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Global spatiotemporal order and induced stochastic resonance due to a locally applied signal

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We study the phenomenon of spatiotemporal stochastic resonance (STSR) in a chain of diffusively coupled bistable oscillators. In particular, we examine the situation in which the *global* STSR response is controlled by a *locally applied signal* and reveal a wave-front propagation. In order to deepen the understanding of the system dynamics, we introduce, on the time scale of STSR, the study of the effective statistical renormalization of a generic lattice system. Using this technique we provide a criterion for STSR, and predict and observe numerically a bifurcationlike behavior that reflects the difference between the most probable value of the local quasiequilibrium density and its mean value. Our results, tested with a chain of nonlinear oscillators, appear to possess some universal qualities and may stimulate a deeper search for more generic phenomena.

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Since the appearance of [1] the phenomenon of stochastic resonance (SR) has become a popular field of research. A great deal of experimental as well as theoretical and mathematical work has been devoted to the study of the phenomenon in different systems (for reviews, see Refs. [2–5]). There has been a particular emphasis of its relevance and importance in biology and medicine [6] where noise in general, and SR in particular, play a surprisingly constructive role [7]. The ability of SR to generate order from disorder [2–7] is especially of relevance in the context of pattern formation mechanisms that are enhanced by noise [8]. The discovery of an enhancement of the effect by the coupling of nonlinear oscillators into an array [9,10] has brought new insight to studies of SR. This effect in a wide sense is known as spatiotemporal SR (STSR) [4]. An explanation of this effect has previously been described as "collective spatiotemporal motion" and "optimal spatiotemporal synchronization." In spite of much progress, the precise kinetic details of such a synchronization remain without an appropriate study.

We conjecture that among the kinetic details of STSR, wave front propagation plays a prominent role (Refs. [11,12] are good background references in this regard). We investigate the related problem of inducing and controlling global spatiotemporal order in a chain of diffusively coupled bistable oscillators by a *locally applied signal*. The ability to induce STSR throughout the chain by applying a local signal to a small part of the chain is precisely shown. This can be regarded as an element of a spatial signal transmission, and possibly gives a new design freedom to modeling biological and biomedical problems. We also investigate the effective statistical renormalization of the steady states of a generic lattice system [24]. This renormalization reflects the difference between the most probable and mean values of the local quasiequilibrium density which is a result of time averaging [25]. This leads to a new observation that the system, on the time scale of STSR, exhibits a bifurcationlike behavior. It also gives a criterion for the noise intensity depending on the coupling in the chain. Both our results appear to have a certain universal quality and may stimulate a deeper search for generic phenomena, e.g., in a chain of FitzHugh-Nagumo equations [15].

Consider a chain of overdamped oscillators with diffusive coupling of constant $K>0$ and a bistable on-site potential $V(y)=-my^2/2+y^4/4$. We assume that the system is influenced both by external random noise of intensity D, which involves a set of independent generalized Gaussian random processes $\{\xi_n(t)\}$ with two characteristic cumulants, $\langle\xi_n(t)\rangle=0$ and $\langle\xi_m(t)\xi_n(t')\rangle=\delta_{mn}\delta(t-t')$, and a deterministic signal $S(t)$ applied locally to a part of the chain, $S_n^{(M_k)}(t)=[s(t)$ if $n\in M_k$, and 0 otherwise]. In what follows we fix the particular and simplest form of the external signal, $s(t)=A\cos(\omega t)$, together with dimensionless amplitude $A=0.025$, and frequency $\omega=5\pi\times10^{-5}$ that actually set one of the time scales, $T_s=2\pi\omega^{-1}=4\times10^4$. The other characteristic time scales are: the relaxation time of the chain to Gaussian fluctuations in the vicinity of one of its stable steady states, T_r [in our case $T_r\sim(2m)^{-1}\sim1$]; the waiting time of the initial birth of an "instanton," T_K (Kramers' time) [9] — here we do not explicitly consider this time scale but, fortunately, there are in-depth studies of the problem [11,16,17] (*a posteriori* one can say that T_K is considerably shorter than the principal time scale here) and also the time scale related to any wave-front propagation in the chain, T_w.

