# Three-dimensional chaotic flows with discrete symmetries

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There are a number of well-known three-dimensional flows with quadratic nonlinearities, which demonstrate a chaotic behavior. The most popular among them are Lorenz and Rössler systems. Using an exhaustive computer search, J. Sprott found 19 examples of chaotic flows with either five terms and two quadratic nonlinearities or six terms and one nonlinearity [Phys. Rev. E **50**, R647 (1994)]. In contrast to this approach, we use symmetry-related considerations to construct types of chaotic flows with an arbitrary dimension. The discussion is based on our previous work devoted to nonlinear dynamics of the physical systems with discrete symmetries [see Physica D **117**, 43 (1998), etc.]. Here, we present all possible chaotic flows with quadratic nonlinearities which are invariant under the action of 32 point groups of crystallographic symmetry. These systems demonstrate a typical chaotic behavior as well as general dynamical properties of nonlinear systems with discrete symmetries. In particular, we found a dynamical system with the point symmetry group $D_2$ which seems to be more simple and more elegant than those by Lorenz and Rössler.

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## I. INTRODUCTION

The theory of nonlinear dynamics of physical systems with discrete symmetries was developed in a number of our previous papers [1–7]. In particular, the fundamental concept of *bushes* of normal modes, which represent *exact* nonlinear excitations in such systems, was introduced in Ref. [1]. Some theorems about the structure of the bushes of modes were stated and proved in Ref. [2]. Group-theoretical methods, based on the theory of irreducible representations of the symmetry groups, were developed in Refs. [1,2]. Bushes of vibrational modes of small dimensions for different classes of discrete symmetry (for all point groups of crystallographic symmetry, for some space groups, for the fullerene $C_{60}$, etc.) were obtained in Refs. [1,3,4]. All ''irreducible'' bushes of modes and symmetry determined similar nonlinear normal modes, introduced by Rosenberg [8], for all mechanical systems with any of 230 space groups, were found in Ref. [5]. The *stability* of the bushes of vibrational modes for the Fermi-Pasta-Ulam chains and for simple octahedral systems with Lennard-Jones potential was investigated in Refs. [6,7].

All of the above-mentioned papers treat the cases of *regular* motion and, as a rule, bushes of vibrational modes only, while this paper is devoted to a discussion of *chaotic* motion in some nonlinear systems with discrete symmetry. Here we consider the class of three-dimensional flows with quadratic nonlinearities. This is precisely the class to which the well-known Lorenz and Rössler systems belong. These two systems demonstrate a complex behavior; in particular, they possess strange (chaotic) attractors for some *range* of their pertinent parameters. The Lorenz system is characterized by seven terms and two quadratic nonlinearities on the right hand side of the corresponding differential equations:

$$\dot{x}=-\sigma x+\sigma y,\quad \dot{y}=rx-xz-y,\quad \dot{z}=xy-bz.$$

The Rössler system possesses seven terms and only one quadratic nonlinearity:

$$\dot{x}=-y-z,\quad \dot{y}=x+ay,\quad \dot{z}=b-cz+xz.$$

In Ref. [9], Sprott posed the question: ''Are there any three-dimensional systems of autonomous ordinary differential equations with one (as in the Rössler case) or two (as in the Lorenz case) quadratic nonlinearities and fewer then seven terms whose solutions are chaotic?'' Using a direct computer search, Sprott found 19 chaotic flows of this type, which appear to be distinct in the sense that there is no obvious transformation of one to another. In this search, Sprott exploited the idea of the ''algebraic simplicity'' of differential equations, and his systems indeed look more simple than those by Lorenz and Rössler. But the simplest case of the considered systems was published later [10]. It can be written as

$$\dot{x}=y,\quad \dot{y}=z,\quad \dot{z}=-az\pm y^2-x. \tag{1}$$

Some additional information on the discussed subject can be found in a review paper [11].

Note that many systems obtained with the aid of the above idea seem to be rather exotic. Indeed, the chaotic behavior in such systems is often observed for quite narrow regions of their intrinsic parameters [for example, the chaos for system (1) occurs only for the interval $2.0168<a<2.0577$], and the basins of the appropriate strange attractors are relatively small.

