the h_l coordinates. Starting with the basic Eq. (7), we replace a_0 by 1 and a_l by $\mathcal{R}_l^p h_p$. This yields

$$h_{n+1,l}=\frac{\lambda_{(l)}h_{n,l}+\Delta_l^{pq}h_{n,p}h_{n,q}}{1+2\Delta_0^{p0}h_{n,p}+\Delta_0^{pq}h_{n,p}h_{n,q}}, \tag{21}$$

with coefficients calculable from Eq. (9). For instance

$$\Delta_l^{pq}=(\mathcal{R}^{-1})_l^{l'}\Gamma_{l'}^{p'q'}\mathcal{P}_{p'}^{p}\mathcal{R}_{q'}^{q}.$$

In general, upper roman indices transform with $\mathcal{R}$ and the lower ones with $(\mathcal{R})^{-1}$. By construction, the linear transformation is diagonal.

We then introduce the expansion

$$h_l=y_l+\sum_{i_1,i_2,\ldots}s_{l,i_1i_2\ldots}y_1^{i_1}y_2^{i_2}\ldots, \tag{22}$$

where the sums over i's run from 0 to infinity in each variable with at least two nonzero indices. In the following, we use the notation $\mathbf{i}$ for $(i_1,i_2,\ldots)$. More generally, vectors will be represented by boldface characters. The unknown coefficients $s_{l,\mathbf{i}}$ in Eq. (22) are obtained by matching two expressions of $h_{n+1,l}$, one obtained from the RG transformation of the h_l given in Eq. (21), the other obtained by evolving the scaling variables according to the exact multiplicative transformation

$$y_{n+1,l}=\lambda_{(l)}y_{n,l}. \tag{23}$$

The matching conditions can be expressed as:

$$h_{n+1,l}[\mathbf{h}_n(\mathbf{y})]=h_{n,l}(\lambda_1 y_{n,1},\lambda_2 y_{n,2},\ldots). \tag{24}$$

and yield the conditions

$$s_{l,\mathbf{i}}=\frac{N_{l,\mathbf{i}}}{D_{l,\mathbf{i}}}. \tag{25}$$

with

$$N_{l,\mathbf{i}}=\sum_{\mathbf{j}+\mathbf{k}=\mathbf{i}}\left(-\Delta_l^{pq}s_{p,\mathbf{j}}s_{q,\mathbf{k}}+s_{l,\mathbf{j}}\prod_m\lambda_{(m)}^{j_m}2\Delta_0^{p0}s_{p,\mathbf{k}}\right)+\sum_{\mathbf{j}+\mathbf{k}+\mathbf{r}=\mathbf{i}}s_{l,\mathbf{j}}\prod_m\lambda_{(m)}^{j_m}\Delta_0^{pq},s_{p,\mathbf{k}}s_{q,\mathbf{r}}. \tag{26}$$

and

$$D_{l,\mathbf{i}}=\lambda_{(l)}-\prod_m\lambda_{(m)}^{i_m}. \tag{27}$$

For a given set of indices $\mathbf{i}$, we introduce the notation

$$\mathcal{I}_q(\mathbf{i})=\sum_m i_m m^q. \tag{28}$$

One sees that I_0 is the degree of the associated product of scaling variables and $\mathcal{I}_1$ its order in the HT expansion (since y_l is also of order β^l). Given that all the indices are positive and that at least one index is not zero, one can see that if $\mathbf{j}+\mathbf{k}=\mathbf{i}$ then $\mathcal{I}_q(\mathbf{j})<\mathcal{I}_q(\mathbf{i})$ and $\mathcal{I}_q(\mathbf{k})<\mathcal{I}_q(\mathbf{i})$. Consequently, Eq. (26) yields a solution order by order in $\mathcal{I}_0$ or in $\mathcal{I}_1$ (since the right-hand side is always contains $s_{l,\mathbf{i}}$ of lower order in $\mathcal{I}_0$ or $\mathcal{I}_1$) provided that none of the denominators $D_{l,\mathbf{i}}$ are exactly zero. The main goal of this paper is to investigate what happens when some of the denominators happen to be exactly zero.