The corresponding chain stochastic differential equation (SDE) in dimensionless variables has the form,

$$\dot{y}_n = K\Delta y_n - V'(y_n) + \sqrt{2D}\xi_n(t) + S_n^{(M_k)}(t), \qquad (1)$$

with n on the chain $\mathbb{N}$ in the integer lattice $\mathbb{Z}$, $n\in\mathbb{N}=\{1,\ldots,k+1,\ldots,k+\mathrm{M},\ldots,\mathrm{N}\}$, $M_k=\{k+1,k+2,\ldots,k+\mathrm{M}\}$, $0\leqslant\mathrm{M}\leqslant\mathrm{N}$; and $\Delta y_i\equiv y_{i+1}-2y_i+y_{i-1}$. Two different topologies are possible for the chain: either the ends are connected or not connected. We take the latter case with Neumann boundary condition.

The corresponding physical context of Eq. (1) is in the Smoluchowski kinetics of a harmonically coupled chain of particles with transverse displacements $\{y_n\}$. The underlying free energy functional and the corresponding deterministic part of the dynamics, without signal, are: $\mathcal{F}(y)=\Sigma_{(n)}\frac{1}{2}K(y_n-y_{n-1})^2+V(y_n)$, $\dot{y}_n=-\partial\mathcal{F}/\partial y_n$. This interpretation can con-

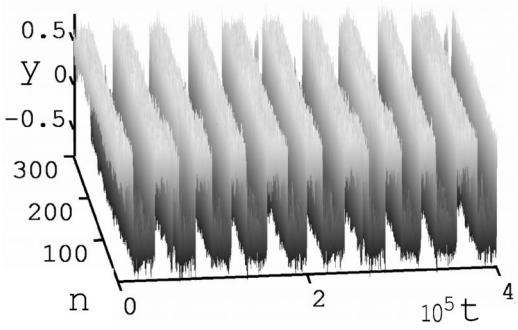

FIG. 1. Locally induced global spatiotemporal pattern in the chain of bistable oscillators; parameters are specified in Fig. 2.

siderably facilitate the understanding of the dynamic behavior of Eq. (1). Especially since even diffusively coupled, chain oscillators appear surprisingly difficult to analyze consistently from a rigorous mathematical perspective [18].

We study a spatially discrete model because chains or array structures are often of relevance in biology. Besides biology there are also crystal lattices [17]. Moreover interesting dynamical effects exist in discrete models that are not present in their continuous analogs, e.g., the propagation failure of traveling waves [18] and breathers [19]. Even in population dynamics it was recently demonstrated that important effects due to the discrete nature of organisms may be entirely missed by continuous models [20].

To introduce induced STSR phenomena, we first present a representative numerical simulation of Eq. (1) that is performed longer than all the characteristic time scales. We use time steps 0.01 and 0.001 that are considerably shorter than the principal time scale. Further shortening of the step does not change the result.

As shown in Fig. 1 the system can indeed represent the well-recognized phenomenon with a local signal applied to only 1/6 part of the chain. The collective variables, $Y = \mathrm{N}^{-1}\Sigma_{n=1}^{\mathrm{N}} y_n$ and $Y' = \mathrm{M}^{-1}\Sigma_{n=k+1}^{k+\mathrm{M}} y_n$, also adequately and legibly reflect the key features of the effect as shown in Fig. 2.

The generic features of STSR, i.e., those related to a diffusively coupled chain of bistable oscillators, have reasonable prototypes going back as far as the pioneering paper [9]. The main result from Ref. [9] is that, for transition times, only the energy of the unstable instantonlike spatially inhomogeneous steady state solutions, and not the total energy barrier for the chain, is of importance cf. Ref. [21]. However, in fact, there has been no discussion at all of the transition

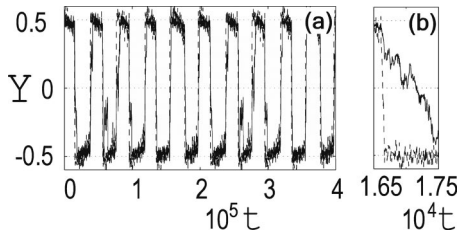

FIG. 2. (a) Dynamics of the collective variables Y (solid) and Y' (dash) with parameters $N=300$, $M=50$, $k=125$, $K=15$, $D=0.1$, and $m=0.25$; (b) fine structure of the transition region in (a) reveals a wave-front propagation.

kinetics, which appear after the birth of the "instantons."

