PHYSICAL REVIEW E **69**, 036608 (2004)

Vector electromagnetic *X* waves

Alessandro Ciattoni*
Istituto Nazionale per la Fisica della Materia, UdR Università dell'Aquila, 67010 L'Aquila, Italy and Dipartimento di Fisica, Università dell'Aquila, 67010 L'Aquila, Italy

Claudio Conti
Istituto Nazionale per la Fisica della Materia, UdR "Roma Tre," Via della Vasca Navale 84, 00147 Roma, Italy

Paolo Di Porto
Dipartimento di Fisica, Università dell'Aquila, 67010 L'Aquila, Italy and Istituto Nazionale per la Fisica della Materia, UdR "La Sapienza," 00185 Roma, Italy

(Received 6 October 2003; published 30 March 2004)

A vector propagation scheme for describing electromagnetic nondiffracting beams (X waves) is introduced. In particular we show that, from the knowledge of the transverse field components on a given transverse plane and at a fixed instant, it is possible to predict the whole electric field everywhere which in particular allows us to investigate the *imaging* properties of nondiffracting beam. Furthermore, we show that the longitudinal field component crucially depends on the pulse velocity and that it can be neglected only if the velocity is slightly greater than c. The proposed formalism is tested by means of two examples, the vector fundamental and Gaussian X waves which admit analytical treatment. As an application of the propagation scheme, we derive in closed form the expressions for the field propagator showing that its transverse component formally coincides with one of the scalar fundamental X wave.

DOI: 10.1103/PhysRevE.69.036608 PACS number(s): 42.25.Bs, 41.20.Jb, 02.90.+p

I. INTRODUCTION

The investigation of nondiffracting three-dimensional waves has attracted the interest of many researchers in the past decades starting from the pioneering paper of Durnin *et al.* [1], in which they reported the first experimental investigation of an optical monochromatic diffraction-free beam whose existence had been theoretically predicted by Stratton [2]. In the polychromatic realm, the most interesting generalization of the original monochromatic Bessel beam are the limited diffraction pulses introduced by Lu and Greenleaf [3,4] (X waves). An X wave is a solution of the wave equation which rigidly travels along a direction (say the z axis) with a fixed velocity; this explains why these fields have been investigated, experimentally and theoretically, both in acoustics [5,6] and in electromagnetism [7–11].

From a theoretical point of view, many approaches have been proposed for describing nondiffracting waves [12–15] and their main propagation features have been understood. The most striking properties are as follows: (a) nondiffracting waves have a velocity V which is greater than the velocity c of plane waves traveling in the medium and (b) that their total energy turns out to be infinite. Both of these properties are direct consequences of the **k**-space structure of diffraction-free fields: they are a superposition of all the plane waves whose wave vectors belong to a fixed cone of semiaperture angle ψ. These plane waves constructively interfere only at the cone axis and it is straightforward to prove that the intersection point of any wavefront with the z axis travels along the axis itself with a velocity equal to $c/\cos\psi > c$. In the case of electromagnetic waves this property is often referred to as the superluminality of nondiffracting pulses. Besides, the fact that the allowed plane waves must belong to a surface (a cone) in the three-dimensional **k**-space immediately implies (due to the Parseval theorem) that the total energy of the field is divergent. At a first glance both (a) and (b) properties of nondiffracting beams seem to be serious shortcomings against practical realizability and usefulness of these rather exotic pulses. Superluminality (a) seems to be a violation of the special theory of relativity and, particularly, of the relativistic causality; the infinite content of energy (b), on the other hand, seems to make these objects definitively not physical. However, it is obvious that no violation of relativity can arise since nondiffracting pulses are *exact* solutions of the relativistic covariant three-dimensional wave equation [8]. As far as (b) is concerned, we note that nondiffracting beams clearly exhibit a *stationary* character. In fact, the only temporal dynamics is a pure translation of the whole packet at velocity V. It is obvious that a stationary state is rigorously achieved by any physical system only after an infinite long transitory during which the sources continuously provide energy to the system. An analogous situation in optics is that of a monochromatic paraxial beam which is obviously stationary and also possesses an infinite energy; however the model is widely employed to describe actual fields. Therefore, we conclude that nondiffracting beams describe limiting cases of actually feasible fields. The analogy between nondiffracting waves and paraxial beams goes a step further upon noting that both the fields posses finite powers over any transverse plane.

In the present paper we analyze electromagnetic nondiffracting beams in the light of two main features. First, the electromagnetic field has an intrinsic vector structure and, in

*Electronic address: alessandro.ciattoni@aquila.infn.it

the case of nondiffracting beams, the coupling among the three Cartesian components of the electric field must be affected by the absence of diffraction. To the best of our knowledge, the vector structure of the X waves has been investigated only by Recami in Ref. [7] where the author imposes the shape invariance condition to the Hertz potential and then deduces the electromagnetic field in the case of the fundamental X wave (which admit an analytical treatment). Second, we are interested in the *imaging* properties of nondiffracting beams. All theoretical approaches describe a nondiffracting beam as a suitable superposition of monochromatic Bessel beams weighted by an arbitrary function playing the role of a spectrum. However, the arbitrariness of the spectrum has never been exploited for relating the field at an *arbitrary* transverse plane to its distribution over another transverse plane or, in other words, to investigate the imaging properties of a nondiffracting beam. The only attempt in this sense is furnished by Saari and Sonajalg in Ref. [16], where the authors employ the pulsed Bessel beams they introduce as a point-spread function which shows a well-constrained support but, at the same time, has an excellent capability to maintain the image focused without any spread over large propagation distances.

Starting from an exact vector solution of Maxwell equations in vacuum, we impose each Cartesian component to fulfill the diffraction-free condition, i.e., to depend on z and t only through the combination $Z=z-Vt$; thus, we obtain an expression representing any vector electromagnetic nondiffracting beam. We show that the field angular spectrum is two dimensional and that the spectrum of the transverse components (x and y components) coincides with the two-dimensional Fourier transform of the field at the plane $Z=0$. On the other hand, the spectrum of the longitudinal z component is easily related to the spectrum of the transverse part. Therefore the knowledge of the transverse part of the field at $Z=0$ is sufficient to predict the nondiffracting beam for all Z and thus we are in the position to investigate its imaging properties. As a second relevant result, we obtain that the longitudinal component has an order of magnitude which is roughly that of the transverse part multiplied by $\sqrt{V^2/c^2-1}$, thus concluding that the packet velocity is crucial in fixing the relevance of the z component. Therefore, the longitudinal component is negligible only if V is slightly greater than c and becomes dominant for velocity much larger than c. As a further result we derive a relation connecting the field longitudinal component to the transverse ones in terms of an integral transform at each transverse plane. As examples to apply the proposed approach we investigate two nondiffracting beams; the first is the vector generalization of the well-known family of multiple temporal derivatives of the fundamental X wave introduced by Lu and Greenleaf [4], whereas the second deals with X waves whose distribution at $Z=0$ shows a Gaussian profile.

Our scheme for describing electromagnetic X waves is, in addition, exploited to introduce a propagator approach. In order to place into evidence the fact the the whole nondiffracting beam is fixed by the boundary distribution of the transverse part of the field at $Z=0$, we derive a general relation expressing the field at an arbitrary transverse plane as an integral Z-dependent transform of the boundary transverse field distribution. More precisely we derive the expression for the field propagators in closed form and we find that the transverse component propagator coincides with the temporal derivative of the fundamental X wave. This appears to be a relevant result since it points out that the role of the fundamental X waves, in the general theory of nondiffracting beam, is more important than an elegant analytical result.