Using the explicit expression of the eigenvalues Eq. (12), we can rewrite the denominators as

$$D_{l,\mathbf{i}}=2\left(\frac{c}{4}\right)^l-2^{\mathcal{I}_0(\mathbf{i})}\left(\frac{c}{4}\right)^{\mathcal{I}_1(\mathbf{i})}. \tag{29}$$

Using the parametrization $c=2^{1-2/D}$, the zero denominators appear when

$$D-l(D+2)=D\mathcal{I}_0(\mathbf{i})-(D+2)\mathcal{I}_1(\mathbf{i}). \tag{30}$$

Given that $\mathcal{I}_q$ are integers, this can only occur at some rational values of D. Ignoring temporarily this set of values, we can say that for generic values of c, the denominator is not zero. Following the basic idea of dimensional regularization, we will then perform, order by order in $\mathcal{I}_1$, the construction of $s_{l,\mathbf{i}}$ for a generic value of c and discuss the limit where c takes some special value at the end of the calculation.

We now determine the range of values of $\mathcal{I}_0$ and $\mathcal{I}_1$ relevant for our problem. In Eq. (22), we have assumed that $h_l \simeq y_l$ for sufficiently small values of the scaling variables. The linear problem is completely solved and we may assume

TABLE I. Values of $Q_{l,\mathbf{i}}(c), c_{crit.}$ and $T_{l,\mathbf{i}}(c)$ defined in the text.

l	$\Pi_m y_m^{i_m}$	$Q_{l,\mathbf{i}}(c)$	$c_{crit.}$	$-T_{l,\mathbf{i}}(c)$
1	${y_1}^2$	$2+c/-2+c$	2	$-2+c$
1	${y_1}^3$	$-(4-20\ c+c^2)/2\ (-2+c)^2$	2	$-4+c^2$
1	$y_1\ y_2$	$-3\ (-40+c^2)/-8+c^2$	$2\ \sqrt{2}$	$-8+c^2$
1	${y_1}^4$	$-120+156\ c-18\ c^2+c^3/2\ (-2+c)^3$	2	$-8+c^3$
1	${y_1}^2\ y_2$	$3(-11520-640\ c+1184\ c^2+288\ c^3+40\ c^4-26\ c^5$ $+3\ c^6)/(-2+c)(-8+c^2)(-16+c^3)$	$2\ 2^{1/3}$	$-16+c^3$
1	${y_2}^2$	$1536/-32+c^3$	$2\ 2^{2/3}$	$-32+c^3$
1	$y_1\ y_3$	15	$2\ 2^{2/3}$	$-32+c^3$
2	${y_1}^3$	$6+c/2\ (-2+c)$	1	$-1+c$
2	$y_1\ y_2$	$14+c/-2+c$	2	$-2+c$
2	${y_1}^4$	$-(-44-28\ c+c^2)/4\ (-2+c)^2$	$\sqrt{2}$	$-2+c^2$
2	${y_1}^2\ y_2$	$-2\ (256+304\ c-112\ c^2-14\ c^3+c^4)/(-2+c)^2\ (-8+c^2)$	2	$-4+c^2$
2	${y_2}^2$	$-3\ (-104+c^2)/-8+c^2$	$2\ \sqrt{2}$	$-8+c^2$
2	$y_1\ y_3$	$240/-8+c^2$	$2\ \sqrt{2}$	$-8+c^2$
3	${y_1}^4$	$10+c/6\ (-2+c)$	$1/2$	$-(1/2)+c$
3	${y_1}^2\ y_2$	$18+c/-2+c$	1	$-1+c$
3	${y_2}^2$	$16/-2+c$	2	$-2+c$
3	$y_1\ y_3$	$22+c/-2+c$	2	$-2+c$

$\mathcal{I}_0(\mathbf{i})>1$. In addition, since both h_l and y_l are of order β^l, we need $\mathcal{I}_1(\mathbf{i})\geqslant l$. At lowest nontrivial order in β, we have $\mathcal{I}_l(\mathbf{i})=l$, and

$$D_{l,\mathbf{i}} = \left(\frac{c}{4}\right)^l (2-2^{\mathcal{I}_0(\mathbf{i})}).$$

In this special case, the only possible poles are at $c=0$. However, the factor $(c/4)^l$ at the denominator is exactly canceled by the same factor appearing in the Δ_l^{pq} in Eq. (21). More precisely

$$h_{n+1,l} = \left(\frac{c}{4}\right)^l \left(2h_{n,l} + \sum_{p+q=l} h_{n,p}h_{n,q}\right) + \mathcal{O}(\beta^{l+1}). \qquad (31)$$

Using this, it is not difficult to prove by induction that if $\mathcal{I}_l(\mathbf{i})=l$,

$$s_{l,\mathbf{i}} = \prod_m \frac{1}{i_m!}. \qquad (32)$$

It is thus clear that at the lowest nontrivial order, the coefficients have no singularities.