In contrast to the search based on the idea of algebraic simplicity, we implement a search based on symmetry-related methods. In this paper, we discuss chaotic flows associated with point groups of crystallographic symmetry. In spite of the fact that these groups, as well as space groups, act in three-dimensional Euclidean space, *multidimensional* dynamical systems can be associated with them, if one uses their matrix representations. Below, we present the general

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TABLE I. Irreducible representations of the group $D_2$.

| Irreps | Symmetry elements | | | |
|---|---|---|---|---|
| | $E$ | $C_2^x$ | $C_2^y$ | $C_2^z$ |
| $\Gamma_1$ | 1 | 1 | 1 | 1 |
| $\Gamma_2$ | 1 | 1 | $-1$ | $-1$ |
| $\Gamma_3$ | 1 | $-1$ | 1 | $-1$ |
| $\Gamma_4$ | 1 | $-1$ | $-1$ | 1 |

method for constructing $N$-dimensional flows associated with group representations, but consider its realization only for the case $N=3$.

## II. DYNAMICAL SYSTEMS INVARIANT UNDER REPRESENTATIONS OF SYMMETRY GROUPS

We consider dynamical systems invariant under the transformations induced by the representations (reducible and irreducible) of the different groups of discrete symmetry. This approach corresponds to the well-known proposition that if a given physical system is characterized by a certain symmetry group $G$ [12], its properties are described by variables which transform according to the representations of the group $G$. In the present paper, we discuss the case where $G$ is one of 32 point groups of crystallographic symmetry.

Let us consider an $N$-dimensional dynamical system described by $N$ variables $\mu_j(t)$:

$$\dot{\mu}_j=f_j(\mu_1,\mu_2,\ldots,\mu_N),\quad j=1,2,\ldots,N. \tag{2}$$

We fix the dimension $N$ of system (2) and construct all different $N$-dimensional representations (in general, they are reducible) of the given group $G$. This procedure is indeed possible since every representation $\Gamma$ of the group $G$ can be written as the direct sum of a number of its irreducible representations (irreps) $\Gamma_i$

$$\Gamma=\sum^{\oplus}\Gamma_i,\quad \dim\Gamma=N, \tag{3}$$

and because all $\Gamma_i$ of the point groups of crystallographic symmetry are well known (see, for example, Ref. [13]). Note that one and the same irrep $\Gamma_i$ can be contained in the direct sum (3) several times. Then we choose, by turn, each of the above representations and demand our dynamical system to be invariant under the transformations of its variables $\mu_j(t)$ $(j=1,2,\ldots,N)$ induced by this representation.

Let us illustrate the above procedure with the following example. We consider the point group $D_2$ (the Schoenflies notations are used throughout) consisting of four symmetry elements: $E$ (identity element) and $C_2^x$, $C_2^y$, $C_2^z$, which are the 180° rotations about the $x$, $y$, $z$ axes, respectively. Being Abelian, the group $D_2$ possesses four one-dimensional irreps listed in Table I.

Let us construct all possible three-dimensional (3D) *reducible* representations of the group $D_2$ by combining the irreps from Table I into the appropriate direct sums (3). There are 20 variants, but many of them turn out to be equal

TABLE II. Images of the three-dimensional representations of the group $D_2$. In square brackets near the symbol $L_j$, the equivalent point group is indicated. All zero elements of the matrices are dropped.

$$L_2[C_s^z]=\begin{pmatrix}1&&\\&1&\\&&-1\end{pmatrix};\quad L_3[C_2^x]=\begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix};$$

$$L_4[C_i]=\begin{pmatrix}-1&&\\&-1&\\&&-1\end{pmatrix};$$

$$L_5[C_{2v}^x]=\begin{pmatrix}1&&\\&1&\\&&-1\end{pmatrix},\begin{pmatrix}1&&\\&-1&\\&&1\end{pmatrix},\begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix};$$

$$L_6[C_{2h}^z]=\begin{pmatrix}1&&\\&1&\\&&-1\end{pmatrix},\begin{pmatrix}-1&&\\&-1&\\&&1\end{pmatrix},\begin{pmatrix}-1&&\\&-1&\\&&-1\end{pmatrix};$$

$$L_7[C_{2h}^x]=\begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix},\begin{pmatrix}-1&&\\&1&\\&&1\end{pmatrix},\begin{pmatrix}-1&&\\&-1&\\&&-1\end{pmatrix};$$