II. VECTOR *X* WAVES

Let us start by considering an exact integral representation of the electromagnetic field propagating in vacuum. For our purpose, it is convenient to write the complex analytic signal $\hat{\mathbf{E}}$ of the electric field $\mathbf{E}=\mathrm{Re}[\hat{\mathbf{E}}]$ in the form

$$\hat{\mathbf{E}}_\perp(\mathbf{r}_\perp,z,t)=\frac{\partial}{\partial z}\mathbf{F}_\perp(\mathbf{r}_\perp,z,t),$$

$$\hat{E}_z(\mathbf{r}_\perp,z,t)=-\boldsymbol{\nabla}_\perp\cdot\mathbf{F}_\perp(\mathbf{r}_\perp,z,t), \tag{1}$$

where Re is the real part, $\mathbf{r}_\perp=x\hat{\mathbf{e}}_x+y\hat{\mathbf{e}}_y$, $\boldsymbol{\nabla}_\perp=\partial_x\hat{\mathbf{e}}_x+\partial_y\hat{\mathbf{e}}_y$, $\hat{\mathbf{E}}_\perp=\hat{E}_x\hat{\mathbf{e}}_x+\hat{E}_y\hat{\mathbf{e}}_y$, and $\mathbf{F}_\perp=F_x\hat{\mathbf{e}}_x+F_y\hat{\mathbf{e}}_y$ is a suitable analytic signal. It is easy to prove that Maxwell's equations are satisfied if

$$\mathbf{F}_\perp(\mathbf{r}_\perp,z,t)=\int_0^\infty d\omega\int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}e^{i(k_z z-\omega t)}\tilde{\mathbf{F}}_\perp(\mathbf{k}_\perp,\omega), \tag{2}$$

where $\mathbf{k}_\perp=k_x\hat{\mathbf{e}}_x+k_y\hat{\mathbf{e}}_y$, $d^2\mathbf{k}_\perp=dk_x dk_y$, $\tilde{\mathbf{F}}_\perp=\tilde{F}_x\hat{\mathbf{e}}_x+\tilde{F}_y\hat{\mathbf{e}}_y$ is an arbitrary vector spatio-temporal spectrum, and $k_z=\sqrt{\omega^2/c^2-|\mathbf{k}_\perp|^2}$ (where we choose the main branch of the complex square root). Note that $\mathbf{F}_\perp$ is a superposition of all the plane waves whose z component of the wave vector is positive (homogeneous waves) or imaginary (evanescent waves) so that, since in Eq. (2) the frequencies ω are positive, the field represents the most general pulse propagating from left to right along the z axis. It is also worth noting that the introduction of $\mathbf{F}_\perp$ allows a complete vector description of the electromagnetic pulse.

Let us now investigate the dynamics of nondiffracting electromagnetic pulses. The standard definition of an X wave is a solution of the three-dimensional wave equation fulfilling the condition $\mathbf{E}(\mathbf{r}_\perp,z,t)=\mathbf{E}(\mathbf{r}_\perp,z-Vt)$, expressing the fact that each Cartesian component of the electric field must be a shape invariant pulse. From Eq. (1) it follows that this condition is attained if the field $\mathbf{F}_\perp$ is nondiffracting, that is to say if

$$\mathbf{F}_\perp(\mathbf{r}_\perp,z,t)=\mathbf{F}_\perp(\mathbf{r}_\perp,z-Vt). \tag{3}$$

Since Eq. (2) describes any pulse propagating along the positive z axis, it is evident that $\mathbf{F}_\perp$ fulfills Eq. (3) only if the relation $k_z=\omega/V$ holds for all its plane wave components, that is, if the spectrum is given by

$$\tilde{\mathbf{F}}_\perp(\mathbf{k}_\perp,\omega)=\tilde{\mathbf{A}}_\perp(\mathbf{k}_\perp,\omega)\,\delta\!\left(k_z(\omega,\mathbf{k}_\perp)-\frac{\omega}{V}\right), \tag{4}$$

where $\delta(\xi)$ is the Dirac delta function and $\tilde{\mathbf{A}}_\perp$ is an arbitrary complex vector. Equation (4) is easily interpreted as the requirement that all the plane waves of $\mathbf{F}_\perp$ share a common phase velocity along the z axis ($v_z = c^2 k_z/\omega$) given by c^2/V. Since $|v_z| < c$, we recover the condition $V > c$ expressing the well-known superluminality of the X waves. Note also that Eq. (4) implies $k_z = \eta|\mathbf{k}_\perp|$, where $\eta = (V^2/c^2 - 1)^{-1/2}$, so that the wave vectors of all the plane waves forming a nondiffracting beam belong to a cone whose semiaperture angle ψ (called in literature Axicon angle) is given by the relation $\cot\psi = \eta$ [17]. Introducing Eq. (4) into Eq. (2) and performing the integral over ω we readily get

$$\mathbf{F}_\perp(\mathbf{r}_\perp, Z) = \int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{i\eta|\mathbf{k}_\perp|Z} \tilde{\mathbf{f}}_\perp(\mathbf{k}_\perp), \tag{5}$$

where $Z = z - Vt$ is the longitudinal distance in the (superluminal) reference frame where the X wave is at rest, $|\mathbf{k}_\perp| = \sqrt{k_x^2 + k_y^2}$ and $\tilde{\mathbf{f}}_\perp(\mathbf{k}_\perp) = (\eta^2/V)\tilde{\mathbf{A}}_\perp(\mathbf{k}_\perp, \eta V|\mathbf{k}_\perp|)$. Note that $\tilde{\mathbf{f}}_\perp$ is an arbitrary complex vector and, in order to underst and its physical meaning, it is sufficient to substitute Eq. (5) into the first of Eqs. (1) [obtaining a shape invariant field $\hat{\mathbf{E}}_\perp(\mathbf{r}_\perp, z, t) = \hat{\mathbf{E}}_\perp(\mathbf{r}_\perp, Z)$], to evaluate it at $Z = 0$ and to invert the resulting Fourier integral, thus getting

$$\tilde{\mathbf{f}}_\perp(\mathbf{k}_\perp) = \frac{1}{i\eta|\mathbf{k}_\perp|}\tilde{\mathbf{E}}_\perp(\mathbf{k}_\perp), \tag{6}$$

where

$$\tilde{\mathbf{E}}_\perp(\mathbf{k}_\perp) = \frac{1}{(2\pi)^2}\int d^2\mathbf{r}_\perp e^{-i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\hat{\mathbf{E}}_\perp(\mathbf{r}_\perp, 0) \tag{7}$$

is the two-dimensional Fourier transform of the analytic signal of the transverse electric field evaluated at $Z = 0$. Substituting Eq. (6) into Eq. (5) and the resulting field into Eqs.(1) we obtain

$$\hat{\mathbf{E}}_\perp(\mathbf{r}_\perp, Z) = \int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{i\eta|\mathbf{k}_\perp|Z}\tilde{\mathbf{E}}_\perp(\mathbf{k}_\perp),$$

$$\hat{E}_z(\mathbf{r}_\perp, Z) = \frac{1}{\eta}\int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{i\eta|\mathbf{k}_\perp|Z}\left(-\frac{\mathbf{k}_\perp}{|\mathbf{k}_\perp|}\right)\cdot\tilde{\mathbf{E}}_\perp(\mathbf{k}_\perp). \tag{8}$$

Equations (8) contain the expression for the most general electromagnetic nondiffracting pulse and they deserve some discussion. First, note that if the transverse field $\hat{\mathbf{E}}_\perp$ is known at $Z = 0$ (or, equivalently, at an arbitrary plane orthogonal to the direction of propagation), Eq. (7) yields $\tilde{\mathbf{E}}_\perp(\mathbf{k}_\perp)$ and hence Eqs. (8) furnish the field at all Z. Thus, the knowledge of $\hat{\mathbf{E}}_\perp$ on a transverse plane *at a given instant* is sufficient to predict the pulse evolution. This property does not generally hold for a pulse propagating in vacuum whose description is possible only if the transverse electric field is known at any time on a given plane. The physical origin of this peculiarity lies in the "rigid" motion characterizing nondiffracting pulses. Therefore the X wave description reduces to a steady-state problem in its rest frame where the knowledge of the field on a transverse plane is sufficient to predict the field everywhere. In this perspective, it is also worth noting that, because of the arbitrariness of the boundary distribution $\hat{\mathbf{E}}_\perp(\mathbf{r}_\perp, 0)$, optical X waves allow to accomplish a sort of diffraction-free transmission of arbitrary two-dimensional images [16].

