Recurrent scattering of spontaneous radiation from a randomly occupied optical lattice

Wei Guo*
Department of Physics and Astronomy, The University of New Mexico, Albuquerque, New Mexico 87131, USA

(Received 21 July 2003; published 19 March 2004)

The problem of recurrent scattering of spontaneous radiation from a randomly occupied optical lattice is discussed herein theoretically. The intensity of the scattered radiation is calculated and averaged over the random distribution of atoms trapped inside the lattice. From the intensity of the scattered radiation, two properties of random lasing are derived. First, the random lasing's strongest oscillation has a wavelength comparable to the lattice constant and, second, its emission spectrum is discontinuous. In the present paper, a quantity, the number of multiple scattering channels based on the long-range regularity possessed by an optical lattice, is defined. This quantity is found to significantly determine the magnitudes of the mean intensity of the scattered radiation and the wave decay constant.

DOI: 10.1103/PhysRevE.69.036602 PACS number(s): 42.25.Dd, 42.25.Fx, 46.65.+g

I. INTRODUCTION

In a random medium, when optical gain is present, spontaneous radiation has a better chance of being rescattered back to the atom from which it was emitted; therefore, a closed-loop path of light can be formed and then functions much like a conventional cavity in an ordinary laser [1]. In such a medium, only those oscillation frequencies that ensure the phase shift along a closed loop being equal to a multiple of 2π are selected. Clearly, multiple scattering of spontaneous radiation is the physical origin of the observed laserlike radiation from random media (also known as random lasing) [2,3]. Although this effect was predicted by Letokhov in 1968 [4], the current research on this subject was largely stimulated by the first observation of laserlike emission from a laser dye solution containing microparticles by Lawandy *et al.* [2]. At present, most theoretical formulations describing such a phenomenon rely on the diffusion approximation [5,6] or the Monte Carlo simulation [7]. In both methods, the phase of the light wave is neglected. The random media used in experiments or modeled after theoretical calculations are usually assumed to be continuous; that is, the constituents of the media are allowed to reach any point in the space claimed by the media.

In addition to continuous random media, there are other types of random media, such as optical lattices, that possess a long range of regularity. An optical lattice is a periodic arrangement of atoms confined in place by laser beams and is the direct result of atom cooling [8]. The randomness of an optical lattice comes from the often uncontrollable distribution of trapped atoms among lattice sites. Previous papers [9] predicted that the light scattering from an optical lattice has many properties which had not been identified from the studies of light scattering from a continuous random medium. Additionally, it was revealed through experiments that amplified spontaneous emission gradually transits to random lasing while the density of scattering particles increases beyond the order of $10^{11}\mathrm{cm}^{-3}$ [3]. It is commonly known that a typical near-resonance optical lattice [10] is usually sparsely occupied with an average occupation fraction around 10% [11]; yet even for such an optical lattice, the average density of the atoms trapped inside can easily surpass the preceding density in a continuous medium as the lattice constants of a near-resonance optical lattice are only of the order of the half wavelength of the lattice-forming light beams. Therefore, it seems that it is not only more likely to generate random lasing from an optical lattice, but also possible to observe new effects from the random lasing.

Since a thorough understanding of random lasing from an optical lattice requires systematic analysis of recurrent scattering of spontaneous radiation from the same lattice, the latter problem is addressed in the present paper by calculating the intensity of spontaneous radiation emitted from and recurrently scattered by all of the atoms trapped inside an optical lattice. The multiple scattering approach is used so that the phase of radiation at each scattering step can be calculated accurately. After the intensity of radiation is averaged over the random distribution of the trapped atoms inside the optical lattice, some properties of random lasing should be able to be obtained from it. Obviously, if each lattice site of an optical lattice is occupied by an atom (achievable in a far-off-resonance optical lattice [11]), then this lattice ceases to be a random medium. In the present paper, only a near-resonance optical lattice is considered, and unless stated, all optical lattices are assumed to be near-resonance optical lattices. It is also assumed that the lattice is a simple cubic lattice with a lattice constant a. The characteristics of a near-resonance optical lattice, including the motion of atoms [12–15], the long range of regularity [16–18], and the number of localized atoms [16,18] are frequently studied by analyzing light scattered from the lattice.

As light propagates inside an optical lattice, its amplitude decays as a result of the destructive multiple scattering by the trapped atoms. The regular distribution of these atoms in space permits a definition of a number of multiple scattering channels (NMSC), which quantitatively represents the scattering ability of the lattice on a wave traveling inside the lattice. The present paper illustrates that both the light decay constant and the intensity of recurrently scattered spontaneous radiation depend on this quantity. It is a well-known fact that when the wavelength of a wave is long enough, the

*Electronic address: weiguoguo@hotmail.com

wave is unable to discern the structure of the medium in which it travels. By examining the numerical value of NMSC, one can quantitatively determine when the regular structure of an optical lattice is invisible to a wave. From its definition, it can be inferred that NMSC is a general concept that applies whenever light propagation is inside a discrete medium with a long range of regularity.

Since the presence of optical gain in a random medium primarily serves to increase the chance that recurrent scattering occurs while not shedding much light on the nature of light propagation inside an optical lattice, such a light-enhancement mechanism is not included in this analysis. Essentially, a stationary scattering formulation in which some dynamic aspects of light propagation inside an active medium such as laser spiking [5,7] and spectral narrowing [19] simply do not appear is presented.

The low occupation of a near-resonance optical lattice makes it a good approximation to ignore the near- and intermediate-zone fields when the recurrent scattering process is formulated. Support for this approximation comes from the fact that in a light multiple scattering process it is the accumulated phase rather than the polarization of light that controls many interesting scattering phenomena (including enhanced back-scattering [20,21] and photon localization [22]) in a random medium. It has been observed that if the incident light is circularly polarized then the results obtained from the scalar- and vector-field approaches are nearly identical [23]. Therefore, in this formulation, a scalar-field approximation is used. The same approximation has also been employed to describe enhanced back scattering of light from a random medium [20,21].

The paper is structured as follows. In Sec. II the problem is formulated, the electric field is evaluated at an observation point in the form of a multiple scattering series, and the definition of NMSC is introduced. The intensity of the recurrently scattered radiation is then averaged over the randomness of the occupation of the lattice in Sec. III. In the following section, the mean intensity of radiation is graphically exhibited and then it is established that light propagation inside an optical lattice can be characterized by NMSC. Two properties of random lasing from an optical lattice are derived in Sec. IV. The results are summarized in Sec. V.