The transition kinetics are most likely related to, but different from, another interesting phenomenon — traveling waves [18]. It is evident from the physical interpretation of Eq. (1) that a traveling wave solution can appear only if the symmetry of the bistable potential is broken and there is an energy gap between the two stable states, and the energy flux is compensated for by dissipative forces. This is not the case for the deterministic part of Eq. (1) with $A=0$. However, if $A \neq 0$ then the external signal periodically breaks the symmetry and creates an absolute minimum at one of the wells of the underlying potential, and a traveling wave front can develop on a sufficiently long time scale T_s. The situation changes favorably in the case of a chain because the translational symmetry is broken. With the chain, starting from the physical interpretation above, we can imagine an unstable instantonlike steady state solution [9]. Its characteristic lifetime introduces a new time scale that is required to be shorter than T_s for the continued existence of the induced STSR effect. Our numerical simulations justify *a posteriori* that this condition is perfectly realizable.

Once the underlying kinetic mechanisms of STSR are understood, we can explore other interesting features. Here we focus on one that has been overlooked in previous studies, viz. the shift of the stable spatially homogeneous steady states.

Let us consider the specific lattice SDE

$$\dot{y}_i = K\Delta y_i - V'(y_i) + \sqrt{2D}\xi_i(t), \quad i \in \mathbb{Z}, \tag{2}$$

and evolve the optimal Gaussian representation of this equation (extending [22] to the case of a lattice system as a starting point for further analysis; alternatively one can start with an averaging principle [21]),

$$\dot{y}_i = K\Delta y_i - [a_i(t) + b_i(t)\delta y_i] + \sqrt{2D}\xi_i(t), \quad i \in \mathbb{Z}, \tag{3}$$

where a_i and b_i are obtained by a minimization procedure in respect of the mean-square error functional $J = \langle [V'(y_i) - (a_i + b_i \delta y_i)]^2 \rangle$ for all $i \in \mathbb{Z}$, where $\delta y_i = y_i - \langle y_i \rangle$. The representation (3) of (2) is especially appropriate since the *ex ante* fluctuations are almost Gaussian near the stable steady states and typical exit paths of Eq. (2). On the time scale T_s the internal deterministic and random oscillations are very fast and can be considered adiabatically following with the external signal.

To actually obtain a_i and b_i, which is an intractable problem since it requires the exact solution of Eq. (2), we consider a self-consistent approximation scheme combining the minimization of J together with the solution of Eq. (3). Thus we replace averaging according to Eq. (2) by its Gaussian approximation according to Eq. (3), $\overline{y} = \langle y \rangle_{\text{Gaussian}} = \langle y \rangle$. As a result, we obtain the self-consistent set

$$\dot{\overline{y_i}} = K\Delta \overline{y_i} - a_i(t), \tag{4}$$

$$\delta\dot{y}_i = K\Delta\,\delta y_i - b_i(t)\,\delta y_i + \sqrt{2D}\xi_i(t), \tag{5}$$

$$a_i(t) = \langle V'[y_i(t)]\rangle, \quad b_i(t) = \langle \delta y_i V'[y_i(t)]\rangle \langle \delta y_i^2\rangle^{-1}.$$

Further, using the Furutsu-Novikov formula (Ref. [23], and references therein) we explicitly obtain $a_i=[3\Re_2(y_i)-m]\overline{y_i}+\overline{y_i^3}$, $b_i=[3\Re_2(y_i)-m]+3\overline{y_i^2}$, where $\Re_2(y_i)=\langle y_i^2\rangle-\overline{y_i}^2$ is the second cumulant. Thus Eq. (4) takes the form $\dot{\overline{y_i}}=K\Delta\overline{y_i}+[m-3\Re_2(y_i)]\overline{y_i}-\overline{y_i^3}$, involving the effective potential function $V_{\rm eff}$, $V'_{\rm eff}(\overline{y_i})=-[m-3\Re_2(y_i)]\overline{y_i}+\overline{y_i^3}$. Since $\Re_2(y_i)\geqslant 0$, $V_{\rm eff}\geqslant V$, always.