A related feature of nondiffracting pulses is a strong space-time coupling [18] characterizing their structure. In order to clarify this point, let us consider one of the Cartesian component of the field in Eqs. (8) (say the x components) at the plane $z = 0$ at any instant t (so that $Z = -Vt$), that is to say

$$\hat{E}_x(\mathbf{r}_\perp, 0, t) = \int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{-i\eta V|\mathbf{k}_\perp|t}\tilde{E}_x(\mathbf{k}_\perp). \tag{9}$$

From this expression we note that the spatial and temporal features of the pulse (such as the transverse width or the time duration) are related to the exponentials containing $\mathbf{r}_\perp$ and t, respectively; both the exponentials contain $\mathbf{k}_\perp$ and are weighted by the spectrum $\tilde{E}_x$. Therefore the spatial and temporal behaviors of the pulse are dictated by $\hat{E}_x(\mathbf{r}_\perp, 0)$ at the same time, resulting in a high correlation between spatial and temporal features (space-time coupling).

The field components in Eqs. (8) are given by Fourier integrals and, as a consequence, they are square integrable. This implies that the power carried by the field is finite. In this sense, the X waves are more realistic than their monochromatic counterpart, the Bessel beams, whose power is infinite. The power of the j-field component ($j = x, y, z$) over a transverse plane at fixed Z is $W_j(Z) = \int d^2\mathbf{r}_\perp |\hat{E}_j(\mathbf{r}_\perp, Z)|^2$ that is, with the help of Eqs. (8) and the Parseval theorem,

$$W_j(Z) \equiv W_j = (2\pi)^2\int d^2\mathbf{k}_\perp |\tilde{E}_j(\mathbf{k}_\perp)|^2, \tag{10}$$

showing that the *power of any X* wave does not depend on Z. In turn, the total energy $U_j = \int_{-\infty}^{+\infty} dZ\, W_j(Z)$ is a divergent quantity as it was to be expected in connection with the stationary nature of the considered fields. In fact, a stationary state is rigorously attained only after an infinite transitory evolution during which the sources continuously provide energy to the field, thus justifying the infinite amount of energy. In this perspective, the X wave situation (which is usually highly nonmonochromatic) is quite close to that of a monochromatic paraxial field in vacuum which is clearly a stationary field and generally possesses an infinite amount of energy (even if its power over any transverse plane is finite and independent of z). In this perspective it is interesting to compare the first of Eqs. (8) with the expression for the analytic signal associated to a monochromatic paraxial beam (propagating in vacuum), namely,

$$\hat{\mathbf{E}}_\perp(\mathbf{r}_\perp, z, t) = e^{i(k_0 z - \omega t)}\int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{-i|\mathbf{k}_\perp|^2/2k_0 z}\tilde{\mathbf{E}}_\perp(\mathbf{k}_\perp), \tag{11}$$

where ω is the frequency of the field whereas $k_0=\omega/c$. The formal analogy between the slowly varying amplitude of Eq. (11) and the first of Eqs. (8) is evident.

From the second of Eqs. (8) we note that the longitudinal component $\hat{E}_z$ of the X wave is completely fixed by the boundary transverse components $\hat{\mathbf{E}}_\perp(\mathbf{r}_\perp,0)$, in agreement with a well-known property of the electromagnetic field in vacuum [19]. The expression for $\hat{E}_z$ reveals that, roughly speaking, $|\hat{E}_z|\sim|\hat{\mathbf{E}}_\perp|/\eta$ so that the faster the X wave the greater is the longitudinal component in comparison with the transverse ones. This feature can be understood bearing in mind that an X pulse is the superposition of all the plane wave whose wave vectors belong to a cone whose semiaperture angle is $\psi=\arctan(1/\eta)=\arctan\sqrt{V^2/c^2-1}$ so that the faster the X wave the more open is the cone. Since in vacuum the electric field of each plane waves is orthogonal to the wave vector (transversality), the more the cone is open the greater is the contribution of each plane wave to the longitudinal component of the electric field. We conclude that the longitudinal component of the electromagnetic X wave can be neglected only if V is slightly greater than c, showing that the scalar approach generally fails. If, for example, $V<1.005c$, we have $\eta>10$ proving that the longitudinal component is negligible; on the other hand, if $V>\sqrt{2}c$ it follows that $\eta<1$ so that the longitudinal component tends to become dominant.

The connection between longitudinal and transverse components can be expressed in a significant way combining the first and the second of Eqs. (8), thus getting (see Appendix A)

$$\hat{E}_z(\mathbf{r}_\perp,Z)=\frac{1}{2\pi i\eta}\int d^2\mathbf{r}'_\perp\frac{\mathbf{r}_\perp-\mathbf{r}'_\perp}{|\mathbf{r}_\perp-\mathbf{r}'_\perp|^3}\cdot\hat{\mathbf{E}}_\perp(\mathbf{r}'_\perp,Z), \quad (12)$$

which is a relation giving $\hat{E}_z$ at an arbitrary transverse plane (i.e., for Z fixed) once the $\hat{\mathbf{E}}_\perp$ is known on the same plane.

III. TWO EXAMPLES OF VECTOR *X* WAVES

We wish now to apply the approach introduced in the above section to the relevant cases of vector fundamental X waves and Gaussian X waves.

A. Vector fundamental *X* waves

Let us consider the family of X waves whose spectrum is given by

$$\tilde{\mathbf{E}}_\perp^{(n)}(\mathbf{k}_\perp)=\frac{E_0\sigma^2}{2\pi}(\sigma|\mathbf{k}_\perp|)^{n-1}e^{-\sigma|\mathbf{k}_\perp|}\hat{\mathbf{e}}_x\equiv\tilde{E}_x^{(n)}(\mathbf{k}_\perp)\hat{\mathbf{e}}_x, \quad (13)$$

where n is a positive integer, whereas E_0 and $\sigma>0$ are two real constants. Substituting Eq. (13) into the first of Eqs.(8) we get (see Appendix B)

$$\hat{E}_x^{(n)}(\mathbf{r}_\perp,Z)=E_0\left(\frac{\sigma}{i\eta}\right)^n\frac{\partial^n}{\partial Z^n}\frac{\sigma}{[(\sigma-i\eta Z)^2+|\mathbf{r}_\perp|^2]^{1/2}}, \quad (14)$$

which are easily recognized to be the well-known family of derivatives of the X wave originally introduced by Lu and Greenleaf [4,17]; we will refer to this family of limited diffraction beams as the fundamental X waves. Therefore, Eq. (13) contains the spectrum of the electromagnetic X waves whose transverse part reproduces the fundamental X waves. The longitudinal component of the electric field is obtained after substituting Eq. (13) into the second of Eqs. (8), namely,

$$\hat{E}_z^{(0)}(\mathbf{r}_\perp,Z)=E_0\frac{\sigma x}{i\eta|\mathbf{r}_\perp|^2}\left\{1-\frac{(\sigma-i\eta Z)}{[(\sigma-i\eta Z)^2+|\mathbf{r}_\perp|^2]^{1/2}}\right\},$$

$$\hat{E}_z^{(n)}(\mathbf{r}_\perp,Z)=\frac{i\sigma}{\eta}\frac{\partial}{\partial x}E_x^{(n-1)}(\mathbf{r}_\perp,Z) \quad \text{for } n\geq 1. \quad (15)$$