II. DECOMPOSITION OF RADIATION FIELDS

For simplicity, the atoms trapped inside an optical lattice are treated as identical electric dipoles characterized by an isotropic polarizability α and a diminishing physical size. Since the total field $E(\mathbf{r}_0)$ at an observation position $\mathbf{r}_0$ is formed by the fields emitted either directly from the atoms or first emitted from the atoms and then multiply scattered by the same atoms, the total field can be expressed in the following integral equation in Gaussian units

$$E(\mathbf{r}_0)=\int G_0(\mathbf{r}_0,\mathbf{r}_1)n(\mathbf{r}_1)d\mathbf{r}_1 + 4\pi k_0^2\alpha\int G_0(\mathbf{r}_0,\mathbf{r}_1)n(\mathbf{r}_1)E(\mathbf{r}_1)d\mathbf{r}_1, \tag{1}$$

where $d\mathbf{r}=dxdydz$ is the volume element. The kernel $G_0(\mathbf{r},\mathbf{r}_1)$ is the Green function corresponding to the differential operator $\nabla^2+k_0^2$, and represents a diverging spherical wave radiated by an atom at $\mathbf{r}_1$ and observed at $\mathbf{r}$:

$$G_0(\mathbf{r},\mathbf{r}_1)=\frac{e^{ik_0|\mathbf{r}-\mathbf{r}_1|}}{4\pi|\mathbf{r}-\mathbf{r}_1|}.$$

The spontaneous radiation from a point dipole at position $\mathbf{r}_1$ is simulated approximately with the Green function $G_0(\mathbf{r},\mathbf{r}_1)$ in this formulation, as the first term on the right-hand side (RHS) of Eq. (1) illustrates. The validity of this approximation comes from the knowledge that the uncertainty of an excited state is usually small compared to the energy separation between the excited state and the ground state. Therefore, the spontaneous radiation can be regarded as monochromatic. A common factor representing the amplitude of spontaneous radiation has been dropped from the RHS of Eq. (1) for simplicity. The randomness of the lattice is described by each variable β_i, which takes on the value 1 when the ith site is occupied and 0 when that site is vacant. By denoting the average occupation fraction of each site by β_0, the microscopic number density of the atoms $n(\mathbf{r})$ can be separated into an average occupation part n_0 and a fluctuation part δn [9]. Moreover, a linear-integral-operator method can be used to express $E(\mathbf{r}_0)$ as a series in ascending powers of the density fluctuations [9]:

$$\begin{aligned}E(\mathbf{r}_0)=&I_0(\mathbf{r}_0;\mathbf{k}_0)+\mu\int I_2(\mathbf{r}_0,\mathbf{r}_1)\,\delta n(\mathbf{r}_1)I_3(\mathbf{r}_1;\mathbf{k}_0)d\mathbf{r}_1\\&+\mu^2\int I_2(\mathbf{r}_0,\mathbf{r}_1)\,\delta n(\mathbf{r}_1)d\mathbf{r}_1\int I_1(\mathbf{r}_1,\mathbf{r}_2)\,\delta n(\mathbf{r}_2)\\&\times I_3(\mathbf{r}_2;\mathbf{k}_0)d\mathbf{r}_2+\cdots,\end{aligned}\tag{2}$$

where μ is used to replace $4\pi k_0^2\alpha$ to simplify notation.

The first term on the RHS of Eq. (2), I_0, reads

$$I_0(\mathbf{r}_0;\mathbf{k}_0)=\beta_0\Bigg[\sum_i G_0(\mathbf{r}_0,\mathbf{R}_i) + \mu\beta_0\sum_{i\neq j} G_0(\mathbf{r}_0,\mathbf{R}_i)G_0(\mathbf{R}_i,\mathbf{R}_j)+\cdots\Bigg], \tag{3}$$

where $\{\mathbf{R}_i\}$ represents the entire set of lattice sites. Under typical experimental conditions [13], the deviation of an atom from the lattice site to which it is trapped is extremely small; by writing I_0 and, therefore, $E(\mathbf{r})$, as an explicit function of lattice sites, built into Eq. (2) is the assumption that the atoms are tightly trapped at the lattice sites. This assumption is known as the Lamb-Dicke limit. Moreover, it has been proven experimentally that the multiple scattering effect is insignificant unless the atoms are so closely trapped to the lattice sites that the Debye-Waller factor is nonvanishing [18]. Equation (3) exhibits how the spontaneous radiation from each atom trapped inside an evenly occupied lattice with an average occupation of β_0 has to be multiply scattered by all the atoms before being added to form a resultant field at the observational position $\mathbf{r}_0$. Throughout this paper, it is assumed that the surface of the optical lattice facing the optical detector is the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane and $\hat{\mathbf{z}}$ is in the direction away from the detector. Since the observation position is usually distant from the lattice sample, the far-zone approximation can be applied to transform the propagator G_0 in each final scattering step in Eq. (3) into a plane wave. The series in Eq. (3) subsequently represents the formal multiple scattering of a plane wave with a wave vector $\mathbf{k}_s \equiv -k_0\mathbf{r}_0/r_0$ by the uniformly occupied lattice and can be formally resummed [9] as

$$I_0(\mathbf{r}_0;k_0) \simeq \beta_0 \frac{e^{ik_0r_0}}{4\pi r_0}\Lambda(\mathbf{k}_s)e^{ik_{s\perp x}(N_x-1)a/2+ik_{s\perp y}(N_y-1)a/2} \times \frac{\sin(k_{s\perp x}N_x a/2)}{\sin(k_{s\perp x}a/2)}\frac{\sin(k_{s\perp y}N_y a/2)}{\sin(k_{s\perp y}a/2)} \times \frac{1-e^{[i\mathrm{Re}\beta(\mathbf{k}_s)-\mathrm{Im}\beta(\mathbf{k}_s)]aN_z}}{1-e^{[i\mathrm{Re}\beta(\mathbf{k}_s)-\mathrm{Im}\beta(\mathbf{k}_s)]a}}, \tag{4}$$

where N_x, N_y, and N_z are the numbers of sites in the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions, respectively,

$$\Lambda(\mathbf{k}_s) \equiv \frac{k_0}{1-w_1s'}\frac{1+(s''w_1N_b)/(1-w_1s')}{\sqrt{k_0^2+(w_1N_b)/(1-w_1s')}},$$

and $\beta(\mathbf{k}_s)=\sqrt{k_{sz}^2+w_1N_b/(1-w_1s')}$. Other quantities appearing in $\Lambda(\mathbf{k}_s)$ and $\beta(\mathbf{k}_s)$ are defined in Ref. [9] and omitted for simplicity. In expression (4), the notation $\mathbf{k}_{s\perp}$ is used to represent the projection of $\mathbf{k}_s$ on the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane.