$$L_8[D_2]=\begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix},\begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix},\begin{pmatrix}-1&&\\&-1&\\&&1\end{pmatrix};$$

to each other. For our purpose, we need to obtain only *images* of these representations, i.e., the matrix groups, corresponding to the representations, independent of the manner of correspondence between their matrices and the elements of the symmetry group. In other words, one must write out only *different* matrices of every representation without taking into account the order of their appearance in the representation. All different images of three-dimensional representations [14] of the group $G=D_2$ are given in Table II. In this table, we omit the identity matrices in all images $L_j(j=1,2,\ldots,8)$, as well as the identity image

$$L_1=\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$$

which does not pose any constraint on the coefficients of the dynamical system. Each of the images $L_2$, $L_3$, and $L_4$ has one generator which is the matrix explicitly indicated in Table II, while each of the other images ($L_5$, $L_6$, $L_7$, $L_8$) possesses two generators (the first two matrices may be chosen as such generators, because the last matrix is simply the product of the former matrices).

It is essential that we can ascribe a certain point symmetry group to each of the *three-dimensional* images (obviously, this is impossible for the case of $N>3$). Indeed, the three-dimensional matrices act on the vectors, say, $\vec{r}=(x,y,z)$, and we can consider every image from Table II as a *vector representation* of a certain point group. The action of such representation on the vector $\vec{r}$ is equivalent to the action on it by the corresponding symmetry group. For example, the matrix

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

acts on the arbitrary vector $\vec{r}$ as an *inversion* $(i)$, the matrix

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

acts on it as a 180° rotation about the $z$ axis, and the matrix

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

describes the reflection in the mirror plane normal to the $z$ axis (all these symmetry elements are determined with respect to the origin of coordinates). Because of this interpretation of the above matrices of the 3D images, it is easy to find the point symmetry groups corresponding to these images. In Table II, the Schoenflies symbol of the appropriate point group is indicated near each $L_j$.

Thus, analyzing the possible 3D representations of the group $D_2$, we obtained the vector representations of a few different symmetry groups:

$$C_s,\ C_2^x,\ C_i,\ C_{2v}^x,\ C_{2h}^x,\ C_{2h}^z,\ D_2. \tag{4}$$

This is not an accidental fact. Indeed, one can find all 3D transformations associated with point groups of crystallographic symmetry by writing out the vector representations of all such groups without using the above-described combinatorial procedure. Since this is impossible for the $N$-dimensional cases with $N>3$, let us return to a discussion of the images of the $N$-dimensional representations of the groups of discrete symmetry.

Note that the groups $C_{2h}^x$ and $C_{2h}^z$ from list (4) are equivalent in the crystallographic sense: they differ from each other only by the orientation of the symmetry elements (the two fold axes of these groups coincide with the $x$ and $z$ coordinate axes, respectively). Such equivalence is the consequence of the existence of a certain unitary transformation which connects the images associated with these groups [15]:

$$L_6=S^{\dagger}L_7S. \tag{5}$$

Here

$$S=\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$$

is the matrix corresponding to a 120° rotation about a space diagonal of the cube.

On the other hand, each image singles out a certain dynamical system which must be invariant under the transformations of the dynamical variables induced by this image. Two dynamical systems are called equivalent if there exists a transformation (linear, in our case) from the variables of the first system to the variables of the second system. Obviously, equivalent images generate equivalent dynamical systems. Therefore, we must leave only one copy of the set of equivalent images in the list of all different images. For this reason, we exclude the image $L_6[C_{2h}^x]$ from list (4).

In the general case, the elimination of the mutually equivalent images is a difficult task. Indeed, ordering of the matrices in each image is inessential and, consequently, we do not know which matrix of the first image may transform to a given matrix of the second image. Nevertheless, a similar problem for the images of the *irreducible* representations, in the framework of phase transition theory, was solved by Gufan and co-workers [16] and by Hatch and Stokes [17] for all 230 space groups.

There are 32 different 3D images for the 32 point groups of crystallographic symmetry, but it turns out that many of them do not induce dynamical systems which can demonstrate a chaotic behavior. Let us consider this question in more detail.