Equations (14) and (15) contain the exact expressions for the electromagnetic fundamental X waves and they represent the vector generalization of the well-known scalar fundamental X waves. The parameter σ is directly related to the width of the pulse at $Z=0$ (waist) and it is worth noting that the fields depend on $\mathbf{r}_\perp$ and Z only through the dimensionless quantities $\mathbf{r}_\perp/\sigma$ and Z/σ. Note that the first X wave of the family ($n=0$) has a diverging power at each transverse plane (at Z fixed) since, for $|\mathbf{r}_\perp|$ large, $\hat{E}_x^{(0)}\sim|\mathbf{r}_\perp|^{-1}$. For $n\geq 1$ the X waves of this family are not affected by the above shortcoming and approach physically realizable fields. In Fig. 1 we plot the normalized Cartesian components $E_x(x,0,Z)/E_0$ and $E_z(x,0,Z)/E_0$ of the electric field [real parts of Eqs. (14) and (15)] of the fundamental $n=1$ vector X waves evaluated for various V/c. The x component obviously shows the typical spreading of the fundamental X waves. The z component, on the other hand, vanishes at $Z=0$ [as a consequence of the reality of the spectrum in Eq. (13)], then rapidly grows for increasing $Z>0$ up to a maximum, and finally vanishes as Z goes to infinity. Note that, as expected, the faster the X wave the greater is the maximum of the longitudinal component. Since at the plane $x=0$ E_z vanishes, the longitudinal component is not peaked at the origin but exhibits two symmetrical lobes. In Fig. 2 we report the level plot of $|E_z|/E_0$ of the $n=1$ vector X wave (characterized by a velocity $V=1.05c$) at three transverse planes, showing the above-mentioned lobes and their evolution.

B. Vector Gaussian *X* wave

Consider now an X wave whose boundary field distribution is

$$\hat{\mathbf{E}}_\perp(\mathbf{r}_\perp,0)=E_0e^{-|\mathbf{r}_\perp|^2/2\sigma^2}\hat{\mathbf{e}}_x, \quad (16)$$

where E_0 and $\sigma>0$ are real constants. This is a Gaussian distribution of waist σ centered at $\mathbf{r}_\perp=\mathbf{0}$ and its spectrum is readily obtained by inserting Eq. (16) into Eq. (7), namely,

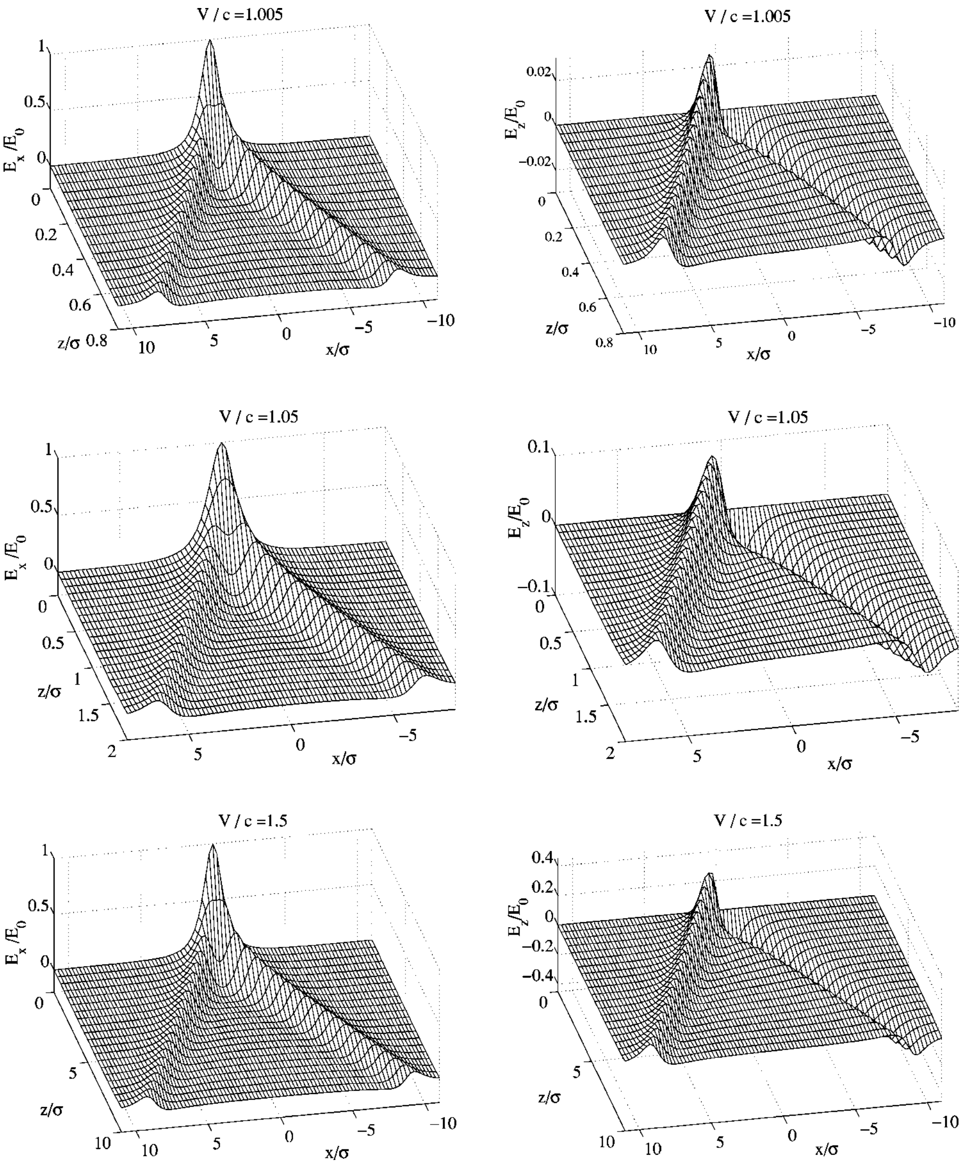

FIG. 1. Normalized Cartesian components $E_x(x,0,Z)/E_0$ and $E_z(x,0,Z)/E_0$ of the electric field [real parts of Eqs. (14) and (15)] of the fundamental $n=1$ vector X waves evaluated for various V/c. Note that the faster the X wave the greater is the longitudinal component E_z.

$$\tilde{E}_x(\mathbf{k}_\perp)=\frac{E_0\sigma^2}{2\pi}e^{-(1/2)\sigma^2|\mathbf{k}_\perp|^2}. \tag{17}$$