We now discuss the case $\mathcal{I}_1(\mathbf{i})>l$. We have in general

$$D_{l,\mathbf{i}} = 2\left(\frac{c}{4}\right)^l (c_{crit.})^{l-\mathcal{I}_1(\mathbf{i})} T_{l,\mathbf{i}}, \qquad (33)$$

with

$$T_{l,\mathbf{i}} = (c_{crit.}^{\mathcal{I}_1(\mathbf{i})-l} - c^{\mathcal{I}_1(\mathbf{i})-l}), \qquad (34)$$

and

$$c_{crit.} = 4 \times 2^{[1-\mathcal{I}_0(\mathbf{i})]/[\mathcal{I}_1(\mathbf{i})-l]}. \qquad (35)$$

One should always keep in mind that $c_{crit.}$ is a function of both l and $\mathbf{i}$. Inspection of Eqs. (9) and (16) shows that the numerator has a factor $c^{\mathcal{I}_1(\mathbf{i})}(c-4)^{l-\mathcal{I}_1(\mathbf{i})}$. Consequently

$$s_{l,\mathbf{i}} = \left(\frac{c}{c-4}\right)^{\mathcal{I}_1(\mathbf{i})-l} Q_{l,\mathbf{i}}(c), \qquad (36)$$

where $Q_{l,\mathbf{i}}(c)$ is a rational function of c with no poles or zeroes at 0 or 4. We do not have a compact formula for these rational functions, however it is easy to calculate them using symbolic manipulation programs.

Naively, we would expect that $Q_{l,\mathbf{i}}(c)$ has a factor $T_{l,\mathbf{i}}(c)$ at the denominator and other poles inherited from the $s_{l,\mathbf{i}}$ of lower orders. The values of $Q_{l,\mathbf{i}}(c)$ up to order β^4 are shown in Table I. The naive expectations concerning the poles are only observed in 9 cases out of the 17 considered. In the 8 other cases, some cancellations occur. For instance, there is no $(c+2)$ at the denominator of $Q_{1,(3,0,\ldots)}$. More importantly, whenever $c_{crit.}<2$, we observe a cancellation of all the factors appearing in $T_{l,\mathbf{i}}(c)$. This occurs, for instance, for $Q_{2,(3,0,\ldots)}$, where the factors $c-1$ cancel. If we do the calculations explicitly using Eq. (26), we obtain five terms at the numerator:

$$N_{2,(3,0,\ldots)} = \frac{-11c^3}{64} - \frac{3c^3}{8(-4+c)} + \frac{c^3}{4(-4+c)(-2+c)} + \frac{15c^4}{64(-4+c)} + \frac{c^4}{8(-4+c)(-2+c)},$$

while the denominator reads

$$D_{2,(3,0,\ldots)} = \frac{c^2}{8}(c-1)$$

After reduction and factorization, the numerator becomes

$$N_{2,(3,0,\ldots)} = \frac{(-1+c)\ c^3\ (6+c)}{16\ (-4+c)\ (-2+c)},$$

canceling the $c-1$ at the denominator. We have pursued the same procedure up to order β^7 and considered the 175 possible terms. In 50 cases, we had $c_{crit.}<2$. In each of these 50 cases, we observed a complete cancellation of $T_{l,\mathbf{i}}(c)$. It seems thus reasonable to conjecture that $Q_{l,\mathbf{i}}(c)$ has no poles for $|c|<2$. If this conjecture is correct, dimensional regularization provides a unique continuous expression for the coefficients for any c with $|c|<2$ and the model is "solvable" using the recursion for the coefficients given by Eq. (26). Note that for values of c real and positive, the correspondence $c=2^{1-2/D}$ implies that the interval $0<c<2$ corresponds to $0<D<+\infty$. The conjecture implies that for any value of c in this interval, we can construct analytical expression of $a_{n,l}$ (which contains all the thermodynamical quantities) in terms of $a_{0,l}$ (which depends on the initial energy density):

$$a_{n,l} = (\mathcal{R}^{-1})_l^r h_r[\lambda_1^n y_1(\mathbf{a}_0), \lambda_2^n y_2(\mathbf{a}_0), \ldots]. \qquad (37)$$

The initial values of $\mathbf{y}(\mathbf{a}_0)$ have a simple interpretation discussed in Sec. VIII.

