Explanation of power law behavior of autoregressive conditional duration processes based on the random multiplicative process

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Autoregressive conditional duration (ACD) processes, which have the potential to be applied to power law distributions of complex systems found in natural science, life science, and social science, are analyzed both numerically and theoretically. An ACD(1) process exhibits the singular second order moment, which suggests that its probability density function (PDF) has a power law tail. It is verified that the PDF of the ACD(1) has a power law tail with an arbitrary exponent depending on a model parameter. On the basis of theory of the random multiplicative process a relation between the model parameter and the power law exponent is theoretically derived. It is confirmed that the relation is valid from numerical simulations. An application of the ACD(1) to intervals between two successive transactions in a foreign currency market is shown.

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Power law distributions are observed in the field of natural science, life science, and social science [1,2]. The power law is utilized to characterize "complex system," which has been explained by self-organized criticality (SOC) [3]. In fact, the SOC has succeeded in explaining power law behavior observed in various fields. Naturally the SOC cannot explain all the power law behavior, such as financial fluctuations. Recently, financial fluctuations have been attracting the attention of many physicists. The movement is called "econophysics" [4] and advances collaborations between economists and physicists. The successive studies make remarkable progress in understanding the mechanism of power law behavior in financial fluctuations [5–7]. Specifically some researchers have been interested in time intervals between two successive transactions. Takayasu *et al.* have examined time interval distribution of the yen/dollar exchange rate. They have indicated that a probability density function (PDF) of the time interval exhibits a power law distribution [8]. Engle and Russell have introduced autoregressive conditional duration (ACD) processes in order to discuss the time interval related to foreign currencies [9]. Actually these processes are useful in analyzing financial data that do not arrive in equal time intervals. However, theoretical investigation has not yet been sufficient.

The purpose of this Brief Report is to analyze the ACD processes by using mathematical techniques developed in the random multiplicative process (RMP). The fundamental idea of the RMP was pointed out by Champernowne [10], and the mathematical formalization is given by Kesten [11]. The RMP is a stochastic process with both multiplicative and additive noises. The effect of the multiplicative noise represents both positive and negative feedback originating from nonlinearity of a system. In the context of statistical mechanics the RMP has been investigated [12–15]. It has been widely applied as a model to understand aspects of the singular behavior in nonlinear dynamics, such as the on-off intermittency [16–20] and conformation of polymers in random velocity fields [21].

The ACD processes are formalized as follows. Let us consider a Poisson point process. Let t_s denote the time interval between the sth event and the $(s+1)$th one. Then the PDF of t_s, $p_s(t)$ follows the exponential distribution:

$$p_s(t) = \frac{1}{\langle T_s \rangle} \exp(-t/\langle T_s \rangle). \qquad (1)$$

Here it is assumed that $\langle T_s \rangle$ is given by a conditional average under past realizations $t_{s-s'}(s'=1,\ldots,K)$, expressed as

$$\langle T_s \rangle = \alpha_0 + \sum_{s'=1}^{K} \alpha_{s'} t_{s-s'}, \qquad (2)$$

where $\alpha_{s'}(s'=0,\ldots,K)$ are positive coefficients. Equation (2) is called ACD(K). It is obvious that Eq. (2) is rewritten as

$$t_s = \left(\alpha_0 + \sum_{s'=1}^{K} \alpha_{s'} t_{s-s'} \right) \epsilon_s, \qquad (3)$$

where ϵ_s is a stochastic variable following an identical and independent exponential distribution with a unit mean. Namely, the PDF of ϵ is given by

$$p_\epsilon(\epsilon) = \exp(-\epsilon). \qquad (4)$$

Of course, calculating the conditional average of Eq. (3) under the realizations $t_{s-s'}(s'=1,\ldots,K)$ one gets Eq. (2). Furthermore, when all the $\alpha_{s'}=1/K$ the ACD(K) is rewritten as

$$t_s = (\alpha_0 + \langle t \rangle_K)\epsilon_s, \qquad (5)$$

where $\langle t \rangle_K$ is a moving average over K steps, which is defined as

$$\langle t \rangle_K \equiv \frac{1}{K} \sum_{s'=1}^{K} t_{s-s'}. \qquad (6)$$

This is the self-modulation process (SMP) which is introduced by M. Takayasu and H. Takayasu [22], namely, the ACD processes include the SMP as the special case.