A definite transformation of the dynamical variables $x(t)$, $y(t)$, $z(t)$ corresponds to each matrix of every 3D image. For example, the first matrix of the image $L_8[D_2]$ (see Table II) generates the transformation

$$x\to x,\quad y\to -y,\quad z\to -z, \tag{6}$$

while the second matrix gives

$$x\to -x,\quad y\to y,\quad z\to -z. \tag{7}$$

The transformation induced by the third matrix is redundant since this matrix is equal to the product of the first two matrices of $L_8[D_2]$.

In the most general form, the 3D dynamical system with quadratic nonlinearities can be written as

$$\begin{aligned}
\dot{x}&=a_1+b_{11}x+b_{12}y+b_{13}z+c_{111}x^2+c_{112}xy\\
&\quad+c_{113}xz+c_{122}y^2+c_{123}yz+c_{133}z^2,\\
\dot{y}&=a_2+b_{21}x+b_{22}y+b_{23}z+c_{211}x^2+c_{212}xy\\
&\quad+c_{213}xz+c_{222}y^2+c_{223}yz+c_{233}z^2,\\
\dot{z}&=a_3+b_{31}x+b_{32}y+b_{33}z+c_{311}x^2+c_{312}xy\\
&\quad+c_{313}xz+c_{322}y^2+c_{323}yz+c_{333}z^2.
\end{aligned} \tag{8}$$

Let us demand that this system be invariant under transformation (6) and then under transformation (7). As a result, we obtain the following dynamical system with six arbitrary coefficients which cannot be found with the aid of the group-theoretical arguments:

$$\dot{x}=b_{11}x+c_{123}yz,$$
$$\dot{y}=b_{22}y+c_{213}xz, \tag{9}$$
$$\dot{z}=b_{33}z+c_{312}xy.$$

The number of these arbitrary coefficients can be reduced to two unknown coefficients by rescaling the dynamical variables *x*, *y*, and *z*, and time *t*. Thus, we must analyze the possibility of chaotic behavior of our dynamical system for all possible values of these two arbitrary coefficients only (see the next section).

As the second example, let us consider the image $L_4[C_i]$. Its single matrix determines the inversion:

$$x\to -x,\quad y\to -y,\quad z\to -z. \tag{10}$$

Performing this transformation on the general system (8), we see that quadratic terms do not change their signs, while the linear terms do change. Multiplying each transformed equation by $-1$ and comparing it with that before the substitution (10), we arrive at the conclusion that all coefficients of the quadratic terms must be zero. In other words, the resulting system turns out to be *linear*. Since chaotic behavior is impossible in linear dynamical systems, we must drop *all images* containing the inversion, such as $C_{2h}$ [see list (4)], $D_{2h}$, $D_{4h}$, $T_h$, $O_h$, etc., from the overall set of the 3D images.

## III. THREE-DIMENSIONAL DYNAMICAL SYSTEMS WITH CHAOTIC ATTRACTORS

Besides the linearity, there are other properties, which lead to the elimination of a given dynamical system from the list of candidates for chaotic behavior.

(1) We must exclude the so-called Onsager dynamical systems [18]:

$$\dot{x}_i=-\frac{\partial U}{\partial x_i}, \tag{11}$$

where $i=1,2,\ldots,N$, and $U(x_1,x_2,\ldots,x_N)$ is a function of all dynamical variables. Indeed, Eqs. (11) are equivalent to those for the continuous variant of the well-known numerical method of steepest descent. The function *U* from Eq. (11) cannot increase during such descent and the phase trajectory leads to a local minimum (in particular, it can be equal to $-\infty$), or to some subspaces of minima. No chaotic behavior can appear in such situation. For example, an Onsager system corresponds to the image [19]

$$L[T]=\begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix},\begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix},\begin{pmatrix}&1&\\&&1\\1&&\end{pmatrix}, \tag{12}$$

and to image $L[O]$ which differs from $L[T]$ by adding the fourth matrix:

$$\begin{pmatrix}&1&\\1&&\\&&1\end{pmatrix}.$$

One and the same dynamical system corresponds to both of the above images:

$$\dot{x}=b_{11}x+c_{123}yz,$$
$$\dot{y}=b_{11}y+c_{123}xz, \tag{13}$$
$$\dot{z}=b_{11}z+c_{123}xy.$$

This is indeed an Onsager system with $U(x,y,z)=-\frac{1}{2}b_{11}(x^2+y^2+z^2)-c_{123}xyz$.