Substituting Eq. (17) into Eqs. (8), we obtain (see Appendix C)

$$\hat{E}_x(\mathbf{r}_\perp,Z)=E_0\sum_{n=0}^{\infty}\left(\frac{\eta Z}{\sigma}\right)^n\frac{(i\sqrt{2})^n}{n!}\Gamma\left(\frac{n+2}{2}\right)\times{}_1F_1\left(\frac{n+2}{2},1;-\frac{|\mathbf{r}_\perp^2|}{2\sigma^2}\right),$$

$$\hat{E}_z(\mathbf{r}_\perp,Z)=\frac{E_0}{i\eta\sqrt{2}}\left(\frac{x}{\sigma}\right)\sum_{n=0}^{\infty}\left(\frac{\eta Z}{\sigma}\right)^n\frac{(i\sqrt{2})^n}{n!}\Gamma\left(\frac{n+3}{2}\right)\times{}_1F_1\left(\frac{n+3}{2},2;-\frac{|\mathbf{r}_\perp^2|}{2\sigma^2}\right), \tag{18}$$

where $\Gamma(\xi)$ is the Euler Gamma function, whereas ${}_1F_1(\alpha,\beta;\xi)$ is the hypergeometric confluent function. Equations (18) contain the expressions for the vector Gaussian X wave. Note that, as for the fundamental X waves, the Gaussian X wave depends on $\mathbf{r}_\perp$ and Z only through the dimensionless quantities $\mathbf{r}_\perp/\sigma$ and Z/σ. Note, in addition, that the power of each component is finite as it is clearly shown by substituting Eq. (17) into Eq. (10). Even if the field components are given by series, Eqs. (18) are suitable for numerical analysis since it can be proven that for $(\eta Z/\sigma)\lesssim 8$ the series can be truncated up to the first two hundred terms and therefore easily computable by a standard computer. In Fig. 3, we plot the normalized electric field components [real parts of Eqs. (18)] of the vector Gaussian X wave, for various Z/σ. The evolutions of both the transverse and longitudinal components are quite similar to that of the $n=1$ fundamental X wave.

The expression of the field on the propagation axis (i.e., for $\mathbf{r}_\perp=0$) can be expressed in closed form as (see Appendix C)

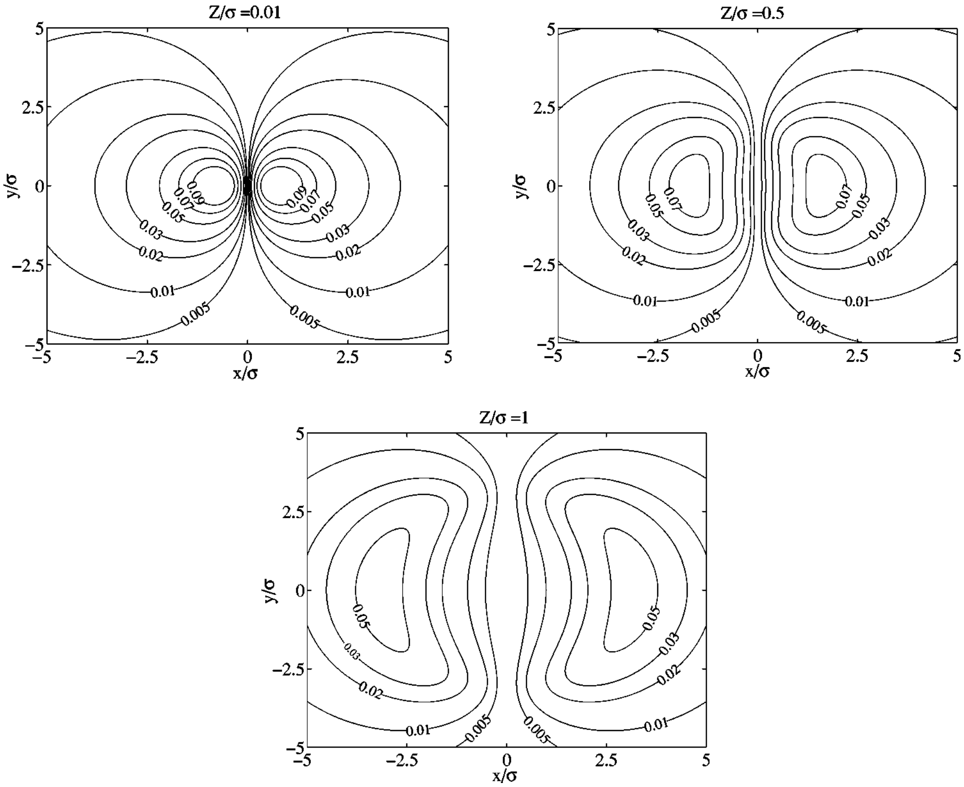

FIG. 2. Level plot of $|E_z|/E_0$ of the $n=1$ vector X waves at three transverse planes for a velocity $V=1.05c$.

$$\hat{E}_x(0,Z)=E_0\left[1+i\sqrt{\pi}\left(\frac{\eta Z}{\sqrt{2}\sigma}\right)e^{-(\eta Z/\sqrt{2}\sigma)^2}\mathrm{erfc}\left(\frac{\eta Z}{i\sqrt{2}\sigma}\right)\right], \tag{19}$$

whereas $\hat{E}_z(0,Z)=0$; here $\mathrm{erfc}(\xi)$ is the analytical continuation of the error function $\mathrm{erfc}(\xi)=(2/\sqrt{\pi})\int_\xi^\infty du\,\exp(-u^2)$. In Fig. 4 we plot the normalized electric field [real part of Eq. (19)] of the vector Gaussian X wave as a function of the normalized propagation distance, together (for comparison) with the analogous field for the $n=1$ fundamental X wave.

IV. PROPAGATOR APPROACH

In Sec. II we have developed a fully vectorial scheme for describing any electromagnetic X wave traveling in vacuum. Our approach allows us to evaluate the field for all $\mathbf{r}_\perp$ and Z, once the transverse part of the field is known at $Z=0$, and is essentially based on the use of the Fourier integral. However, although intuitive and efficient, the use of the Fourier space can be avoided, as usual in optics, by introducing a suitable propagator directly relating the propagated field to its boundary distribution. This goal is achieved by substituting Eq. (7) into Eq. (6) and the resulting spectrum into Eq. (5), thus getting

$$\hat{\mathbf{F}}_\perp(\mathbf{r}_\perp,Z)=\int d^2\mathbf{r}'_\perp\,G(\mathbf{r}_\perp-\mathbf{r}'_\perp|Z)\hat{\mathbf{E}}_\perp(\mathbf{r}'_\perp,0), \tag{20}$$

where

$$G(\mathbf{R}_\perp|Z)=\frac{1}{(2\pi)^2 i\eta}\int d^2\mathbf{k}_\perp\frac{e^{i\mathbf{k}_\perp\cdot\mathbf{R}_\perp}}{|\mathbf{k}_\perp|}e^{i\eta|\mathbf{k}_\perp|Z} \tag{21}$$

with $\mathbf{R}_\perp=\mathbf{r}_\perp-\mathbf{r}'_\perp$). The integral in Eq. (21) turns out to be given by (see Appendix D)

$$G(\mathbf{R}_\perp|Z)=\frac{1}{2\pi i\eta}\frac{1}{[(\epsilon-i\eta Z)^2+|\mathbf{R}_\perp|^2]^{1/2}}, \tag{22}$$

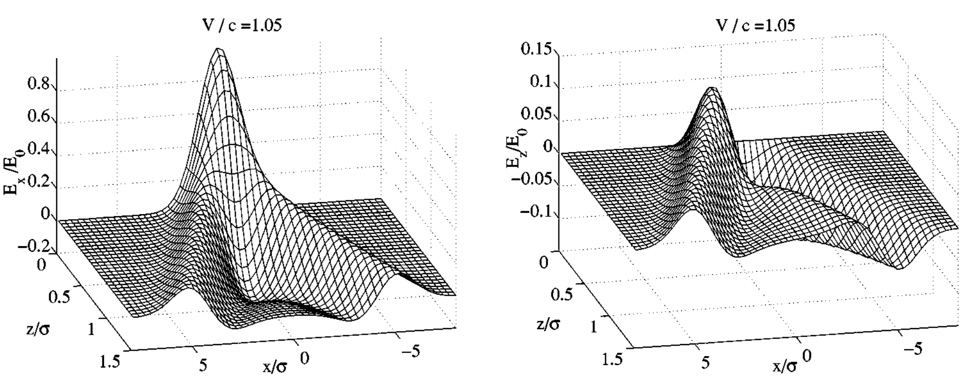

FIG. 3. Normalized Cartesian components $E_x(x,0,Z)/E_0$ and $E_z(x,0,Z)/E_0$ of the electric field [real parts of Eqs. (18)] of the vector Gaussian X waves evaluated for various $V=1.05c$.