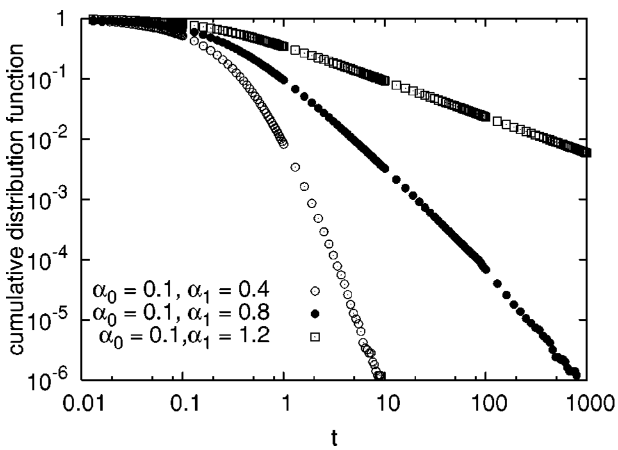

FIG. 1. Typical examples of cumulative distribution functions in a log-log scale of the ACD(1) fixing $\alpha_0=0.1$. Unfilled circles represent a cumulative distribution function at $\alpha_1=0.4$, filled circles at $\alpha_1=0.8$ and unfilled squares at $\alpha_1=1.2$. The calculation step to obtain these CDFs is 5×10^6.

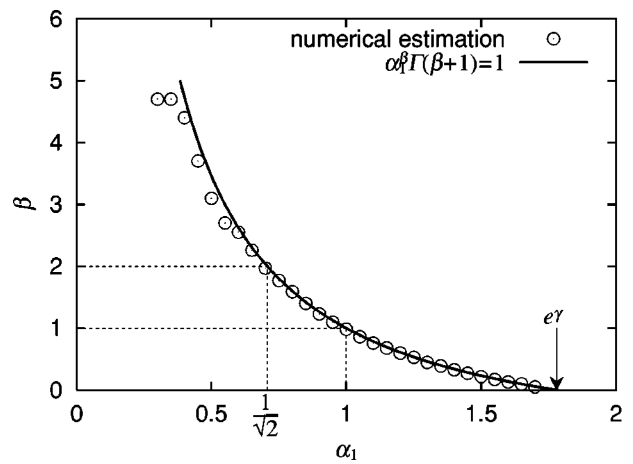

FIG. 2. The relation between the model parameter α_1 and the power law exponent β at $\alpha_0=0.1$. Unfilled circles represent the power law exponents obtained from the slopes of the cumulative distribution function in the log-log scale and a solid curve the theoretical relation $\alpha_1^\beta\Gamma(\beta+1)=1$. Note that $\beta=2$ when $\alpha_1=1/\sqrt{2}$, and that $\beta=1$ when $\alpha_1=1$. For $\alpha_1>e^\gamma$, where γ is Euler's constant, no stationary distribution exists.

For simplicity ACD(1) is considered, namely, $K=1$ on Eq. (2),

$$t_s = (\alpha_0 + \alpha_1 t_{s-1})\epsilon_s. \tag{7}$$

If a stationary average satisfying Eq. (7) is assumed then one has

$$\langle T\rangle = \frac{\alpha_0}{1-\alpha_1}. \tag{8}$$

Because it is clear that $\langle T\rangle>0$ from the definition, Eq. (8) shows a singularity for $\alpha_1\geqslant 1$. Similarly, calculating a stationary second order moment of Eq. (7), one has

$$\langle T^2\rangle = \frac{2\alpha_0^2(1+\alpha_1)}{(1-2\alpha_1^2)(1-\alpha_1)}. \tag{9}$$

From the definition of $\langle T^2\rangle$ it is obvious that Eq. (9) suggests a singularity for $\alpha_1>1/\sqrt{2}$, namely, it is expected that a power law distribution appears. In order to verify that the PDF follows the power law the cumulative distribution function (CDF) corresponding to the PDF of t, $p_t(t)$, which is defined as

$$P_t(\geqslant t) = \int_t^\infty p_t(t')dt', \tag{10}$$

is numerically calculated.