In Ref. [9], the entire spatial frequency space was classified into two groups: $\{\mathbf{K}_B\}$ consisting of those reciprocal vectors $\mathbf{K}$'s for which $|\mathbf{k}_1+\mathbf{K}|\sim k_0$, and the rest $\{\mathbf{K}_G\}$, where $\mathbf{k}_1$ is any vector in the fundamental Brillouin zone. It was noted that the vectors $\mathbf{K}+\mathbf{k}_1$, with $\mathbf{K}$ in the former group represent all possible wave vectors that can result from the scattering of radiation with wave vector $\mathbf{k}_1$ from a regular lattice of atoms, while satisfying Bragg scattering conditions. The number of Brillouin zones in $\{\mathbf{K}_B\}$ is the number of multiple scattering channels and is denoted in this paper as N_b. Since, in Ref. [9], both the wavelength and the lattice constant were chosen to be constants, N_b became a fixed number and its effects on light propagation inside an optical lattice were not revealed. In the present paper, the properties of NMSC are discussed in detail in Sec. IV when the mean intensity of spontaneous radiation is analyzed. In a one-dimensional situation, $\{G_z\}$, the counterpart of $\{\mathbf{K}_G\}$ can be defined as those one-dimensional Brillouin zones which do not satisfy the condition $|G_z|\sim k_{sz}$.

Between two successive scattering events induced by the density fluctuations, the radiation field travels inside an evenly occupied lattice. Such a process, one example of which is sandwiched between two density fluctuations in the second line in Eq. (2), is denoted as $I_1(\mathbf{r}_1,\mathbf{r}_2)$, and reads

$$I_1(\mathbf{r}_1,\mathbf{r}_2)=G_0(\mathbf{r}_1,\mathbf{r}_2)+\mu\beta_0\sum_i G_0(\mathbf{r}_1,\mathbf{R}_i)G_0(\mathbf{R}_i,\mathbf{r}_2)+\cdots, \tag{5}$$

which can be summed up into closed form [9]

$$I_1(\mathbf{r},\mathbf{r}_1)=s_0(k_0)\frac{e^{[i\mathrm{Re}\alpha(k_0)-\mathrm{Im}\alpha(k_0)]|\mathbf{r}-\mathbf{r}_1|}}{|\mathbf{r}-\mathbf{r}_1|}, \tag{6}$$

where $s_0=1/[4\pi(1-w_1s')^2]$ and $\alpha(k_0)=\sqrt{k_0^2+(w_1N_b)/(1-w_1s')}$. Equation (6) illustrates that a wave usually gets its wave vector renormalized after being multiply scattered and the degree of renormalization, as noted from the expression of $\alpha(k_0)$, is largely dictated by NMSC. In Sec. IV, the dependence of the decay constant $2\mathrm{Im}\alpha(k_0)$ on N_b is discussed.

After leaving a density fluctuation, the radiation has to be scattered by the evenly occupied lattice before arriving at $\mathbf{r}_0$. This multiple scattering process, represented by $I_2(\mathbf{r}_0,\mathbf{r}_1)$ in Eq. (2), can also be summed up [9] as

$$I_2(\mathbf{r}_0,\mathbf{r}_1)=\frac{e^{ik_0r_0}}{4\pi r_0}\Lambda(\mathbf{k}_s)e^{i\mathbf{k}_\perp\cdot\mathbf{r}_i}e^{i[\mathrm{Re}\beta(\mathbf{k}_s)+i\mathrm{Im}\beta(\mathbf{k}_s)]z_i}. \tag{7}$$

One outcome of separating the number density $n(\mathbf{r})$ into an average part n_0 and a fluctuation part $\delta n(\mathbf{r})$ is that the spontaneous radiation from each trapped atom formally travels inside a uniform optical lattice before it encounters a density fluctuation. Since the atom at the location of the density fluctuation also radiates, the total radiation emitted from a density fluctuation should, resultingly, include the radiation which originated from the fluctuation, plus the radiation approaching the fluctuation. This sum, designated as $I_3(\mathbf{r};k_0)$, can be expressed as a multiple scattering series:

$$I_3(\mathbf{r};k_0)=\frac{1}{\mu}\left[1+\mu\beta_0\sum_i I_1(\mathbf{r},\mathbf{R}_i)\right]. \tag{8}$$

Since a typical near-resonance optical lattice is sparsely occupied with an occupation fraction around 10%, the dependence of $I_3(\mathbf{r},k_0)$ on the location of density fluctuation $\mathbf{r}$ is not sensitive, and I_3, can, therefore, be treated as a constant. For this reason, I_3 is henceforth referred to as uniform radiation. In this approximation, the summation in expression (8) is calculated in the Appendix, leading I_3 to the following expression after μ^{-1} is dropped

$$I_3(\mathbf{r};k_0)\propto 1+\mu\beta_0 s(k_0)k_0A(2\pi^2)^{-1}. \tag{9}$$

With the physical meanings of I_0, I_1, I_2, and I_3 explained, it is clear that spontaneous radiation from a randomly occupied optical lattice has to be alternatively scattered by the uniform lattice and the density fluctuations before being measured.

III. MEAN INTENSITY OF RECURRENTLY SCATTERED RADIATION

The mean intensity of spontaneous radiation recurrently scattered from the lattice and observed at $\mathbf{r}_0$, denoted as I_{Re}, is obtained by multiplying $E(\mathbf{r}_0)$ (2) with its complex conjugate and then averaging the product over the density fluctuations. When averaging the intensity over the density fluctuations, the general expression of the density fluctuation correlation function [9] has to be specialized to include only those configurations for which the phase of the light in each multiple-scattering term of Eq. (2) is exactly canceled by the phase of the complex conjugate of that term. To achieve this requirement, a field and its complex conjugate can only travel along the same path or the time-reversed path. Therefore, only even-order moments of density fluctuations are needed in the correlation function. It is also required that the initial scattering steps and the last scattering steps occur at the same density fluctuation positions so that closed loops can be formed. What is implied in the second requirement is that four density fluctuations should share the same lattice site. In the general expression of the density fluctuation correlation function, the density fluctuations are grouped according to a permutation notation $P_{\nu,\kappa,\varrho,\ldots}$, which indicates a permutation of a set of ν lattice sites an atom can reside in, a different set of κ lattice sites an atom can reside in, and so on, until all n positions are exhausted. With the above-mentioned specialization requirements in mind, the permutation notation needs to be specialized to $P_{4,2,2,\ldots}$, so that the nth-order density fluctuation correlation function becomes