V. THE CONNECTED PARTS

The generating function of the connected parts of the average values of the total field reads

$$\ln[\mathcal{R}_n(k)] = a_{n,1}^c k^2 + a_{n,2}^c k^4 + \cdots, \qquad (38)$$

with

$$a_{n,l}^c = \sum_{\mathbf{i}:\mathcal{I}_1(\mathbf{i})=l} (-1)^{\mathcal{I}_0(\mathbf{i})-1}[\mathcal{I}_0(\mathbf{i})-1]!\prod_m \frac{a_m^{i_m}}{i_m!}. \qquad (39)$$

We repeat that we are working exclusively in the HT phase and that we do not need to subtract powers of the magnetization. Using Eq. (14) and the construction discussed in the previous section, we can then calculate $a_{n,l}^c(\mathbf{y}_n)$. In addition, we have

$$a_{n,l}^c = (-\beta)^l \frac{1}{2l!}\left(\frac{c}{4}\right)^{ln}\left\langle\left(\sum_{2^n sites}\phi_x\right)^{2l}\right\rangle^c, \qquad (40)$$

with the connected part of the average values $\langle\rangle^c$ defined in the usual way. For instance

$$a_{n,2}^c = a_{n,2} - (1/2)a_{n,1}^2 = (-\beta)^2\frac{1}{4!}\left(\frac{c}{4}\right)^{2n}\left\langle\left(\sum\nolimits_{2^n sites}\phi_x\right)^4\right\rangle^c$$

with

$$\left\langle\left(\sum_{2^n sites}\phi_x\right)^4\right\rangle^c = \left\langle\left(\sum_{2^n sites}\phi_x\right)^4\right\rangle - 3\left\langle\left(\sum_{2^n sites}\phi_x\right)^2\right\rangle^2. \qquad (41)$$

We define the finite volume susceptibility and their analog for the higher order derivatives of the free energy (zero momentum renormalized couplings),

$$\chi_n^{(q)} \equiv \frac{\left\langle\left(\sum_{2^n sites}\phi_x\right)^q\right\rangle^c}{2^n}. \qquad (42)$$

We restrict our considerations to the set of initial values such that the infinite volume limit of $\chi_n^{(2l)}$ exists and is finite for every positive l. This means that we are not at another critical point or more generally not on a critical hypersurface at the boundary of the HT phase. We emphasize that the existence of the infinite volume limit requires $0<c<2$. For $c>2$, the energy of a constant field configuration scales faster than the number of sites and the model has no interest from a statistical mechanics point of view.

In the following, we assume that the initial values $a_{0,l}$ are such that

$$\lim_{n\to\infty}\chi_n^{(q)} = \chi^{(q)} \qquad (43)$$

is finite. From Eq. (40), it is then clear that for n large enough, we have the leading scaling

$$a_{n,l}^c \propto \left[2\left(\frac{c}{4}\right)^l\right]^n = \lambda_{(l)}^n. \qquad (44)$$

It is thus tempting to find a simple relationship between $a_{n,l}^c$ and $y_{n,l}$. Indeed, such relation can be found at lowest nontrivial order from Eq. (32) which implies that

$$a_l^c = y_l + \mathcal{O}(\beta^{l+1}). \qquad (45)$$

This can be seen either by using the Möbius inversion formula [24]

$$y_l = \sum_{\mathbf{i}:I_1(\mathbf{i})=l}(-1)^{\mathcal{I}_0(\mathbf{i})-1}\times[\mathcal{I}_0(\mathbf{i})-1]!\times\prod_m\left(\sum_{\mathbf{r}:\mathcal{I}_1(\mathbf{r})=m}\frac{\prod_j y_j^{r_j}}{r_j!}\right)^{i_m}\frac{1}{i_m!}, \qquad (46)$$

or more simply by noticing that

$$e^{\sum_{l=1}^{\infty} y_l k^{2l}} = \sum_{\mathbf{r}} k^{2\mathcal{I}_1(\mathbf{r})}\prod_j\frac{y_j^{r_j}}{r_j!}. \qquad (47)$$

Similar formulas are used in multiparticle scattering theory [25,26].