Figure 1 shows the CDF of t_s for various α_1 at fixed α_0. If $p_t(t)$ has a power law tail then $P_t(\geqslant t)$ is written as

$$P_t(\geqslant t) \propto t^{-\beta}, \tag{11}$$

where β is a power law exponent, $\beta>0$. If $0<\beta<2$ then the PDF is in the stable regime. When the power law exponent is lower than 2 the second order moment diverges. Moreover, when the exponent is lower than 1 the first order moment diverges.

From Fig. 1 it is found that each CDF has a straight-line part in the log-log scale, namely, the CDF follows the power law distribution. It is also found that β is a function of α_1 because each slope of the CDF depends on α_1. Furthermore, Fig. 1 shows that the value of a cut in the CDF is nearly equal to α_0.

Here a relation between α_1 and β is theoretically derived. From Eq. (7) one can immediately write the expressions of $b_s=\alpha_1\epsilon_s$ and $f_s=\alpha_0\epsilon_s$ on the basis of the RMP,

$$t_s = b_s t_{s-1} + f_s, \tag{12}$$

where b_s and f_s are a multiplicative noise and an additive one, respectively. In fact it is obvious that the multiplicative noise and the additive noise have cross correlation. However, here it is assumed that it is 0 for instance. From the equation of the power law exponent β [12] one has

$$\langle(\alpha_1\epsilon)^\beta\rangle = 1. \tag{13}$$

Substituting Eq. (4) into Eq. (13) one gets the relation between β and α_1,

$$\alpha_1^\beta\Gamma(\beta+1) = 1, \tag{14}$$

where $\Gamma(\cdot)$ represents the gamma function. In order to confirm the relation between α_1 and β, one estimates β from the slope of the CDF in log-log scale for various α_1. As shown in Fig. 2 the theoretical equation shows good agreement with estimation of α_1 versus β. From Fig. 2 it is found that for $\alpha_1>1/\sqrt{2}$ $P_t(\geqslant t)$ follows a power law with the exponent less than 2. For $\alpha_1>1$ it follows a power law with the exponent less than 1.

Furthermore, one considers the necessary and sufficient condition for t to have a stationary PDF [12]. Then the inequality is required,

$$\langle \log \alpha_1 \epsilon \rangle = \log \alpha_1 - \gamma < 0, \tag{15}$$

namely, a stationary PDF exists when

$$\alpha_1 < e^{\gamma} = 1.78107\ldots, \tag{16}$$

where γ is Euler constant [23]. From Fig. 2 it is clarified that the CDF of t no longer has a stationary solution for $\alpha_1 > e^{\gamma}$.

In time series analysis of financial fluctuations it has been reported that the PDF both of time series and of their multiplicative inverse has a power law tail [7,24]. That can be also explained by the ACD(1). By inserting $\tau_s = 1/t_s$ into Eq. (7) one obtains

$$\tau_s = \frac{\tau_{s-1}}{\alpha_0 \tau_{s-1} + \alpha_1} \xi_s, \tag{17}$$

where $\xi_s = 1/\epsilon_s$. By transforming Eq. (4) expressed in term of ϵ into of ξ one has the PDF of ξ,

$$p_{\xi}(\xi) = \frac{1}{\xi^2} \exp(-1/\xi). \tag{18}$$

For large τ_s Eq. (17) can be approximated by $\tau_s \approx (1/\alpha_0)\xi_s$. Therefore by using $p_{\xi}(\xi)$ the PDF of τ for large τ is described as $p_{\tau}(\tau) \approx \alpha_0 p_{\xi}(\alpha_0 \tau)$. Here the CDF of τ for large τ is given by

$$P_{\tau}(\geqslant \tau) \approx P_{\xi}(\geqslant \alpha_0 \tau), \tag{19}$$

where $P_{\xi}(\geqslant \xi)$ represents the CDF of ξ, which can be calculated by

$$P_{\xi}(\geqslant \xi) = 1 - \exp\left(-\frac{1}{\xi}\right). \tag{20}$$

Substitution of Eq. (20) into Eq. (19) yields

$$P_{\tau}(\geqslant \tau) \approx 1 - \exp\left(-\frac{1}{\alpha_0 \tau}\right) \propto \tau^{-1}, \tag{21}$$

namely, the CDF of τ has the power law tail of which the exponent is 1.