(2) Some images induce systems of differential equations for which there exists a first integral of motion. The order of such a system can be reduced by 1, and the chaotic behavior turns out to be impossible according to the Poincaré-Bendixon theorem.

For example, the image $L[C_{4v}]$, as well as the image $L[C_{6v}]$ generates one and the same dynamical system [20]:

$$\dot{x}=b_{11}x+c_{113}xz,$$
$$\dot{y}=b_{11}y+c_{113}yz, \tag{14}$$
$$\dot{z}=a_3+b_{33}z+c_{311}(x^2+y^2)+c_{333}z^2.$$

It is easy to check that the value $I=x/y$ turns out to be the first integral of motion for Eq. (14) and, therefore, this system must be excluded from our consideration.

A similar analysis of all dynamical systems generated by the images of the 3D representations of the 32 point groups of crystallographic symmetry leads us to the list of *six* nontrivial candidates containing flows with chaotic attractors:

$$C_s, C_2, D_2, C_3, C_{3v}, S_4. \tag{15}$$

Actually, each image determines a certain *class* of dynamical system, since a number of arbitrary coefficients enters into the appropriate differential equations [see, for example, Eqs. (9), (13), and (14)]. A chaotic behavior appears only for *specific values* of these coefficients.

We use a numerical procedure similar to that by Sprott [9] for finding the coefficients of dynamical systems with chaotic attractors. That is, for each coefficient, a definite range $[-A,+A]$ and appropriate increment *h* are chosen. Then, for every site of the resulting grid, we employ the fourth-order Runge-Kutta integration procedure for the corresponding dynamical system, and select only systems with chaotic behavior as evidenced by a decidedly positive Lyapunov exponent.

Below, for every case, we present the image, the general form of the flow generated by this image, an example of the flow with the specific choice of the appropriate coefficients,

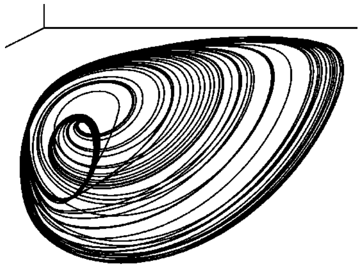


FIG. 1. Chaotic attractor for the $C_s$ system described by Eqs. (17).

and a picture of the corresponding chaotic attractor. Note that in all our pictures the initial transients are removed from the plots of the chaotic attractors.

### A. Point group $C_s$

*Image*:

$$\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

General form of the flow:

$$\begin{aligned}\dot{x}&=b_{11}x+c_{112}xy+c_{113}xz,\\ \dot{y}&=a_2+b_{22}y+b_{23}z+c_{211}x^2+c_{222}y^2+c_{223}yz+c_{233}z^2,\\ \dot{z}&=a_3+b_{32}y+b_{33}z+c_{311}x^2+c_{322}y^2+c_{323}yz+c_{333}z^2.\end{aligned} \tag{16}$$

*Example* of the flow (Fig. 1):

$$\begin{aligned}\dot{x}&=-xy-xz,\\ \dot{y}&=-2-z+x^2,\\ \dot{z}&=-1-x^2-yz.\end{aligned} \tag{17}$$

### B. Point group $C_2$

*Image*:

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

General form of the flow:

$$\begin{aligned}\dot{x}&=b_{11}x+b_{12}y+c_{113}xz+c_{123}yz,\\ \dot{y}&=b_{21}x+b_{22}y+c_{213}xz+c_{223}yz,\\ \dot{z}&=a_3+b_{33}z+c_{311}x^2+c_{312}xy+c_{322}y^2+c_{333}z^2.\end{aligned} \tag{18}$$

*Example* of the flow (Lorenz's system, Fig. 2):

$$\dot{x}=-\sigma x+\sigma y,$$

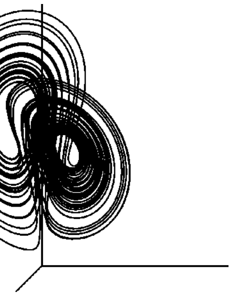


FIG. 2. Chaotic attractor for the $C_2$ system described by Eqs. (19) (Lorenz's system).

$$\dot{y}=rx-xz-y, \tag{19}$$

$$\dot{z}=xy-bz,$$

$$\sigma=10,\quad r=28,\quad b=8/3.$$

### C. Point group $D_2$

*Image*:

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix},\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}.$$