Finally we consider the principal problem of an explicit calculation of $\Re_2(y_i)$ and solve it under certain hypotheses. Consider the spatial correlations, $\kappa_{mn}(t)=\langle \delta y_m(t)\delta y_n(t)\rangle$ and suppose that the fluctuations tend to their steady state via a stage of spatial homogenization with the ansatz $\kappa_{mn}(t)=\kappa_{m-n}(t)=\kappa_r(t)$. The known equilibrium solution of Eq. (2) with probability density $\rho_\infty(y)\propto\exp[-\mathcal{F}(y)/D]$ has the property of spatial homogeneity, but does the property still persists as time evolves? To facilitate understanding, consider the process of the formation of ρ_∞ as a result of time averaging, and fix two limiting cases related to low and high levels of the noise intensity. As $D\downarrow 0$ the system spends most time in a potential trough, occasionally passing from one to the other. Being in a trough it has time to form a local quasiequilibrium density that reflects the local asymmetry of the underline potential. As $t\to+\infty$, a sum of the local densities is formed to satisfy the global symmetry condition. Under a high level of noise, the rate of passage from one trough to the other is frequent enough in order to rapidly form the mean value $y=0$. The qualitative picture described above is linked to the particular time scale of the problem in question. Favorably for the ansatz, we are interested in averaging over a time scale about T_s that becomes apparent in the local quasiequilibrium density. Test numerical simulations are also essential in order to reinforce the ansatz.

Using Eq. (5) together with the Furutsu-Novikov formula, we obtain a dynamical equation for $\kappa_r(t)$: $\dot{\kappa}_r=2K\Delta\kappa_r-2b(t)\kappa_r+2D\delta_{r0}$, $r\in\mathbb{Z}$—and as a result, the equation for the steady state correlation function, $K\Delta\kappa_r-b\kappa_r+D\delta_{r0}=0$, $r\in\mathbb{Z}$, with the natural asymptotic conditions $\lim_{r\to\pm\infty}\kappa_r=0$; b is still unknown. Substituting $\kappa_r=At^{-|r|}$, $t>1$, we obtain the set of algebraic equations, corresponding to $r=0$ and $r\geqslant 1$: $[2K(1-t^{-1})+b]A=D$, $t^2-2(1+b/2K)t+1=0$. The last equation has two different roots, $t_\pm=(2K)^{-1}[2K+b\pm\sqrt{b(4K+b)}]$, connected by the relation, $t_+t_-=1$. Since $t\equiv t_+>1$, this means that $t_-=t^{-1}<1$. Therefore finally we obtain

$$\kappa_r=\frac{D}{\sqrt{b(4K+b)}}\left[\frac{2K+b+\sqrt{b(4K+b)}}{2K}\right]^{-|r|}.$$

In particular, $\kappa_0=D/\sqrt{b(4K+b)}$. Further, we can use κ_0 in combination with $V'_{\rm eff}(\overline{y})=0$ to characterize the effective spatially homogeneous steady states. Excluding b $(=3\kappa_0-m+3\overline{y}^2)$ from this set, we arrive at two cases: (a) $\overline{y}=0$, $(3\kappa_0-m)(4K+3\kappa_0-m)=D^2/\kappa_0^2$, (b) $\overline{y}^2=m-3\kappa_0$, $(m-3\kappa_0)(2K+m-3\kappa_0)=D^2/4\kappa_0^2$. Observe first that while (a) always gives $\overline{y}=0$ and $\kappa_0\geqslant m/3$, the set (b) has no real roots $\overline{y}$ $(\kappa_0>0)$ for a certain range of values of D and K (see Fig. 3). In other words, the stable steady states disappear and only $\overline{y}=0$ remains in this range. We could claim that this occurs with a bifurcationlike behavior. To identify this observation, we carry out numerical experiments on the system (1) varying the noise intensity D over a wide range. The response to the external signal provides evidence of the effect as shown in Fig. 4, which is enhanced further with larger N. In the case (c) the stochastic resonance pattern, adjusted for the renormalization, is recognizable, but in the case (a) only simple oscillations with the frequency of the external signal are visible around $\overline{y}=0$.