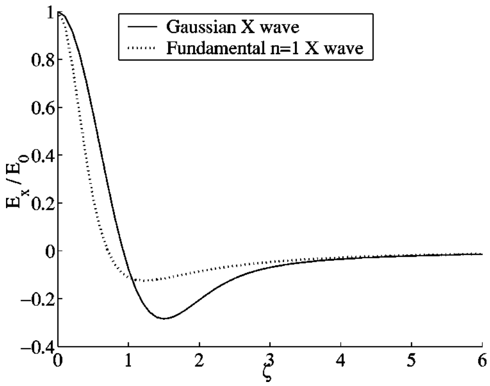

FIG. 4. On axis normalized transverse component $E_x(0,Z)/E_0$ of the electric field as a function of $\zeta = \eta Z/(\sqrt{2}\sigma)$ for the vector Gaussian and fundamental $n=1$ X waves.

where the limit $\epsilon \to 0^+$ has to be taken after G has been convolved with $\hat{\mathbf{E}}_\perp(\mathbf{r}_\perp,0)$ in Eq. (20). Inserting the resulting field $\hat{\mathbf{F}}_\perp$ into Eqs. (1) we readily get

$$\hat{\mathbf{E}}_\perp(\mathbf{r}_\perp,Z) = \int d^2\mathbf{r}'_\perp G_\perp(\mathbf{r}_\perp - \mathbf{r}'_\perp|Z)\hat{\mathbf{E}}_\perp(\mathbf{r}'_\perp,0),$$

$$\hat{E}_z(\mathbf{r}_\perp,Z) = \int d^2\mathbf{r}'_\perp \mathbf{G}_z(\mathbf{r}_\perp - \mathbf{r}'_\perp|Z)\cdot\hat{\mathbf{E}}_\perp(\mathbf{r}'_\perp,0), \qquad (23)$$

where

$$G_\perp(\mathbf{R}_\perp|Z) = \frac{1}{2\pi}\frac{(\epsilon - i\eta Z)}{[(\epsilon - i\eta Z)^2 + |\mathbf{R}_\perp|^2]^{3/2}},$$

$$\mathbf{G}_z(\mathbf{R}_\perp|Z) = \frac{1}{2\pi i\eta}\frac{\mathbf{R}_\perp}{[(\epsilon - i\eta Z)^2 + |\mathbf{R}_\perp|^2]^{3/2}}. \qquad (24)$$

Equations (24) contain the *exact expressions* for the X wave propagators.

The physical interpretation of the propagators is easily obtained by considering an X wave whose boundary field is of the form $\hat{\mathbf{E}}_\perp(\mathbf{r}_\perp,0) = E_0\delta(\mathbf{r}_\perp)\hat{\mathbf{e}}_x$ which, substituted in Eq. (20), yields $\hat{\mathbf{E}}_\perp(\mathbf{r}_\perp,Z) = E_0 G_\perp(\mathbf{r}_\perp|Z)\hat{\mathbf{e}}_x$ and $\hat{E}_z(\mathbf{r}_\perp,Z) = E_0\mathbf{G}_z(\mathbf{r}_\perp|Z)\cdot\hat{\mathbf{e}}_x$; therefore the propagators represent an X wave whose boundary distribution is a δ peak around $\mathbf{r}_\perp = 0$. This is consistent with the fact that $G_\perp(\mathbf{r}_\perp|0) = \epsilon/(2\pi)(\epsilon^2 + |\mathbf{r}_\perp|^2)^{-3/2}$ is, in the limit $\epsilon \to 0^+$, a representation of the two-dimensional Dirac δ function. For $Z>0$ ϵ is obviously unessential and we directly have $G_\perp(\mathbf{r}_\perp|Z) = (\eta Z)/(2\pi i)[|\mathbf{r}_\perp|^2 - (\eta Z)^2]^{-3/2}$ and $\mathbf{G}_z(\mathbf{r}_\perp|Z) = \mathbf{R}_\perp/(2\pi i\eta)[|\mathbf{r}_\perp|^2 - (\eta Z)^2]^{-3/2}$. We observe that the order of magnitude of the longitudinal component is that of the transverse one divided by η, in agreement with the considerations of the Sec. II.

It is worth noting that the fields $G(\mathbf{r}_\perp|Z)$ and $G_\perp(\mathbf{r}_\perp|Z)$ essentially coincide with the zeroth and first fundamental X waves (see Eq. (14)). Therefore, the fundamental X waves are more relevant than simple analytical examples since, in the general theory, they play the role of propagators (once their width, proportional to ϵ, is assumed to go to zero).

V. CONCLUSIONS

Starting from an exact solution of Maxwell's equations describing an arbitrary pulse propagating along the positive z axis in vacuum, with the aid of the propagation invariant requirement, we have obtained a vectorial expression describing any electromagnetic limited diffraction beam (X wave) in free space. Our scheme predicts that, once the distributions of the x- and y-field components are known at a transverse plane of the reference rest frame of the pulse, the vector X wave is known everywhere and at every time. The physical origin of this peculiar feature turns out to be related to the essential stationary character of X waves. Furthermore, we have shown that the longitudinal z- field component can be obtained from its transverse counterparts through a suitable integral transform whose kernel is derived in closed form. In addition, the X wave velocity has been shown to affect in a relevant way the magnitude of the longitudinal component; the faster the X wave the greater the z component in comparison with the transverse part. In particular, we have demonstrated that the longitudinal component cannot be neglected if the X wave "velocity" is greater than $\sqrt{2}c$. The vector propagation scheme has been employed for describing two examples of vector X waves, the family of fundamental X waves and the Gaussian one: the former represents the electromagnetic vector generalization of the well-known X waves of Lu and Greenleaf, whereas the latter example deals with the description of an X wave whose transverse part distribution is Gaussian over a transverse plane. Both these examples exhibit the well-known X shaped profile in the transverse part of the field, whereas the shape of the longitudinal component exhibits two symmetrical lobes (with respect to the plane containing the pulse polarization direction and the z axis) diverging from the z axis. As a further application of the proposed propagation scheme, we have set up a propagator approach. More precisely, we have derived closed-form expressions directly joining the vector X wave in the rest frame to the distribution of its transverse part over a transverse plane. Apart from the conceptual value of this approach, it is worth noting that the propagator kernel for the transverse part is deeply connected to the $n=1$ fundamental X wave.

ACKNOWLEDGMENTS

We thank Stefano Trillo for fruitful discussions. The research of A.C. and P.Di P. was funded by the Italian Ministry of Research through the "Space-Time Effects" Project Nos. PRIN 2001 and FIRB 2002, and by the Italian Istituto Nazionale per la Fisica della Materia (INFM). C.C. acknowledges the Fondazione Tronchetti Provera for financial support.