Equation (45) means that there are no nonlinear contributions of order β^l to a_l^c. For instance, there are no y_1^3 or y_1y_2 terms in a_3^c. This is expected because the nonlinear terms of order β^l scale faster than y_l, (assuming $0<c<2$). We we say that a term "scale faster," we mean that it goes to zero at a slower rate when n becomes large. In general, at each RG step, a term $\Pi_m\, y_m^{i_m}$ of order β^l is multiplied by

$$2^{\mathcal{I}_0(\mathbf{i})}\left(\frac{c}{4}\right)^l > \lambda_{(l)} = 2\left(\frac{c}{4}\right)^l .$$

The strict inequality comes from the fact that for the nonlinear terms $\mathcal{I}_0(\mathbf{i})>1$. It is thus clear that nonlinear terms of order β^l would spoil the HT scaling of Eq. (44) and contradict the existence of a infinite volume limit.

For higher order terms, the sign of the denominator $D_{l,\mathbf{i}}$ introduced in Eq. (27) tells us whether or not the term scales faster or slower than the linear term. With our sign convention, $c>c_{crit.}(l,\mathbf{i})$, means $D_{l,\mathbf{i}}<0$ and the term spoils the HT scaling Eq. (44). Since the coefficients are rational functions of c, they cannot vanish suddenly when c becomes larger than $c_{crit.}(l,\mathbf{i})$. Consequently if $0<c_{crit.}(l,\mathbf{i})<2$, the coefficient of the corresponding term is expected to vanish identically.

We have checked that this argument is consistent with our previous explicit calculations. We have used Eqs. (39) and (14) and the already calculated coefficients in Eq. (22) to calculate

$$a_l^c = y_l + \sum_{\mathbf{i}:\mathcal{I}_1(\mathbf{i})>l} t_{l,\mathbf{i}} y_1^{i_1} y_2^{i_2} \ldots , \tag{48}$$

up to order 7. For all the 50 terms with $0<c_{crit.}<2$, we found that the corresponding $t_{l,\mathbf{i}}$ are identically zero.

VI. THE HT PHASE

In the preceding section, we have argued (and checked explicitly up to order 7) that terms that scale faster than the linear term for $c_{crit.}<c<2$ have a zero coefficient. In this section, we discuss more carefully some aspects of the argument and explain that having such terms nonzero would result in serious inconsistency.

First of all, the existence of a HT phase is well established. The existence of a infinite volume limit [22] and the absence of spontaneous magnetization for sufficiently high temperature [8] can be shown rigorously for $0<c<2$ and a Ising measure. Bounds on the free energy density [23], can be established for $0<c<2$ and measures with a compact support. The argument should also apply to measures that can be well approximated by measures with a compact support {see Eq. (3) and the argument [27] that for Landau-Ginzburg measures, the restriction to $|\phi|<\phi_{max}$ leads to exponentially controllable errors}.

It is thus reasonable to assume that there exists some neighborhood of the HT fixed point where the infinite volume limit of the susceptibility and higher order derivatives [see Eq. (43)] exist. Terms scaling faster than the linear term seem to contradict the existence of these infinite volume limits. However, we should exclude the possibility that several terms (scaling identically) cancel each others. The existence of universal ratio of amplitudes means that we cannot in general pick arbitrary initial values for the scaling variables. However, such constraints apply for large values of the HT scaling variables. On the other hand, for arbitrarily small values of the HT scaling variables, one should be able to make independent variations of each variable while staying in the HT phase. This prevents the fine tuning required to obtain cancellations. The HT fixed point $R^\star=1$ corresponds to a local measure $W(\phi)\propto\delta(\phi)$ for which the correlations are zero. It is intuitively clear that by taking measures narrowly peaked at zero, one can avoid long range correlations. This continuity argument can probably be made rigorous by using Banach spaces as in Refs. [6,22]. We conclude that the coefficients $t_{l,\mathbf{i}}$ in Eq. (48) of the terms with $0<c_{crit.}(l,\mathbf{i})<2$ must vanish identically.

VII. THE ABSENCE OF POLES FOR $0<c<2$

We are now in position to show that the small denominator problem can be evaded for any c such that $0<c<2$ and that the solution of the RG flows problem suggested in Eq. (37) can be constructed safely order by order. In Sec. V, we have constructed the a_l^c in terms of the previously calculated a_l. However we could have proceeded directly, writing $a_{n+1,l}^c$ in terms of the $a_{n,l}^c$:

$$a_{n+1,l}^c = \mathcal{M}_l^k a_{n,k}^c + \sum_{k+q\geqslant l} v_l^{kq} a_{n,k}^c a_{n,q}^c + \cdots . \tag{49}$$