Here an application to financial time series is shown. Takayasu *et al.* empirically investigated time intervals between two successive transactions of the yen/dollar exchange rate [8]. They reported that the CDF of the time intervals has a power law tail with the exponent 1.8. From Eq. (14) α_1 is given by

$$\alpha_1 = \Gamma(\beta + 1)^{-1/\beta}. \tag{22}$$

Equation (22) yields that the parameter α_1 can be estimated from the power law exponent β. From the power law exponent $\beta = 1.8$ one gets $\alpha_1 = 0.75$. From the cut in value of the CDF one gets $\alpha_0 = 2.0$. Figure 3 shows the CDF calculated from the ACD(1) at $\alpha_0 = 2.0$ and $\alpha_1 = 0.75$. The CDF calculated from the ACD(1) agrees with that estimated from the real data. Therefore it is found that Eq. (22) is useful in obtaining the model parameter α_1 from the power law exponent β.

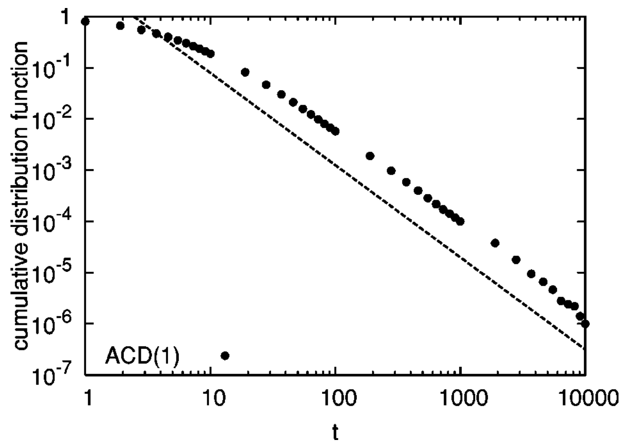

FIG. 3. Log-log plots of the cumulative distribution function obtained from the ACD(1). Filled circles represent the cumulative distribution function calculated from the ACD(1) at $\alpha_0 = 2.0$ and $\alpha_1 = 0.75$. The dashed line shows a power law with the exponent 1.8.

The autoregressive conditional duration processes were analyzed both numerically and theoretically. It is confirmed that the CDF of the dynamical variable of the ACD(1) has a power law tail from numerical simulations. On the basis of the theory of the random multiplicative process, the relation between the model parameter α_1 and the power law exponent, β, is theoretically derived. It is verified that the theoretical relation between α_1 and β fits the estimation obtained from the CDF of the ACD(1). The necessary and sufficient condition for a PDF to have a stationary distribution is given by $\alpha_1 < e^{\gamma}$. It was analytically verified that a PDF of the multiplicative inverse of the dynamical variable of the ACD(1) has the power law tail of which the exponent is 1. It was confirmed that the CDF of time intervals between two successive transactions of the yen/dollar exchange rate has a power law. These model parameters of the ACD(1) were estimated from the CDF calculated from the time intervals of the real data. It was shown that the CDF obtained from the ACD(1) is consistent with that estimated from the real data.

It is coincidence that the theory of the RMP with the identical and independent noises is applicable to the ACD(1) because it is obvious that both the multiplicative and additive noises have the same noise source. I think that the reason is a challenging open problem. The method based on the RMP is only available to the ACD(1). Hence for $K \geqslant 2$ it is expected that development of an alternative method to derivate the same relation as $K = 1$.

The ACD processes can be applied to a power law distribution in various fields since they are very simple. Specifically, I believe that they should be useful in explaining the power law distribution in complex systems found in natural science, life science, and social science. The ACD allows us to make a theoretical model of time series with an arbitrary exponent and to compare statistical properties of their dynamics.

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