APPENDIX A: DERIVATION OF EQ. (12)

Inverting the two-dimensional Fourier integral of the first of Eqs. (8) we get

$$e^{i\eta|\mathbf{k}_\perp|Z}\tilde{\mathbf{E}}_\perp(\mathbf{k}_\perp)=\frac{1}{(2\pi)^2}\int d^2\mathbf{r}'_\perp e^{-i\mathbf{k}_\perp\cdot\mathbf{r}'_\perp}\hat{\mathbf{E}}_\perp(\mathbf{r}'_\perp,Z), \tag{A1}$$

which, substituted into the second of Eq. (8), yields (after interchanging the integration order)

$$\hat{E}_z(\mathbf{r}_\perp,Z)=\int d^2\mathbf{r}'_\perp \mathbf{W}_\perp(\mathbf{r}_\perp-\mathbf{r}'_\perp)\cdot\hat{\mathbf{E}}_\perp(\mathbf{r}'_\perp,Z), \tag{A2}$$

where

$$\mathbf{W}_\perp(\mathbf{R}_\perp)=-\frac{1}{\eta(2\pi)^2}\int d^2\mathbf{k}_\perp \frac{e^{i\mathbf{k}_\perp\cdot\mathbf{R}_\perp}}{|\mathbf{k}_\perp|}\mathbf{k}_\perp \tag{A3}$$

and $\mathbf{R}_\perp=\mathbf{r}_\perp-\mathbf{r}'_\perp$. Exploiting a well-known property of the Fourier integral, we change Eq. (A3) into

$$\mathbf{W}_\perp(\mathbf{R}_\perp)=\frac{i}{\eta(2\pi)^2}\nabla_\perp\int d^2\mathbf{k}_\perp\frac{e^{i\mathbf{k}_\perp\cdot\mathbf{R}_\perp}}{|\mathbf{k}_\perp|}$$

$$\equiv\frac{i}{\eta(2\pi)^2}\nabla_\perp\Theta(\mathbf{R}_\perp). \tag{A4}$$

The integral defining Θ can be easily evaluated by introducing polar coordinates according to $\mathbf{k}_\perp=k(\hat{\mathbf{e}}_x\cos\theta+\hat{\mathbf{e}}_y\sin\theta)$ and $\mathbf{R}_\perp=R(\hat{\mathbf{e}}_x\cos\varphi+\hat{\mathbf{e}}_y\sin\varphi)$, thus getting

$$\Theta(\mathbf{R}_\perp)=\int_0^\infty dk\int_0^{2\pi}e^{ikR\cos(\theta-\varphi)}=2\pi\int_0^\infty dk\,J_0(kR)=\frac{2\pi}{R}, \tag{A5}$$

where $J_0(\xi)$ is the Bessel function of the first kind of zeroth order. Substituting Eq. (A5) into Eq. (A4) and the consequent result into Eq. (A2), we readily obtain Eq. (12).

APPENDIX B: TRANSVERSE AND LONGITUDINAL COMPONENTS OF FUNDAMENTAL *X* WAVES

Inserting Eq. (13) into the first of Eqs. (8) we obtain

$$E_x^{(n)}(\mathbf{r}_\perp,Z)=\frac{E_0s^{n+1}}{2\pi(i\eta)^n}\frac{\partial^n}{\partial Z^n}\int d^2\mathbf{k}_\perp\frac{e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}}{|\mathbf{k}_\perp|}e^{-(\sigma-i\eta Z)|\mathbf{k}_\perp|}$$

$$\equiv\frac{E_0s^{n+1}}{2\pi(i\eta)^n}\frac{\partial^n}{\partial Z^n}T(\mathbf{r}_\perp,Z,\sigma). \tag{B1}$$

In order to evaluate the integral defining T, we use polar coordinates, according to $\mathbf{k}_\perp=k(\hat{\mathbf{e}}_x\cos\theta+\hat{\mathbf{e}}_y\sin\theta)$ and $\mathbf{r}_\perp=r(\hat{\mathbf{e}}_x\cos\varphi+\hat{\mathbf{e}}_y\sin\varphi)$, thus getting

$$T(\mathbf{r}_\perp,Z,\sigma)=2\pi\int_0^\infty dk\,e^{-(\sigma-i\eta Z)k}J_0(kr), \tag{B2}$$

where the representation $J_0(\xi)=(1/2\pi)\int_0^{2\pi}d\theta\exp[i\xi\cos(\theta-\varphi)]$ for the the Bessel function of the first kind of order 0 has been exploited. The integral appearing in Eq. (B2) is the Laplace transform of the Bessel function, namely $\int_0^\infty dke^{-sk}J_0(kr)=(s^2+r^2)^{-1/2}$, whose convergence domain is $\mathrm{Re}(s)>0$ (the square root being evaluated on its main branch). Equation (B2) thus yields

$$T(\mathbf{r}_\perp,Z,\sigma)=2\pi\frac{1}{[(\sigma-i\eta Z)^2+r^2)]^{1/2}}, \tag{B3}$$

which, inserted into Eq. (B1), furnishes the transverse part of the electric field as in Eq. (14). In order to derive the expression for the longitudinal component, let us substitute Eq. (13) into the second of Eqs. (8). For $n\geqslant 1$ we get

$$E_z^{(n)}(\mathbf{r}_\perp,Z)=-\frac{E_0s^{n+1}}{2\pi(i\eta)^n}\frac{\partial}{\partial x}\frac{\partial^{n-1}}{\partial Z^{n-1}}$$

$$\times\int d^2\mathbf{k}_\perp\frac{e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}}{|\mathbf{k}_\perp|}e^{-(\sigma-i\eta Z)|\mathbf{k}_\perp|}. \tag{B4}$$

Comparing this expression with Eq. (B1) we readily get the second of Eqs. (15). For $n=0$ we obtain, after passing to polar coordinates,

$$E_z^{(0)}(\mathbf{r}_\perp,Z)=-\frac{E_0s}{2\pi\eta}\int_0^\infty dke^{-(\sigma-i\eta Z)k}$$

$$\times\int_0^{2\pi}d\theta e^{ikr\cos(\theta-\varphi)}\cos\theta. \tag{B5}$$

The integral over θ can be performed by exploiting the Anger-Jacobi relation $\exp[ikr\cos(\theta-\varphi)]=\sum_{n=-\infty}^{n=\infty}i^nJ_n(kr)\exp[in(\theta-\varphi)]$ thus obtaining

$$E_z^{(0)}(\mathbf{r}_\perp,Z)=-\frac{E_0s}{i\eta}\cos\varphi\int_0^\infty dke^{-(\sigma-i\eta Z)k}J_1(kr). \tag{B6}$$

Equation (B6) becomes the first of Eqs. (15) after recalling that $\cos\varphi=x/r$ and that the Laplace transform of $J_1(kr)$ can be evaluated through the relation

$$\int_0^{+\infty}dke^{-sk}J_1(kr)=\frac{1}{r}\frac{(s^2+r^2)^{1/2}-s}{(s^2+r^2)^{1/2}}. \tag{B7}$$