General form of the flow:

$$\begin{aligned}\dot{x}&=b_{11}x+c_{123}yz,\\ \dot{y}&=b_{22}y+c_{213}xz,\\ \dot{z}&=b_{33}z+c_{312}xy.\end{aligned} \tag{20}$$

*Example* of the flow (Fig. 3):

$$\begin{aligned}\dot{x}&=-4x+yz,\\ \dot{y}&=-y+xz,\\ \dot{z}&=z-xy.\end{aligned} \tag{21}$$

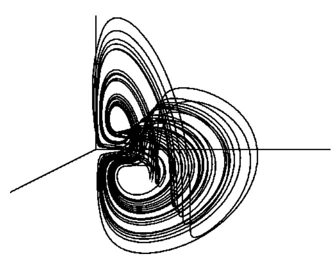


FIG. 3. Chaotic attractor for the $D_2$ system described by Eqs. (21).

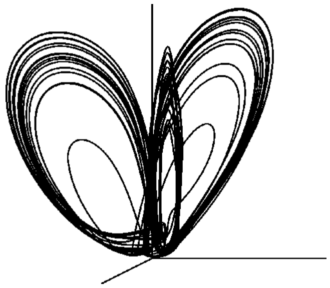


FIG. 4. Chaotic attractor for the $C_3$ system described by Eqs. (23).

### D. Point group $C_3$

*Image*:

$$\begin{pmatrix} -1/2 & \sqrt{3}/2 & \\ -\sqrt{3}/2 & -1/2 & \\ & & 1 \end{pmatrix}.$$

General form of the flow:

$$\dot{x}=b_{11}x+b_{12}y+c_{111}(x^2-y^2)+2c_{211}xy+c_{113}xz+c_{123}yz,$$
$$\dot{y}=-b_{12}x+b_{11}y+c_{211}(x^2-y^2)-2c_{111}xy-c_{123}xz+c_{113}yz, \tag{22}$$
$$\dot{z}=a_3+b_{33}z+c_{311}(x^2+y^2)+c_{333}z^2.$$

*Example* of the flow (Fig. 4):

$$\dot{x}=x-y+2xy-xz,$$
$$\dot{y}=x+y+(x^2-y^2)-yz, \tag{23}$$
$$\dot{z}=-z+(x^2+y^2).$$

### E. Point group $C_{3v}$

*Image*:

$$\begin{pmatrix} -1/2 & \sqrt{3}/2 & \\ -\sqrt{3}/2 & -1/2 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

General form of the flow:

$$\dot{x}=b_{11}x+c_{111}(x^2-y^2)+c_{113}xz,$$
$$\dot{y}=b_{11}y-2c_{111}xy+c_{113}yz, \tag{24}$$
$$\dot{z}=a_3+b_{33}z+c_{311}(x^2+y^2)+c_{333}z^2.$$

An *example* of such flow with chaotic behavior is unknown.

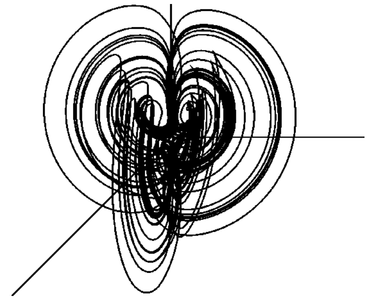


FIG. 5. Chaotic attractor for the $S_4$ system described by Eqs. (26).

### F. Point group $S_4$

*Image*:

$$\begin{pmatrix} & 1 & \\ -1 & & \\ & & -1 \end{pmatrix}.$$

General form of the flow:

$$\dot{x}=b_{11}x+b_{12}y+c_{113}xz+c_{123}yz,$$
$$\dot{y}=-b_{12}x+b_{11}y+c_{123}xz-c_{113}yz, \tag{25}$$
$$\dot{z}=b_{33}z+c_{311}(x^2-y^2)+c_{312}xy.$$

*Example* of the flow (Fig. 5):

$$\dot{x}=-2x+y-xz,$$
$$\dot{y}=-x-2y+yz, \tag{26}$$
$$\dot{z}=z+(x^2-y^2)+xy.$$

Note that among the above flows, there is a case with a $C_{3v}$ point group for which we could neither find any example of the chaotic behavior, nor prove that such a behavior is impossible for this type of dynamical system.