$$\begin{aligned}\langle \delta n(\mathbf{r}_1)\cdots\delta n(\mathbf{r}_n)\rangle &= \langle(\beta-\beta_0)^4\rangle[\beta_0(1-\beta_0)]^{(n-4)/2}\sum_{\{P\}} P_{4,2,2,\ldots}\\ &\times \sum_{i\neq j\neq k\neq \cdots =1}^{N} \delta(\mathbf{r}_1-\mathbf{R}_i)\,\delta(\mathbf{r}_2-\mathbf{R}_i)\\ &\times \delta(\mathbf{r}_3-\mathbf{R}_i)\,\delta(\mathbf{r}_4-\mathbf{R}_i)\,\delta(\mathbf{r}_5-\mathbf{R}_j)\\ &\times \delta(\mathbf{r}_6-\mathbf{R}_j)\,\delta(\mathbf{r}_7-\mathbf{R}_k)\,\delta(\mathbf{r}_8-\mathbf{R}_k)\cdots, \end{aligned}\tag{10}$$

where N denotes the total number of the lattice sites. With the help of Eq. (10), after dropping a common factor $|\mu|^6\langle(\beta-\beta_0)^4\rangle[\beta_0(1-\beta_0)]$ and defining $C_0=\beta_0(1-\beta_0)$, it can be seen that

$$\begin{aligned} I_{\mathrm{Re}} \propto &\sum_{i\neq j} |I_2(\mathbf{r}_0,\mathbf{R}_i)|^2|I_3(\mathbf{R}_i;\mathbf{k}_0)|^2|I_1(\mathbf{R}_i,\mathbf{R}_j)|^4+2C_0|\mu|^2\\ &\times \sum_{i\neq j,j\neq l,l\neq i} |I_2(\mathbf{r}_0,\mathbf{R}_i)|^2|I_3(\mathbf{R}_i;\mathbf{k}_0)|^2|I_1(\mathbf{R}_i,\mathbf{R}_j)|^2\\ &\times|I_1(\mathbf{R}_j,\mathbf{R}_l)|^2|I_1(\mathbf{R}_l,\mathbf{R}_i)|^2+2C_0^2|\mu|^4\\ &\times \sum_{i\neq j,j\neq l,l\neq m,m\neq i} |I_2(\mathbf{r}_0,\mathbf{R}_i)|^2|I_3(\mathbf{R}_i;\mathbf{k}_0)|^2|I_1(\mathbf{R}_i,\mathbf{R}_j)|^2\\ &\times|I_1(\mathbf{R}_j,\mathbf{R}_l)|^2|I_1(\mathbf{R}_l,\mathbf{R}_m)|^2|I_1(\mathbf{R}_m,\mathbf{R}_i)|^2+\cdots. \end{aligned}\tag{11}$$

Since I_0 depends critically on the observation direction [refer to Eq. (4)] in any experiment, by a suitable specification of the observation direction $\hat{\mathbf{k}}_s$, it is possible to make the magnitude of I_0 null. For this reason, the contribution from I_0 is excluded in the calculation of I_{Re}. Also used in the derivation of the expression (11) is the assumption that light travels along each closed loop only once.

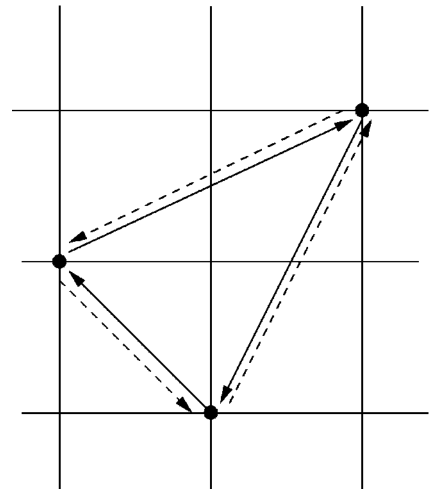

FIG. 1. The third-order recurrent scattering diagram, where solid and dashed arrows indicate, respectively, a light path and its time-reversed version. Scatterers are represented by closed circles.

It is noted that the first term on the RHS of Eq. (11) represents the second-order recurrent scattering event in which the spontaneous radiation, when arriving at an atom at $\mathbf{R}_i$, is first emitted from $\mathbf{R}_i$ to another atom at $\mathbf{R}_j$, gets scattered back to $\mathbf{R}_i$ from $\mathbf{R}_j$, and is then transmitted to the detector from $\mathbf{R}_i$. The following terms in Eq. (11) represent higher-order scattering events. Refer to Fig. 1 for an illustration of the third-order recurrent scattering event. Since the waves traveling on either a path or its time-reversed path contribute equally to the measured light intensity, the multiplier 2 appears in each of the third- and higher-order recurrent scattering events on the RHS of Eq. (11) to account for this effect. If recurrent scattering events are ignored, the time-reversed paths produce so-called cross terms in a theoretical discussion of light scattering from random media. These cross terms are responsible for the enhanced back scattering of light, no matter if the random media have optical gain [24] or not [9,20,21].

From the expression I_1 (6), the sums of I_1's in Eq. (11) can be transformed into integrals in the spatial frequency domain. Three of these sums are listed as follows:

$$\sum_{i\neq j}\frac{e^{-2\alpha_m(k_0)R_{ij}}}{R_{ij}^4}=\frac{1}{4\pi^4}\left(\frac{2\pi}{a}\right)^3\int_{\Omega_3} g^2(\mathbf{k})\,d\mathbf{k},\tag{12}$$

$$\begin{aligned}\sum_{i\neq j,j\neq l,l\neq i}&\frac{e^{-\alpha_m(k_0)R_{ij}}}{R_{ij}^2}\frac{e^{-\alpha_m(k_0)R_{jl}}}{R_{jl}^2}\frac{e^{-\alpha_m(k_0)R_{li}}}{R_{li}^2}\\ &=\frac{1}{8\pi^6}\left(\frac{2\pi}{a}\right)^6\int_{\Omega_3} g^3(\mathbf{k})\,d\mathbf{k},\end{aligned}\tag{13}$$