APPENDIX C: DERIVATION OF THE VECTOR GAUSSIAN *X* WAVE

Substituting Eq. (17) into Eqs. (8) and passing to polar coordinates we obtain

$$E_x(\mathbf{r}_\perp,Z)=\frac{E_0\sigma^2}{2\pi}\int_0^\infty dk\,ke^{i\eta kZ}e^{-k^2\sigma^2/2}\int_0^{2\pi}d\theta e^{ikr\cos(\theta-\varphi)},$$

$$E_z(\mathbf{r}_\perp,Z)=-\frac{E_0\sigma^2}{2\pi\eta}\int_0^\infty dk\,ke^{i\eta kZ}e^{-k^2\sigma^2/2}$$

$$\times\int_0^{2\pi}d\theta e^{ikr\cos(\theta-\varphi)}\cos\theta. \tag{C1}$$

The integrals over θ can be performed by again exploiting the Anger-Jacobi relation thus obtaining

$$E_x(\mathbf{r}_\perp, Z) = E_0 \sigma^2 \int_0^\infty dk\, k e^{i\eta k Z} e^{-\frac{k^2\sigma^2}{2}} J_0(kr),$$

$$E_z(\mathbf{r}_\perp, Z) = \frac{E_0\sigma^2}{i\eta} \cos\varphi \int_0^\infty dk\, k e^{i\eta k Z} e^{-k^2\sigma^2/2} J_1(kr), \tag{C2}$$

where $J_n(\xi)$ is the Bessel function of the first kind of order n. The above integrals cannot be analytically evaluated. However we con convert both of them into two series after noting that $\exp(i\eta kZ)=\Sigma_{n=0}^{\infty}(i\eta kZ)^n/n!$ which, inserted into Eqs. (C2), yields

$$E_x(\mathbf{r}_\perp, Z) = E_0\sigma^2 \sum_{n=0}^{+\infty} \frac{(i\eta Z)^n}{n!} \int_0^\infty dk\, k^{n+1} e^{-k^2\sigma^2/2} J_0(kr),$$

$$E_z(\mathbf{r}_\perp, Z) = \frac{E_0\sigma^2}{i\eta}\cos\varphi \sum_{n=0}^{+\infty} \frac{(i\eta Z)^n}{n!} \times \int_0^\infty dk\, k^{n+1} e^{-k^2\sigma^2/2} J_1(kr). \tag{C3}$$

Both the integrals appearing in these equations can be analytically evaluated from the relation

$$\int_0^\infty dt t^{\mu-1} e^{-p^2t^2} J_\nu(at) = \frac{\Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}\right)}{2p^\mu \Gamma(\nu+1)} \left(\frac{a}{2p}\right)^\nu {}_1F_1\left(\frac{\nu}{2}+\frac{\mu}{2}, \nu+1, -\frac{a^2}{4p^2}\right), \tag{C4}$$

valid for $\mathrm{Re}(p)>0$ and $\mathrm{Re}(\mu+\nu)$ [20]. Here $\Gamma(\xi)$ is the Euler Gamma function and ${}_1F_1(\alpha,\beta;\xi)$ is the hypergeometric confluent function defined by

$${}_1F_1(\alpha,\beta;\xi) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \frac{\xi^n}{n!}, \tag{C5}$$

where $(\gamma)_0=1$ and, for $n \geqslant 1$, $(\gamma)_n = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)$ (Pocchammer symbol). Exploiting Eq. (C4), and the relation $\cos\varphi = x/r$, Eqs. (C3) become Eqs. (18). The expression for the transverse part of the field on the axis $\mathbf{r}_\perp=\mathbf{0}$ can be explicitly evaluated. From the first of Eqs. (C2) we obtain

$$E_x(0,Z) = \frac{E_0\sigma^2}{i\eta} \frac{\partial}{\partial Z} \int_0^\infty dk\, e^{i\eta k Z} e^{-k^2\sigma^2/2}. \tag{C6}$$

The integral appearing in this expression can be evaluated through the relation

$$\int_0^{+\infty} dt\, e^{-(at^2+2bt+c)} = \frac{1}{2}\sqrt{\frac{\pi}{a}} e^{(b^2-ac)/a} \operatorname{erfc}\left(\frac{b}{\sqrt{a}}\right), \tag{C7}$$

valid for $\mathrm{Re}(a)>0$, where $\operatorname{erfc}=(2/\sqrt{\pi})\int_\xi^\infty du \exp(-u^2)$ is the error function. Exploiting this integral, Eq. (C6) becomes Eq. (19).

APPENDIX D: DERIVATION OF EQ. (22)

Since the integral in Eq. (21) is not always convergent, we replace it with the expression

$$G(\mathbf{R}_\perp|Z) = \frac{1}{(2\pi)^2 i\eta} \int d^2\mathbf{k}_\perp \frac{e^{i\mathbf{k}_\perp\cdot\mathbf{R}_\perp}}{|\mathbf{k}_\perp|} e^{-(\epsilon - i\eta Z)|\mathbf{k}_\perp|}, \tag{D1}$$

where $\epsilon>0$ and the limit $\epsilon\to 0^+$ is meant to be taken at the end of the calculation. The introduction of the factor $\exp(-\epsilon|\mathbf{k}_\perp|)$ inside the integral follows a standard regularization procedure, usually employed to deal with distributions. Note that, from the definition of the field T in Appendix B [see Eq. (B1)], Eq. (D1) can be rewritten as

$$G(\mathbf{R}_\perp|Z) = \frac{1}{(2\pi)^2 i\eta} T(\mathbf{r}_\perp, Z, \epsilon). \tag{D2}$$

Therefore, inserting Eq. (B3) into Eq. (D2) we obtain Eq. (22).

[1] J. Durnin, J.J. Miceli, and H.H. Eberly, Phys. Rev. Lett. **58**, 1499 (1987).

[2] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).

[3] J. Lu and J.F. Greenleaf, IEEE Trans. Ultrason. Ferroelectr. Freq. Control **37**, 438 (1990).

[4] J. Lu and J.F. Greenleaf, IEEE Trans. Ultrason. Ferroelectr. Freq. Control **39**, 19 (1992).

[5] J. Lu, IEEE Trans. Ultrason. Ferroelectr. Freq. Control **44**, 181 (1997).

[6] J. Salo, J. Fagerholm, A.T. Friberg, and M.M. Salomaa, Phys. Rev. Lett. **83**, 1171 (1999).

[7] E. Recami, Physica A **252**, 586 (1998).

[8] P. Saari, http://xxx.lanl.gov/abs/physics/0103054.

[9] P. Saari and K. Reivelt, Phys. Rev. Lett. **79**, 4135 (1997).

[10] I. Alexeev, K.Y. Kim, and H.M. Michberg, Phys. Rev. Lett. **88**, 073901 (2002).

[11] D. Mugnai, A. Ranfagni, and R. Ruggeri, Phys. Rev. Lett. **84**, 4830 (2000).

[12] J. Lu, H. Zou, and J.F. Greenleaf, IEEE Trans. Ultrason. Fer-

roelectr. Freq. Control **42**, 850 (1995).
[13] J. Fagerholm, A.T. Friberg, J. Huttunen, D.P. Morgan, and M.M. Salomaa, Phys. Rev. E **54**, 4347 (1996).
[14] P.R. Stephanishen and J. Sun, J. Acoust. Soc. Am. **102**, 3308 (1997).
[15] J.F.J. Salo, A.T. Friberg, and M.M. Salomaa, Phys. Rev. E **62**, 4261 (2000).
[16] P. Saari and H. Sonajalg, Laser Phys. **7**, 32 (1997).
[17] M. Zamboni-Rached, E. Recami, and H.E. Hernandez-Figueroa, Eur. Phys. J. D **21**, 217 (2002).
[18] A. Ciattoni, C. Conti, and P.D. Porto, J. Opt. Soc. Am. A **21**, 451 (2004).
[19] A. Ciattoni, B. Crosignani, and P.D. Porto, Opt. Commun. **177**, 9 (2000).
[20] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Gordon and Breach Science, New York, 1986).