## IV. SOME GENERAL PROPERTIES OF THE CHAOTIC ATTRACTORS

Let us consider the very simple and elegant dynamical system (20) with point group $D_2$. We can turn four coefficients of this system to $\pm 1$ by rescaling each of the four variables $(x, y, z, t)$, and reduce our flow to the forms

$$\dot{x}=ax+yz,$$
$$\dot{y}=by+xz, \tag{27}$$
$$\dot{z}=z-xy.$$

Note that in this way, we can obtain some other forms of the same equations (20), for example, the form with the *positive* nonlinear term in the last equation in Eqs. (27), but such

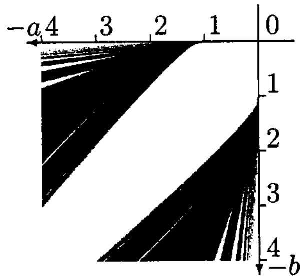



FIG. 6. Domain of the coefficients $a$ and $b$ for system (27) where chaotic behavior is possible.

a dynamical system happens to be of Onsager type with $U = -\frac{1}{2}(ax^2+by^2+z^2)-xyz$, and, therefore, it cannot demonstrate any chaotic behavior.

Our computer simulation shows that the chaotic behavior of system (27) appears for some *negative* values of both parameters: $a<0$, $b<0$. Example (21) belongs precisely to this two-parametric family of the flows with $D_2$ symmetry.

The domain of the $a-b$ plane where chaotic attractors of the system (27) can exist is represented in Fig. 6 by black color. Actually, this domain is strongly riddled with windows of periodicity where cascades of bifurcations are observed.

According to the general theory [2], different dynamical regimes in a nonlinear physical system with discrete symmetry group $G$ can be classified by the subgroups $G_j$ of this group ($G_j \subseteq G$). As a consequence, we expect that the chaotic attractors, as well as the ordinary attractors, in the above listed flows are associated with certain subgroups of the symmetry groups of their dynamical equations. Indeed, the chaotic attractor for the flows with $D_2$ symmetry [Eqs. (21)] turns out to be of $C_2$ symmetry (as can be seen from Fig. 3), and $C_2 \subset D_2$. We checked this fact not only visually, but also with the aid of the computer program, which utilized the following algorithm. Let us imagine that our attractor is completely located in a big cube which, in turn, is divided into a large number of little cubic cells. Then we integrate the dynamical system and store in each of the abovementioned cells the number of times the phase trajectory passes through it. The action of a given symmetry element $g$ on the cube transforms its cells into each other, and we can compare the numbers stored in the cells before and after the action of $g$. If these numbers coincide for all cells with good accuracy, one can conclude that $g$ is indeed a symmetry element of the chaotic attractor. Moreover, proceeding in such a way we can calculate the *probability* that $g$ is a symmetry element of our attractor. Certainly, the correct probability can be obtained only for the limit of infinite partitioning of the cube into the cells and for $t\to\infty$. However, it is possible, in principle, to estimate the probability of the fact that the chaotic attractor possesses the given symmetry group.

According to the general theory [1,2], the elements $C_2^x$ and $C_2^z$ of the parent group $D_2$, which disappear when the symmetry is lowered $D_2 \to C_2^y$, must generate the twins (or so-called "dynamical domains" [21]) of the dynamical regime with the subgroup $C_2^y$. In our case, the rotations $C_2^x$ and $C_2^z$ produce only one new twin of the chaotic attractor with the same symmetry group $C_2^y$. Both twins are depicted in Fig. 7. These twins possess different basins of attraction, and the phase trajectory leads to one or the other chaotic attractor twin, depending on the choice of initial conditions.

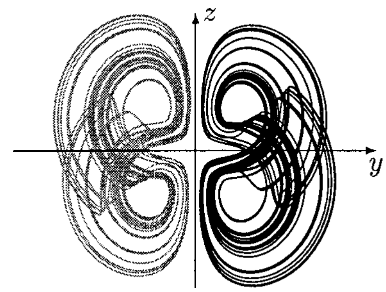



FIG. 7. Twins of the chaotic attractors for the dynamical system (21).