and

$$\begin{aligned}\sum_{i\neq j,j\neq l,l\neq m,m\neq i}&\frac{e^{-\alpha_m(k_0)R_{ij}}}{R_{ij}^2}\frac{e^{-\alpha_m(k_0)R_{jl}}}{R_{jl}^2}\frac{e^{-\alpha_m(k_0)R_{lm}}}{R_{lm}^2}\\ &\times\frac{e^{-\alpha_m(k_0)R_{mi}}}{R_{mi}^2}=\frac{1}{16\pi^8}\left(\frac{2\pi}{a}\right)^9\int_{\Omega_3} g^4(\mathbf{k})\,d\mathbf{k},\end{aligned}\tag{14}$$

where $\alpha_m(k_0)=2\,\mathrm{Im}\,\alpha(k_0)$ is defined. The notation $g(\mathbf{k})$ (appearing in the preceding expressions) is defined [9] as

$$g(\mathbf{k})=\sum_{\mathbf{G}}\frac{1}{|\mathbf{k}-\mathbf{G}|}\arctan\frac{|\mathbf{k}-\mathbf{G}|}{\alpha_m(k_0)}-\left(\frac{a}{2\pi}\right)^3\int d\mathbf{k}_g\frac{1}{k_g}\arctan\frac{k_g}{\alpha_m(k_0)}, \qquad (15)$$

where $\{\mathbf{G}\}$ is used to represent the set of three-dimensional reciprocal lattice vectors. Integration over the fundamental Brillouin zone in the three-dimensional spatial frequency space is symbolized by subscript Ω_3 following an integration sign.

From the expressions (12) to (14), it is found that the series (11), when the first term is excluded, is actually a geometrical series in the spatial frequency domain. Based on this observation, I_{Re} yields:

$$I_{Re}\propto|\Lambda(\mathbf{k}_s)|^2|I_3|^2|s(k_0)|^4a^{-3}\frac{1-e^{-2\,\mathrm{Im}\beta(\mathbf{k}_s)aN_z}}{1-e^{-2\,\mathrm{Im}\beta(\mathbf{k}_s)a}}\times\int_{\Omega_3}d\mathbf{k}\left[g^2(\mathbf{k})+\frac{C_0|\mu|^2|s(k_0)|^2 8\pi a^{-3}g^3(\mathbf{k})}{1-C_0|\mu|^2|s(k_0)|^2 4\pi a^{-3}g(\mathbf{k})}\right], \qquad (16)$$

where all proportionality constants that do not depend on a are ignored. The expression of I_{Re} is complicated and a numerical treatment inevitably needs further analysis, which is presented in Sec. IV.

IV. GRAPHICAL REPRESENTATION AND DISCUSSION

When light is scattered from an optical lattice, sidebands are generated in the scattered light because of the harmonic oscillation of the atoms in the potential wells that hold the atoms to the lattice sides [14]. However, under the assumption of Lamb-Dicke limit adopted in this paper the frequency of the scattered light from an atom becomes identical to that of the light incident on the same atom. Therefore, the process of recurrent scattering of spontaneous radiation can be treated as a process of resonant scattering; that is, an incident wave can have a frequency identical to the internal oscillation frequency of the atoms. The polarizability α of the atoms accordingly becomes

$$\alpha=\frac{3i}{2k_0^3}. \qquad (17)$$

After the radiation wavelength is fixed at $\lambda_0=2\pi/k_0$, the average radiation intensity is measured as a function of the lattice constant a in the direction normal to the lattice surface: $\mathbf{k}_s=(0,0,-k_0)$. This specific choice of the observation direction makes $\beta(\mathbf{k}_s)=\alpha(k_0)$. To make the analysis as close to experimental realities as possible, $\beta_0=0.1$ is selected and I_{Re} is plotted from $a=1.03\lambda_0$ because the approximation method used to derive the expressions of I_0, I_1,

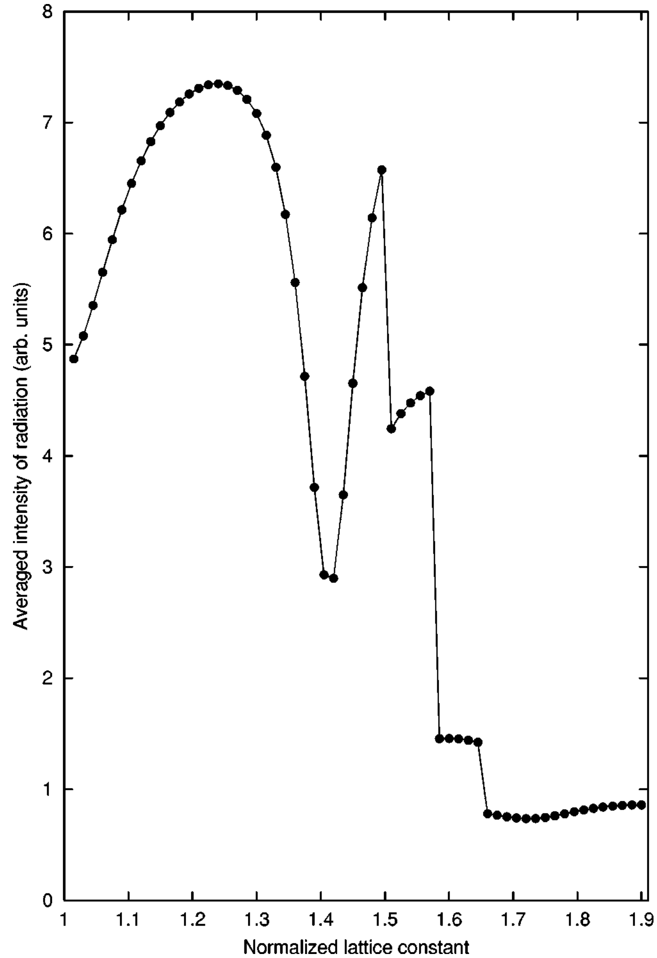

FIG. 2. Averaged intensity of recurrently scattered spontaneous radiation I_{Re} (in arbitrary units) against the normalized lattice constant a/λ_0. The data points are represented by closed circles.

and I_2 in Sec. II works best when $\lambda_0 \ll a$ [9]. The number of the lattice sites in $\hat{z}$ direction N_z is set to 80.