The coefficients v_l^{kq} and the higher order ones can be obtained by using the expansion of Eq. (38) in the logarithm of Eq. (6) and expanding order by order in $\mathbf{a}_n^c$. The series does not terminate. The linear transformation is the same as before because a_l^c and a_l only differ by nonlinear terms. Using

$$a_{n,l}^c = \mathcal{R}_l^r h_{n,r}^c , \tag{50}$$

we obtain

$$h_{n+1,l}^c = \lambda_{(l)} h_{n,l}^c + \sum_{k+q\geqslant l} w_l^{kq} h_{n,k}^c h_{n,q}^c + \cdots . \tag{51}$$

We then introduce the expansion

$$h_l^c = y_l + \sum_{\mathbf{i}:\mathcal{I}_1(\mathbf{i})>l} s_{l,\mathbf{i}}^c \prod_m y_m^{i_m} , \tag{52}$$

and obtain

$$s_{l,\mathbf{i}}^c = \frac{N_{l,\mathbf{i}}^c}{D_{l,\mathbf{i}}} . \tag{53}$$

with $N_{l,\mathbf{i}}^c$ given by a formula similar to Eq. (26), except that it does not terminate. A detailed analysis shows that the two formulas have in common that the numerator depends only on coefficients of strictly lower orders in β, and Eq. (53) can be used order by order to construct the $s_{l,\mathbf{i}}^c$ for generic values of c.

Since $\mathcal{R}^{-1}$ is upper triangular, we see from Eq. (50) that h_l^c is equal to a_l^c plus terms which go to zero faster. Consequently, for large n, the leading scaling is

$$h_{n,l}^c \propto \lambda_{(l)}^n . \tag{54}$$

Following reasonings used before, this implies that terms in the expansion Eq. (52) that scale faster than y_l for any $0<c<2$ should have a vanishing coefficient. In other words:

$$0 < c_{crit.}(l,\mathbf{i}) < 2 \Rightarrow s^c_{l,\mathbf{i}} = 0.$$

Given the specific form of the $s^c_{l,\mathbf{i}}$ given in Eq. (53), the h^c_l have no poles for $0<c<2$. The a^c_l being linear combinations of h^c_l and the a_l being linear combinations of products of a^c_l, we conclude that the expansion of the a_l in terms of the scaling variables have also no poles for $0<c<2$, in agreement with the conjecture stated in Sec. VI.

Again we see that there exists a unique continuous definition of the scaling variables that can be used at particular values of c where the denominator is exactly zero. From a practical point of view, the calculation at fixed c of the $s^c_{l,\mathbf{i}}$ is easier than the calculation of the $s_{l,\mathbf{i}}$, because no limit needs to be taken explicitly. The $s^c_{l,\mathbf{i}}$ being rational function of c they cannot be zero everywhere except at isolated values. Consequently, we can set to zero the $s^c_{l,\mathbf{i}}$ having $c_{crit.}(l,\mathbf{i})<2$ even at values of c where $D_{l,\mathbf{i}}=0$.

VIII. THE INITIAL VALUE PROBLEM

We now return to Eq. (37). In order to complete our solution of the problem, namely, expressing $\mathbf{a}_n$ in terms of their initial values $\mathbf{a}_0$, we need to calculate $\mathbf{y}(\mathbf{a}_0)$.

Before doing this, we want to show that the initial values $\mathbf{y}_0$ have a very simple interpretation. We have learned in the preceding sections that $y_{n,l}$ is the *only* leading term of $a^c_{n,l}$ when n becomes large. If at a given $0<c<2$, a nonlinear terms scales exactly like $y_{n,l}$, then by increasing c slightly (but keeping $c<2$), we can make this term dominant in contradiction with the existence of the infinite volume limit. Consequently,

$$\lim_{n\to\infty}\lambda_l^{-n}a^c_{n,l} = \lim_{n\to\infty}\lambda_l^{-n}y_{n,l} = y_{0,l}. \tag{55}$$

From Eq. (40), we see that

$$y_{0,l} = (-\beta)^l \frac{1}{2l!}\chi^{(2l)}. \tag{56}$$

This means that the infinite volume limit of the susceptibility and of the higher derivatives of the free energy density completely determine the RG flows in the HT phase. This also means that the calculation of $y_{0,l}$ given $a_{0,l}$ is nontrivial far away from the HT fixed point. However, we can take advantage of the fact that

$$\lambda_{(l)}^{-n}y_{n,l} = y_{0,l} \tag{57}$$

to estimate $y_{0,l}$ using expansions valid at intermediate values of n.