Let us note that system (27) possesses the following critical points (they are determined by the conditions $\dot{x}=0$, $\dot{y}=0$, $\dot{z}=0$):

(1) $(0,0,0)$, (4) $(-\mu,\nu,-\mu\nu)$,
(2) $(\mu,\nu,\mu\nu)$, (5) $(-\mu,-\nu,\mu\nu)$,
(3) $(\mu,-\nu,-\mu\nu)$,

where $\mu=\sqrt{-b}$, $\nu=\sqrt{-a}$. The first twin envelops the critical points (2) and (4), and the second twin envelops the critical points (3) and (5). The chaotic attractor of flow (23) with the parent symmetry group $G_0=C_3$ demonstrates precisely this full symmetry group (i.e., $G=G_0$, where $G$ is the symmetry group of the attractor), as it is evident from Fig. 4.

Another interesting phenomenon is demonstrated by flow (25) with the point symmetry group $S_4$. This system can be rewritten in a form with four arbitrary coefficients (parameters) by rescaling the variables $x$, $y$, $z$, and $t$, and one of the possible reduced forms is

$$\begin{aligned}\dot{x}&=ax+by+cxz+dyz,\\ \dot{y}&=-bx+ay+dxz-cyz,\\ \dot{z}&=z+xy,\end{aligned} \tag{28}$$

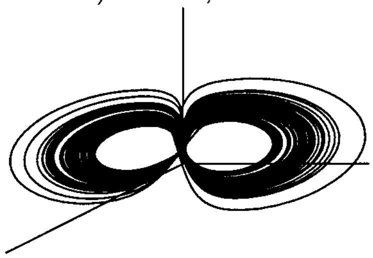
By setting $c=d=-1$, we come to the equations

$$\begin{aligned}\dot{x}&=ax+by-xz-yz,\\ \dot{y}&=-bx+ay-xz+yz,\\ \dot{z}&=z+xy.\end{aligned} \tag{29}$$

There is a number of domains in the $a-b$ plane to which *different* chaotic attractors correspond. Two such attractors for system (29) with $[a=-2,\ b=1]$ and $[a=-3,\ b=1]$ are shown in Fig. 8. The symmetry groups of these attractors are different subgroups of the symmetry group ($S_4$) of Eqs. (29) ($C_2$ and $S_4$, respectively).

Note that a *six-dimensional* dynamical system representing a certain coupling of two three-dimensional systems by Lorenz and Rössler was proposed in Ref. [22]. This system produces the chaotic attractors similar to those discovered by Lorenz and Rössler, depending on the values of its intrinsic

a) $a = -2$, $b = 1$.



b) $a = -3$, $b = 1$.

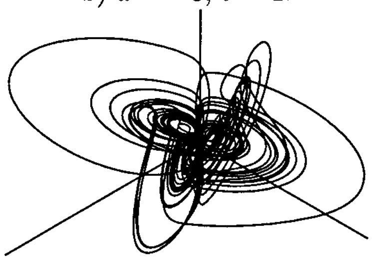


FIG. 8. Two different chaotic attractors for the $S_4$ dynamical system determined by Eqs. (29).

parameters. In contrast to the dynamical system from Ref. [22], we have revealed an analogous phenomenon for very simple three-dimensional and two-parametric flow [Eq. (29)] with the point symmetry group $G = S_4$.

## V. SUMMARY

In this paper, we discussed the method for constructing of $N$-dimensional dynamical systems invariant with respect to the representations of discrete symmetry groups. Three-dimensional flows with quadratic nonlinearities, which are invariant under the action of all point groups of crystallographic symmetry, were found, and the possibility of their chaotic behavior was analyzed. We considered some symmetry-related properties of their chaotic attractors in the frame of the general theory of nonlinear dynamics of systems with discrete symmetry [2]. Of special interest is the elegant two-parametric dynamical system (27) corresponding to the point group $D_2$. We will discuss its chaotic behavior in more detail elsewhere.

We want to emphasize that the three-dimensional $S_4$ flow with the different attractors in the different domains of its parameters seems to be a typical (rather than exotic) case for other types of nonlinearities, as well as for the flows with $N > 3$. Indeed, different chaotic attractors correspond to *different subgroups* of the parent symmetry group $G_0$ of the considered dynamical system. In general, there exist various subgroups of the group $G_0$, and one can hope to find a number of different chaotic attractors for a fixed system or for a fixed class of such systems. In conclusion, let us note that our list (15) is exhaustive, i.e. there are no other three-dimensional chaotic flows with quadratic nonlinearities and point groups of crystallographic symmetry.

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