In Fig. 2 it is noted that the intensity I_{Re} initially increases with the lattice constant a to its first maximum around $a=1.24\lambda_0$, and then decreases to its minimum, followed by another increase to its secondary maximum. The variation of I_{Re} with a in this region is well matched by the variation of $|I_3|^2$ with a; see Fig. 3. This variation is understandable since $|I_3|^2$ by definition, determines the intensity of radiation immediately before a recurrent scattering process begins. On the other hand, the fact that the variation of the uniform radiation I_3 (8) with the lattice constant occurs is expected since I_3 is formed as superposition of spontaneous radiation waves that have traveled through an evenly occupied lattice from the emitting atoms. Therefore, I_3 must rely on the phases of these waves to determine its own magnitude. The variation of $|I_3|^2$ in the following region is also reflected in that of I_{Re}.

Still, in Fig. 2, it is easy to determine that there are three positions, $a=1.50\lambda_0$, $a=1.59\lambda_0$, and $a=1.65\lambda_0$, at which the magnitude of I_{Re} plummets and the curve of I_{Re} becomes discontinuous. To explain this phenomenon, one needs to realize that in a recurrent scattering process, it is the decay

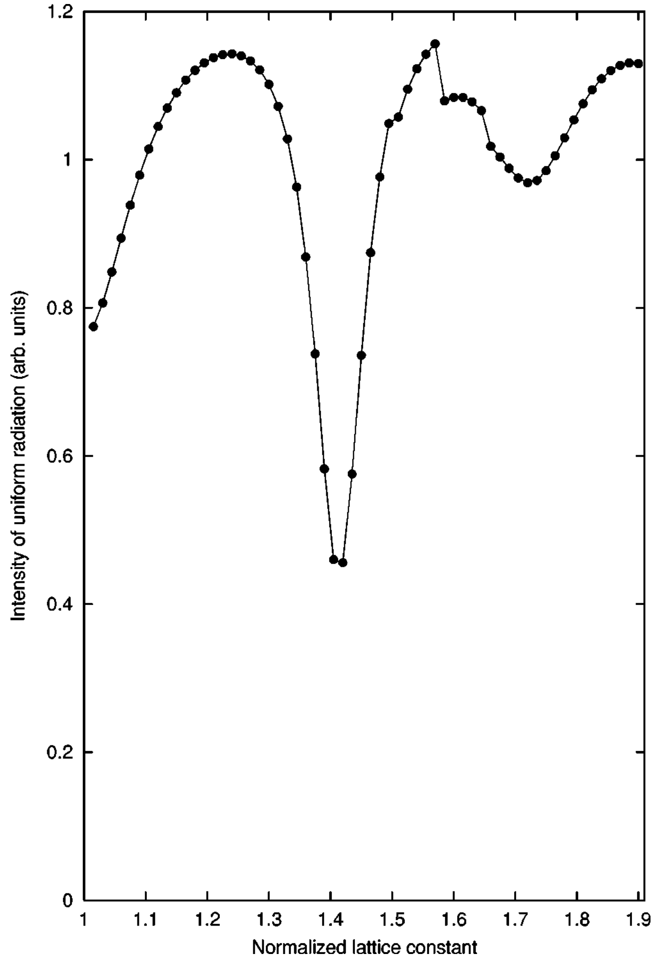

FIG. 3. Intensity of uniform radiation $|I_3|^2$ (in arbitrary units) against the normalized lattice constant a/λ_0. The data points are represented by closed circles.

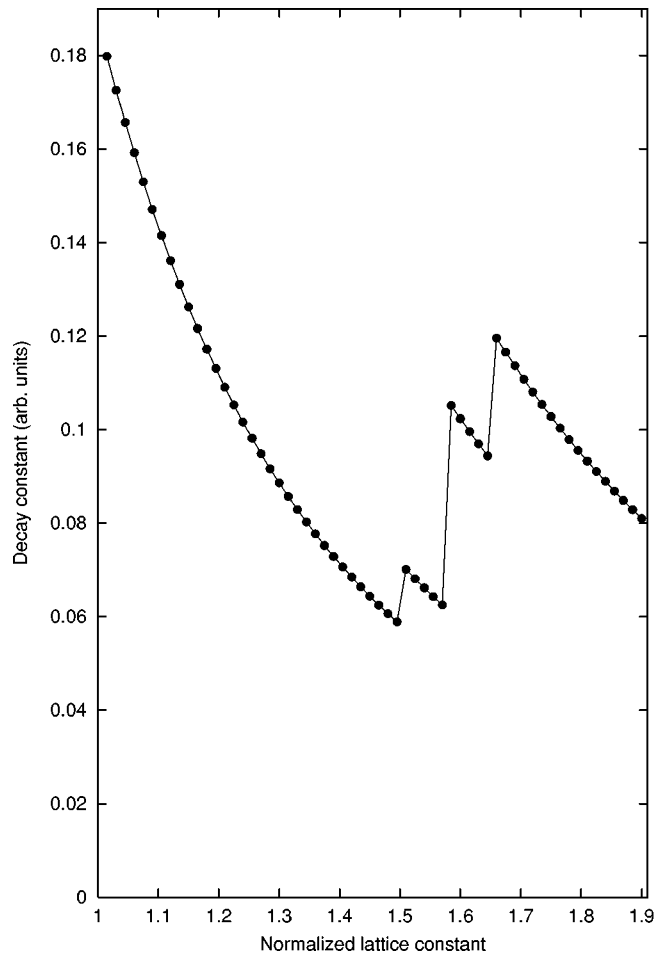

FIG. 4. Decay constant $2\mathrm{Im}\alpha$ (in arbitrary units) against the normalized lattice constant a/λ_0. The data points are represented by closed circles.

constant that determines the strength of the radiation scattered back to the atom from which the radiation was previously scattered. The phase of the radiation Re $\alpha(k_0)$ in Eq. (6) is canceled by its complex conjugate and does not play an essential role in determining the intensity of radiation scattered back to the starting point of a recurrent scattering process. An examination of the decay constant $2\mathrm{Im}\alpha$ shows, in Fig. 4, that at $a=1.50\lambda_0$, $a=1.59\lambda_0$, and $a=1.65\lambda_0$, the decay constant is swiftly raised. Therefore, the dramatic reduction of the magnitude of I_{Re} can be regarded as a direct result of the sudden increment of $2\mathrm{Im}\alpha$. Additionally, since the decay constant represents the rate at which energy is taken away from the radiation as it propagates, the three sudden increments in $2\mathrm{Im}\alpha$ clearly mean that, at these positions, the radiation experiences much stronger scattering.

