Minimal speed of fronts of reaction-convection-diffusion equations

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We study the minimal speed of propagating fronts of convection-reaction-diffusion equations of the form $u_t+\mu\phi(u)u_x=u_{xx}+f(u)$ for positive reaction terms with $f'(0)>0$. The function $\phi(u)$ is continuous and vanishes at $u=0$. A variational principle for the minimal speed of the waves is constructed from which upper and lower bounds are obtained. This permits the *a priori* assessment of the effect of the convective term on the minimal speed of the traveling fronts. If the convective term is not strong enough, it produces no effect on the minimal speed of the fronts. We show that if $f''(u)/\sqrt{f'(0)}+\mu\phi'(u)<0$, then the minimal speed is given by the linear value $2\sqrt{f'(0)}$, and the convective term has no effect on the minimal speed. The results are illustrated by applying them to the exactly solvable case $u_t+\mu u u_x=u_{xx}+u(1-u)$. Results are also given for the density dependent diffusion case $u_t+\mu\phi(u)u_x=[D(u)u_x]_x+f(u)$.

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I. INTRODUCTION

The reaction-diffusion equation $u_t=u_{xx}+f(u)$ has been employed as a simple model of phenomena in different areas, population growth, chemical reactions, flame propagation, and others. In the classical Fisher case [1], $f(u)=u(1-u)$, a front propagating with speed $c_{\rm kpp}=2$ joins the two equilibrium points [2]. The time evolution for general reaction terms was solved by Aronson and Weinberger [AW] [3] who showed that sufficiently localized initial conditions evolve into a front which propagates with speed c_* such that $2\sqrt{f'(0)}\leqslant c_*\leqslant 2\sqrt{\sup[f(u)/u]}$. The asymptotic speed of propagation is the minimal speed for which a monotonic front joining the stable to unstable equilibrium point exists. Existence proofs give limited quantitative information on the dependence of the speed of the front on the parameters of the problem [4]. For this reason different variational methods have been developed. For the one-dimensional case it has been shown that this minimal speed can be derived either from a local variational principle of the minimax type [5], or from an integral variational principle [6,7]. Minimax variational principles for the speed of fronts in several dimensions and for inhomogeneous environments have also been established [4,8].

In many processes, in addition to diffusion, motion can also be due to advection or convection. Nonlinear advection terms arise naturally, for example, in the motion of chemotactic cells. In a simple one-dimensional model, denoting by ρ the density of bacteria, chemotactic to a single chemical element of concentration $s(x,t)$ the density evolves according to

$$\rho_t=[D\rho_x-\rho\chi s_x]_x+f(\rho),$$

where diffusion, chemotaxis and growth have been considered. There is some evidence [9] that, in certain cases, the rate of chemical consumption is due mainly to the ability of the bacteria to consume it. In that case

$$s_t=-k\rho,$$

where diffusion of the chemical has been neglected (arguments to justify this approximation, together with the choice of constant D and χ are given in Ref. [9]). If we now look for traveling wave solutions $s=s(x-ct)$, $\rho=\rho(x-ct)$, then $s_t=-cs_x$, therefore $s_x=k\rho/c$, and the problem reduces to a single differential equation for ρ, namely,

$$\rho_t=D\rho_{xx}-\frac{\chi k}{c}(\rho^2)_x+f(\rho). \qquad (1)$$

The more elaborate models of Keller and Segel for chemotaxis [10], which include diffusion of the chemical and other effects, have been considered to explain chemotactic collapse (see Refs. [11,12], and references therein) and other phenomena. The derivation of these equations from transport theory and the assumptions involved in them have been studied recently [13]. In addition to these biological processes, equations analogous to Eq. (1) appear when modeling the Gunn effect in semiconductors and in other physical phenomena [14,15]. Equation (1) for a Fisher type reaction term $f(u)=u(1-u)$ has been studied in Ref. [16]. An extensive study of the existence of traveling waves of nonlinear diffusion-reaction-convection equations which includes a review of many results to which we refer for additional references is contained in Ref. [17].

In this work we concentrate on the equation with a general convective term which, suitably scaled, we write as

$$u_t+\mu\phi(u)u_x=u_{xx}+f(u), \qquad (2)$$

where the reaction term $f(u)$ is a continuous function with continuous derivative in [0,1] and satisfies

$$f(0)=f(1)=0, \quad f'(0)>0, \quad f'(1)<0$$

and

$$f > 0 \quad \text{in} \quad (0,1).$$

The function $\phi(u)$ is a continuous function with continuous derivative in $[0,1]$. Without loss of generality we may assume that in addition $\phi(0)=0$, since otherwise only a uniform shift in the speed is introduced. The parameter μ is positive.