We now discuss the inversion question. We need to determine the coefficients $r_{l,\mathbf{i}}$ of the expansion

$$y_l = h_l + \sum_{\mathbf{i}} r_{l,\mathbf{i}}\prod_m h_m^{i_m}. \tag{58}$$

This can be done by replacing the y_l appearing in the expansion of h_l in Eq. (22) by Eq. (58). This yields an equation of the form

$$r_{l,\mathbf{i}} + s_{l,\mathbf{i}} + X_{l,\mathbf{i}} = 0,$$

with $X_{l,\mathbf{i}}$ linear in $\mathbf{s}$ and multilinear in $\mathbf{r}$ of strictly lower order. One can then construct $r_{l,\mathbf{i}}$ order by order without ever creating a pole in the range $0<c<2$. At lowest nontrivial order, we have

$$h_l = \sum_{\mathbf{i}:\mathcal{I}_1(\mathbf{i})=l} (-1)^{\mathcal{I}_0(\mathbf{i})-1}[\mathcal{I}_0(\mathbf{i})-1]!\prod_m \frac{y_m^{i_m}}{i_m!} + \mathcal{O}(\beta^{l+1}). \tag{59}$$

A more detailed analysis shows that for higher orders

$$r_{l,\mathbf{i}} = \left(\frac{c}{c-4}\right)^{\mathcal{I}_1(\mathbf{i})-l} Y_{l,\mathbf{i}}(c), \tag{60}$$

with $Y_{l,\mathbf{i}}(c)$ having poles only for $2\leqslant c<4$. The values of $Y_{l,\mathbf{i}}(c)$ up to order 4 are given in Table II.

IX. THE HT EXPANSION

A simple application of the method presented here is the calculation of the high-temperature expansion at finite volume. As a simple example, we consider the first order correction to the susceptibility for a Ising measure $[R_0(k) = \cos(\sqrt{\beta}k)]$. Using the results found in the preceding sections, we obtain

$$\begin{aligned}\chi_n^{(2)} &= \frac{-2}{\beta}a_{n,1}\left(\frac{c}{4}\right)^n = \frac{-2}{\beta}\left[y_{0,1} + \frac{2c}{2-c}\left(\frac{c}{2}\right)^n (y_{0,1})^2 \right.\\ &\left. + \frac{6c}{4-c}\left(\frac{c}{4}\right)^n y_{0,2}\right] + \mathcal{O}(\beta^2)\\ &\equiv 1 + \beta b_{1,n} + \mathcal{O}(\beta^2).\end{aligned} \tag{61}$$

Using $a_{0,1}=-\beta/2$ and $a_{0,2}=\beta^2/24$, we obtain

$$y_{0,1} = -\frac{\beta}{2} - \frac{\beta^2 c}{4(4-c)} - \frac{\beta^2 c(2+c)}{4(4-c)(2-c)}$$

$$y_{0,2} = -\frac{\beta^2}{12},$$

and consequently

$$b_{1,n} = \frac{2c}{(4-c)(2-c)} - \frac{c}{2-c}\left(\frac{c}{2}\right)^n + \frac{c}{4-c}\left(\frac{c}{4}\right)^n. \tag{62}$$

This is in agreement with results obtained [19] using graphical methods.

X. RG INVARIANTS

In Hamiltonian mechanics, integrable systems with q degrees of freedom have q constants of motions and q periodic variables with independent periods depending on the constants of motion. In the present case, time is discrete and exponential decays replace the quasiperiodic behavior. For a truncation of dimension l_{max}, it is nevertheless possible to construct $l_{max}-1$ constants of motion:

TABLE II. Values of $Y_{l,\mathrm{i}}(c)$, and $T_{l,\mathbf{i}}(c)$ defined in the text.