To understand the physics behind this change of scattering strength, one has to realize that near $a=1.50\lambda_0$, $a=1.59\lambda_0$, and $a=1.65\lambda_0$, the number N_b also changes rapidly as Fig. 5 shows: its value jumps first from 26 to 32, then from 32 to 56, and, finally, from 56 to 74. Since N_b represents the number of a wave's dominant multiple scattering channels that satisfy the Bragg scattering conditions in a lattice and are formed because of the regularity of the lattice, it can, in some way, characterize the scattering ability of an optical lattice on the wave traveling in the optical lattice. The increase in the magnitude of N_b means that the radiation will be scattered more intensively by the lattice resulting in more energy taken away from the radiation. Consequently, the energy loss will produce a larger decay constant. In brief, it is NMSC that controls the main features of light propagation inside an optical lattice.

Since its definition is only based on the Bragg scattering conditions and the regularity of the medium, it can be inferred that NMSC is, in fact, a general concept that applies as long as wave propagation in a medium with regularity and discreteness (not merely an optical lattice) is concerned. It is interesting to compare a continuous medium with an optical lattice. In a continuous medium, since all nontrivial values of reciprocal vectors may be considered infinitely large, there is only one scattering channel available. On the other hand, in an optical lattice, when $\lambda_0>2a$, the vector $\mathbf{K}=0$ also becomes the only element in $\{\mathbf{K}_B\}$ since k_0 is already in the fundamental Brillouin zone. Physically, this means if a wave has a wavelength larger than $2a$, then the discrete structure of the lattice is invisible to the wave. In general, for a continuous medium, $N_b=1$; for an optical lattice, $N_b\geqslant 1$. One can state when $N_b=1$, a wave cannot distinguish a continu-

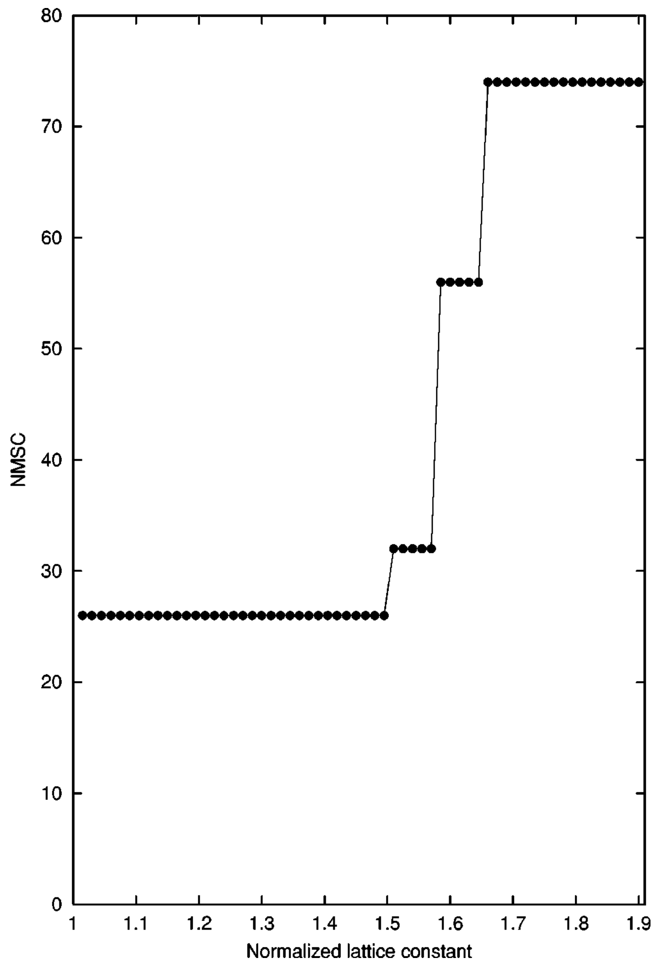

FIG. 5. The number of multiple scattering channels N_b as a function of the normalized lattice constant a/λ_0. The data points are represented by closed circles.

ous medium from an optical lattice; yet when $N_b > 1$, it can. The stepwise dependence of N_b on λ_0 and a can be understood to be caused by the finite size of a Brillouin zone. Either λ_0 or a has to change a nonzero amount in magnitude before, in the spatial frequency space, a different Brillouin zone can be reached.

Since random lasing is closely related to the recurrent scattering of spontaneous radiation, two properties of random lasing from an optical lattice can be inferred which are based on the properties of I_{Re} discussed in this section. It is worth noting that even though optical gain is excluded in this analysis requiring the spontaneous radiation with a specific frequency $\omega_0 = k_0 c$ to form a closed loop makes the formulation equivalent to an experimental reality in that the frequency ω_0 is close to the maximum of the gain spectra. Otherwise, the spontaneous radiation would have a negligible chance of being rescattered back to its starting point.

(1) From Fig. 2, one can infer that the strongest laser oscillation from an optical lattice will have a wavelength close to $0.8a$. This result is dramatically different from random lasing from a continuous medium where the strongest laser oscillation is noted to occur at a wavelength which is significantly smaller than the average interparticle separation [3].

(2) Also from Fig. 2, a spectral analysis of random lasing from an optical lattice reveals a discontinuous spectrum where the discontinuities are caused by N_b increments. Since, in a continuous medium, N_b is always equal to the value 1, the random lasing spectrum from a continuous medium is noted to be continuous [3].

V. CONCLUSION

In this paper, the recurrent scattering of spontaneous radiation from a randomly occupied optical lattice was discussed by following a density fluctuation approach. After averaging the intensity of the recurrently scattered radiation over the random distribution of the atoms trapped inside the lattice, it was found that if random lasing is generated on an optical lattice, its strongest oscillation will have a wavelength comparable to the lattice constant and its spectrum will no longer be continuous. It was also noted that both the wave decay constant and the mean intensity of radiation depend on the value of NMSC. Clearly, the NMSC studied in this paper is relevant to additional phenomena regarding wave propagation inside a medium with regularity and discreteness.