For Eq. (2), the existence of monotonic decaying traveling fronts $u(x-ct)$ for any wave speed greater than a critical value c_* has been proved recently Ref. [18]. Moreover, in Ref. [18] the following estimate for the threshold value c_* is obtained,

$$2\sqrt{f'(0)} \leqslant c_* \leqslant 2\sqrt{\sup_{u\in(0,1]} \frac{f(u)}{u}} + \max_{u\in[0,1]} \mu\phi(u). \tag{3}$$

Analogous results for density dependent diffusion are also established in Ref. [18]. The convergence of some initial conditions to a monotonic traveling front has been proved [19] for systems in which the minimal speed is strictly greater than the linear value $c_L = 2\sqrt{f'(0)}$.

We will show that the minimal speed c_* for the existence of a monotonic decaying front $u(x-ct)$ joining the stable equilibrium $u=1$ to the unstable equilibrium $u=0$ obeys the variational principle

$$c_* = \sup_{g\in\mathcal{S}} \mathcal{E}(g), \tag{4}$$

with

$$\mathcal{E}(g) = \frac{\int_0^1 \{2\sqrt{f(u)g(u)[-g'(u)]} + \mu\phi(u)g(u)\}du}{\int_0^1 g(u)du} \tag{5}$$

and the supremum is taken over the set $\mathcal{S}$ of all positive, monotonic decreasing functions $g(u)$ for which the integrals in Eq. (5) exist and $g(1)=0$. From here it will follow that

$$2\sqrt{f'(0)} \equiv c_L \leqslant c_* \leqslant \inf_{\alpha>0} \sup_{u\in[0,1]} \left(\alpha + \frac{1}{\alpha}f'(u) + \mu\phi(u)\right). \tag{6}$$

From the variational expression (4) one may obtain the value of the minimal speed with any desired accuracy, and the inequalities (6) enable us to characterize the reaction terms for which the speed is the linear value c_L. More precisely, if

$$\frac{f''(u)}{\sqrt{f'(0)}} + \mu\phi'(u) < 0, \quad \text{then} \quad c_* = 2\sqrt{f'(0)}.$$

The bound (3) is also derived from the variational principle. The generalization to density dependent diffusion is given as a direct extension of the previous results.

II. MINIMAL SPEED OF TRAVELLING FRONTS

Traveling monotonic decaying fronts $u(x-ct)$ of Eq. (2) satisfy the ordinary differential equation

$$u_{zz} + [c - \mu\phi(u)]u_z + f(u) = 0,$$

$$\lim_{z\to-\infty} u = 1, \quad \lim_{z\to\infty} u = 0, \quad u_z < 0,$$

where $z = x - ct$. It is convenient to work in phase space; defining as usual $p(u) = -u_z$, the problem reduces to finding the solutions of

$$p(u)\frac{dp(u)}{du} - [c - \mu\phi(u)]p(u) + f(u) = 0, \tag{7}$$

with

$$p(0) = p(1) = 0, \quad \text{and} \quad p > 0.$$

We first perform the linear analysis around the endpoints $u=0$ and $u=1$, which may provide restrictions on the allowable speed. These results will also be needed when proving the existence of a variational principle. Near $u=0$, $p(u)=mu$, where m is the larger root of $m^2 - F(0)m + f'(0) = 0$. For convenience, we have defined $F(u) = c - \mu\phi(u)$. This root is given by

$$m = \frac{F(0) + \sqrt{F(0)^2 - 4f'(0)}}{2}.$$

The condition that m be real imposes the restriction $F^2(0) \geqslant 4f'(0)$. Written explicitly this bound is

$$c \geqslant 2\sqrt{f'(0)} \equiv c_L. \tag{8}$$

Near $u=1$, $p = r(1-u)$, where r is the positive root of $r^2 + F(1)r + f'(1) = 0$, namely,

$$r = \frac{-F(1) + \sqrt{F^2(1) - 4f'(1)}}{2}. \tag{9}$$

No additional restriction on the range of allowable speeds is imposed from the expression above, since by hypothesis $f'(1) < 0$.