l	$\Pi_m\, h_m^{i_m}$	$Y_{l,\mathbf{i}}(c)$	$-T_{l,\mathbf{i}}(c)$
1	${h_1}^2$	$-\left(\frac{2+c}{-2+c}\right)$	$-2+c$
1	${h_1}^3$	$\frac{160-224c+44c^2+4c^3+c^4}{(-2+c)^2(-8+c^2)}$	$-4+c^2$
1	h_1h_2	$\frac{3(-40+c^2)}{-8+c^2}$	$-8+c^2$
1	${h_1}^4$	$-\left(\frac{-229376+638976c-364544c^2-160768c^3+109056c^4+11648c^5+1664c^6-5952c^7+392c^8+148c^9+6c^{10}+c^{11}}{(-2+c)^3(-8+c^2)(-32+c^3)(-16+c^3)}\right)$	$-8+c^3$
1	${h_1}^2h_2$	$\frac{-6(24576-53248c-19456c^2+10496c^3+3968c^4+800c^5-576c^6-40c^7+2c^8+c^9)}{(-2+c)(-8+c^2)(-32+c^3)(-16+c^3)}$	$-16+c^3$
1	${h_2}^2$	$\frac{-1536}{-32+c^3}$	$-32+c^3$
1	h_1h_3	-15	$-32+c^3$
2	${h_1}^3$	$\frac{6+c}{-2+c}$	$-1+c$
2	h_1h_2	$-\left(\frac{14+c}{-2+c}\right)$	$-2+c$
2	${h_1}^4$	$\frac{-(608-1504c+356c^2+36c^3+3c^4)}{2(-2+c)^2(-8+c^2)}$	$-2+c^2$
2	${h_1}^2h_2$	$\frac{-2(-704+1216c-316c^2-20c^3+c^4)}{(-2+c)^2(-8+c^2)}$	$-4+c^2$
2	${h_2}^2$	$\frac{3(-104+c^2)}{-8+c^2}$	$-8+c^2$
2	h_1h_3	$\frac{-240}{-8+c^2}$	$-8+c^2$
3	${h_1}^4$	$-\left(10+c/-2+c\right)$	$-\left(\frac{1}{2}\right)+c$
3	${h_1}^2h_2$	$\frac{2(18+c)}{-2+c}$	$-1+c$
3	${h_2}^2$	$\frac{-16}{-2+c}$	$-2+c$
3	h_1h_3	$-\left(\frac{22+c}{-2+c}\right)$	$-2+c$

$$G_l \equiv -(2l)!\,\frac{y_{n,l}}{(-2y_{n,1})^{(l-1)(D/2)+l}}. \tag{63}$$

These quantities are n independent and we call them RG invariants. We can calculate them at $n=0$. Using Eq. (56), we obtain

$$G_l = (-1)^{l+1}\frac{\beta^{(D/2)(1-l)}\chi^{(2l)}}{(\chi^{(2)})^{(l-1)(D/2)+l}}. \tag{64}$$

We can also calculate them at large enough values of n where the HT expansion works well. The minus sign has been introduced in order to have $G_l>0$ for $D=3$.

We now concentrate on the unstable direction of Wilson's fixed point. We set the relevant scaling variable associated with this fixed point to a value u which becomes our coordinate along the unstable direction and we set all the irrelevant ones to zero. We call $G_l(u)$ the corresponding value of the ratio. Given that u is a scaling variable and that G_l is RG invariant, we have

$$G_l(\lambda_w u) = G_l(u), \tag{65}$$

with λ_w the eigenvalue corresponding to the unstable direction of Wilson's fixed point. Consequently, we have the Fourier expansion:

$$G_l(u) = \sum_r A_{l,r}u^{ir\omega}, \tag{66}$$

with

$$\omega = \frac{2\pi}{\ln \lambda_w}. \tag{67}$$

If the oscillatory terms are very small, as noticed in Refs. [16,28,29], we have the approximate universal ratios

$$G_l(u) \simeq A_{l,0}. \tag{68}$$

These constants can be calculated in an intermediate region where the expansions in both scaling variables are valid [7].

XI. CONCLUSIONS

We have shown that the scaling variables corresponding to the HT fixed point of Dyson's hierarchical model can be constructed order by order without small denominator problems. The ambiguity noticed before [17] for calculations at fixed values of c can be raised by requiring the continuity in c. Practical calculations at finite c are most easily done by following the explicit construction sketched in Sec. VII for the connected part where no complicated limit is required. The remaining poles for $2\leqslant c\leqslant 4$ reflect the degeneracy of the linear spectrum at $c=4$ or the fact that some finite size corrections become leading effects for some value of $2\leqslant c<4$ (where the infinite volume limit does not exist).

We have solved the linear problem in compact form but at this point no compact form is available for the nonlinear problem. Even though we have "constants of motion" (the RG invariants), we do not have simple expressions for them (as the integrals of integrable models). The question of the convergence of the series remains to be addressed and should result in the construction of the boundary of the HT phase. Finally, it would be desirable to extend the method to models with $\eta \neq 0$. One way to achieve this goal would be to develop methods to systematically improve the hierarchical approximation.

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