APPENDIX: SUMMATION OF I_3

By using the Fourier transform relation

$$\frac{e^{i\alpha(k_0)R_{ij}}}{4\pi R_{ij}} = \frac{1}{8\pi^3}\int d\mathbf{k}_1 \frac{e^{i\mathbf{k}_1\cdot(\mathbf{R}_i-\mathbf{R}_j)}}{k_1^2-\alpha^2(k_0)}, \tag{A1}$$

the serial representation of I_3 (12) becomes

$$I_3(\mathbf{R}_i;k_0) \propto 1 + \frac{\beta_0 \mu s(k_0)}{2\pi^2}\sum_{i\neq j}\int d\mathbf{k}_1 \frac{e^{i\mathbf{k}_1\cdot(\mathbf{R}_i-\mathbf{R}_j)}}{k_1^2-\alpha^2(k_0)}, \tag{A2}$$

where a proportionality constant μ^{-1} has been dropped. A term with $i=j$ can be added to and subtracted from Eq. (A2), so that the sum in the equation can be expressed in terms of unrestricted sum over all lattice sites:

$$I_3(\mathbf{R}_i;k_0) \propto 1 + \frac{\beta_0 \mu s(k_0)}{2\pi^2}\int d\mathbf{k}_1 \frac{e^{i\mathbf{k}_1\cdot\mathbf{R}_i}}{k_1^2-\alpha^2(k_0)} \times\left[\sum_j e^{-i\mathbf{k}_1\cdot\mathbf{R}_j} - e^{-i\mathbf{k}_1\cdot\mathbf{R}_i}\right]. \tag{A3}$$

The use of the Poisson summation formula

$$\sum_i e^{i\mathbf{k}\cdot\mathbf{R}_i} = \left(\frac{2\pi}{a}\right)^3 \sum_{\mathbf{G}} \delta(\mathbf{k}-\mathbf{G}), \tag{A4}$$

where $\mathbf{G}$'s are the reciprocal lattice vectors simplifies the RHS of Eq. (A3) into a form involving an integral over δ functions. By evaluating this integral, the following expression for I_3 may be derived:

$$I_3(\mathbf{R}_i;k_0) \propto 1 + \frac{\beta_0 \mu s(k_0) k_0}{2\pi^2}\left[\left(\frac{2\pi}{ak_0}\right)^3 \sum_{n_x,n_y,n_z} \frac{1}{\left(\frac{2\pi}{ak_0}\right)^2 (n_x^2+n_y^2+n_z^2) - \left(\frac{\alpha(k_0)}{k_0}\right)^2} - 4\pi \int_0^\infty ds \frac{s^2}{s^2 - \left(\frac{\alpha(k_0)}{k_0}\right)^2}\right]$$
$$\equiv 1 + \beta_0 \mu s(k_0) k_0 A (2\pi^2)^{-1}. \tag{A5}$$

[1] Since the propagation of fluorescent light is diffusive in a random medium, a random laser does not possess well-defined cavity modes while a conventional laser does. Therefore, a random laser usually has a sizable β factor. See G. van Soest and A. Lagendijk, Phys. Rev. E **65**, 047601 (2002).

[2] N.M. Lawandy, R.M. Balachandran, A.S.L. Gomes, and E. Sauvian, Nature (London) **368**, 436 (1994).

[3] H. Cao, J.Y. Xu, S.-H. Chang, and S.T. Ho, Phys. Rev. E **61**, 1985 (2000).

[4] V.S. Letokhov, Sov. Phys. JETP **26**, 835 (1968).

[5] D.S. Wiersma and D.A. Lagendijk, Phys. Rev. E **54**, 4256 (1996).

[6] S. John and G. Pang, Phys. Rev. A **54**, 3642 (1996).

[7] G.A. Berger, M. Kempe, and A.Z. Genack, Phys. Rev. E **56**, 6118 (1997).

[8] P.S. Jessen and I.H. Deutsch, Adv. At., Mol., Opt. Phys. **39**, 95 (1996).

[9] W. Guo and S. Prasad, Phys. Rev. E **67**, 056614 (2003); **67**, 056615 (2003).

[10] G. Grynberg and C. Mennerat-Robilliard, Phys. Rep. **355**, 335 (2001).

[11] A much larger occupation fraction can usually be obtained on far-off-resonance optical lattices. See M. T. DePue, C. McCormick, S. Lukman Winoto, S. Oliver, and D. S. Weiss, Phys. Rev. Lett. **82**, 2262 (1999); S. Lukman Winoto, M. T. DePue, N.E. Bramall, and David S. Weiss, Phys. Rev. A **59**, R19 (1999); V. Vuletić, C. Chin, A.J. Kerman, and S. Chu, Phys. Rev. Lett. **81**, 5768 (1998).

[12] F.-R. Carminati, L. Sanchez-Palencia, M. Schiavoni, F. Renzoni, and G. Grynberg, Phys. Rev. Lett. **90**, 043901 (2003).

[13] C.I. Westbrook, C. Jurczak, G. Birkl, B. Desruelle, W.D. Phillips, and A. Aspect, J. Mod. Opt. **44**, 1837 (1997).

[14] M. Gatzke, G. Birkl, P.S. Jessen, A. Kastberg, S.L. Rolston, and W.D. Phillips, Phys. Rev. A **55**, R3987 (1997).

[15] G. Grynberg, B. Lounis, P. Verkerk, J.-Y. Courtois, and C. Salomon, Phys. Rev. Lett. **70**, 2249 (1993).

[16] M. Weidemüller, A. Görlitz, Theodor W. Hänsch, and A. Hemmerich, Phys. Rev. A **58**, 4647 (1998).

[17] M. Weidemüller, A. Hemmerich, A. Görlitz, T. Esslinger, and T.W. Hänsch, Phys. Rev. Lett. **75**, 4583 (1995).

[18] G. Birkl, M. Gatzke, I.H. Deutsch, S.L. Rolston, and W.D. Phillips, Phys. Rev. Lett. **75**, 2823 (1995).

[19] G. Zacharakis, G. Heliotis, G. Filippidis, D. Anglos, and T.G. Papazoglou, Appl. Opt. **38**, 6087 (1999).

[20] L. Tsang and A. Ishimaru, J. Opt. Soc. Am. A **2**, 1331 (1985); M.B. van der Mark, M.P. van Albada, and A. Lagendijk, Phys. Rev. B **37**, 3575 (1988).

[21] W. Guo and S. Prasad, Opt. Commun. **212**, 1 (2002).

[22] D.S. Wiersma, P. Bartolini, A. Lagendijk, and R. Righini, Nature (London) **390**, 671 (1997).

[23] F.C. Mackintosh and S. John, Phys. Rev. B **37**, 1884 (1988).

[24] W. Deng, D.S. Wiersma, and Z.-Q. Zhang, Phys. Rev. B **56**, 178 (1997); P.C. de Oliveira, A.E. Perkins, and N.M. Lawandy, Opt. Lett. **21**, 1685 (1996); D.S. Wiersma, M.P. van Albada, and A. Lagendijk, Phys. Rev. Lett. **75**, 1739 (1995).