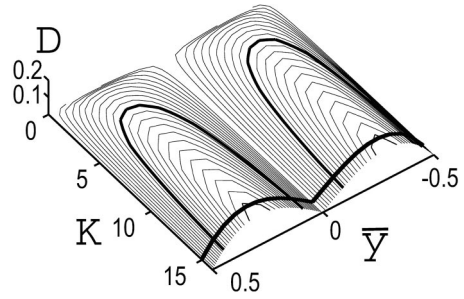

FIG. 3. Spatially homogeneous steady states $\overline{y}\neq 0$ as points of intersection of a characteristic surface $\overline{y}=\overline{y}(D,K)$ and two coordinate planes of the control parameters ($D=0.1$ and $K=15$ are marked). Instead of the two original steady states ±0.5 there are now four for specific values of D and K, and zero otherwise.

For further understanding of this observation, one can consider the equilibrium probability density of Eq. (2). As $D\downarrow 0$, $\rho_\infty(y)$ is concentrated at two spatially homogeneous absolute minima of the potential and forms the most improbable configurations around unstable steady state $y=0$. The homogeneous equilibrium mean value $\langle y\rangle_\infty=0$ is fixed by the invariance of $\rho_\infty(y)$ under the transformation $y\to -y$, and eventually coincides with the unstable steady state. By increasing D we can change the sharpness of the equilibrium density profile, but not the topology. It should be observed here that $\rho_\infty(y)$ represents the average over all sample paths of Eq. (2). If we consider a single sample path then the local density profile of Eq. (2) is formed by time averaging, initially in the vicinity of an absolute minimum of the potential

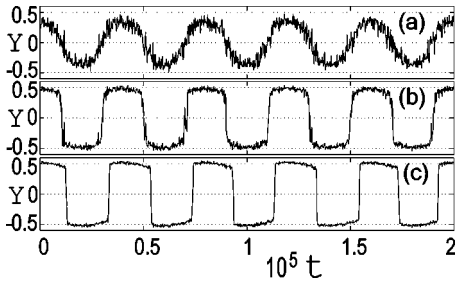

FIG. 4. Three output signals (collective variable Y, $N=M=500$) with (a) $D=0.5$; (b) $D=0.24$; and (c) $D=0.1$; correspondingly, above, near, and below the characteristic surface $\overline{y}=\overline{y}(D,K)$ at $K=15$ (see Fig. 3); effective stable steady states: (a) 0 and (c) ±0.43. The difference appears to be not only quantitative but also, more importantly, qualitative.

reflecting the local potential asymmetry. After a sufficiently long period, the system with probability 1 will pass through the low-probability range to the vicinity of the other absolute minimum of the potential, and so on. The rate of this process rapidly increases with D [21]. The rate of development of a difference between the most probable and mean values of the local density is also increasing with D since, within the same time interval, more different states are realizable. It is remarkable that the time interval related to the signal as well as the inherent nature of SR is favorable to observe the mean values or, in other words, to feel the effective potential. The collective variable Y, which is a sort of site average, allows us to visualize this effect if the chain is long enough.

Finally, if we set $D=0.1$ and $\sqrt{m}=0.5$, we obtain $\overline{y} \approx \pm 0.43$. This is close to the observed value obtained by direct simulation of Eq. (1) (with $N=M=300$ and 500). The agreement improves with larger values of N. There also exists a pair of steady states close to $\overline{y}=0$: $\overline{y} \approx \pm 0.1$ — but these are not clearly identified in the simulation of STSR.

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[24] In contrast to Ref. [13], our approach is dynamic by its nature and the spatially homogeneous steady states appear as the result of temporal evolution and provide more detailed results, cf. Ref. [13].

[25] The point is likely applicable to allied effects [7,14].