In addition to the linear constraint (8), a simple constraint is found from direct integration of Eq. (7). Dividing by $p(u)$ and integrating between 0 and 1, we have

$$c = \mu\int_0^1 \phi(u)du + \int_0^1 \frac{f(u)}{p(u)}du.$$

Since f and p are positive in $(0,1)$, we obtain

$$c > \mu\int_0^1 \phi(u)du.$$

A. Variational principle

In this section we construct a variational principle from which the exact speed of the front may be calculated. Let g be any positive function in (0,1) such that $h=-dg/du>0$. Multiplying Eq. (7) by g/p and integrating with respect to u we find that

$$c\int_0^1 g\,du=\int_0^1\left(hp+\frac{f}{p}g\right)du+\mu\int_0^1\phi g\,du$$

where the first term on the right-hand side is obtained after integration by parts. However since p, h, f, and g are positive, we have that for every fixed u

$$hp+\frac{fg}{p}\geqslant 2\sqrt{fgh},$$

so that

$$c\geqslant\frac{\int_0^1[2\sqrt{fgh}+\mu\phi g]du}{\int_0^1 g\,du}=\mathcal{E}(g). \tag{10}$$

To show that this is a variational principle we must prove that there exists a function $g=\hat{g}$ for which equality holds. Equality is attained for $g=\hat{g}$ such that

$$ph=-p\hat{g}'=\frac{\hat{g}f}{p}.$$

Using Eq. (7) to eliminate $f(u)$ we have that

$$\frac{\hat{g}'(u)}{\hat{g}(u)}-\frac{p'(u)}{p(u)}=-\frac{F(u)}{p(u)},$$

which can be integrated to obtain

$$\hat{g}(u)=p(u)\exp\left[-\int_{u_0}^{u}\left(\frac{F(t)}{p(t)}\right)dt\right]\quad\text{with } 0<u_0<1. \tag{11}$$

Since p vanishes at 0 and 1, we must analyze the behavior of $\hat{g}$ at these points in order to ensure the convergence of the integrals in Eq. (10).

At $u=1$, since $p(u)=r(1-u)$, and since $F(u)$ is continuous at 1, we obtain that

$$\hat{g}\sim r(1-u)^{[1+F(1)/r]}.$$

From the expression for r, Eq. (9), we see that, since $f'(1)<0$, for any value of $F(1)$ the exponent $1+F(1)/r$ is positive, hence $\hat{g}(1)=0$.

Near $u=0$, since $p=mu$ and since F is continuous at zero, we find that

$$\hat{g}\sim\frac{m}{u^{[F(0)/m]-1}}.$$

The integrals in Eq. (10) converge provided

$$\frac{F(0)}{m}-1<1.$$

This condition is satisfied whenever $c_*>c_L$, that is, whenever $c_*>c_L$ there exists a function $g=\hat{g}$ for which equality holds in Eq. (10) or, equivalently, $c_*=\max\mathcal{E}(g)=\mathcal{E}(\hat{g})$.

On the other hand, when $c_*=c_L$, there does not exist a function $\hat{g}$ in the set $\mathcal{S}$ of admissible functions for which equality holds in Eq. (10). Consider however the trial functions

$$g_\alpha(u)=u^{\alpha-1}-1.$$

Clearly $g_\alpha\in\mathcal{S}$ for any $0<\alpha<1$. Moreover one can check that (see the Appendix)

$$\lim_{\alpha\to 0}\mathcal{E}(g_\alpha)=c_L=c_*,$$

therefore

$$c_*=\sup_{g\in\mathcal{S}}\mathcal{E}(g)$$

in this case. Notice that in this last case the maximum is not attained since the limiting function g_0 does not belong to $\mathcal{S}$. This concludes the proof of our variational principle.

B. Upper and lower bounds

The variational principle provides lower bounds with suitably chosen trial functions, which can be arbitrarily close to the exact value of the speed. The fact that it is a variational principle for which equality holds, enables one to obtain also an upper bound to the speed.

To obtain an upper bound we use the fact that

$$2ab\leqslant\alpha a^2+\frac{1}{\alpha}b^2\quad\text{with }\ \alpha>0. \tag{12}$$

Then the following inequality holds:

$$2\sqrt{fgh}=2g\sqrt{fh/g}\leqslant g\left(\alpha+\frac{1}{\alpha}\frac{fh}{g}\right),$$

where we used the inequality above with $a=1,b=\sqrt{fh/g}$.

Then

$$c_* = \sup_g \left[\frac{\int_0^1 (2\sqrt{f\,g\,h} + \mu\phi g)du}{\int_0^1 g\,du} \right]$$

$$\leqslant \sup_g \frac{\int_0^1 g[\alpha + fh/(\alpha g) + \mu\phi]du}{\int_0^1 g\,du}.$$

The second term in the last expression can be integrated by parts. The boundary term $fg|_0^1$ vanishes and we obtain

$$c_* \leqslant \sup_g \frac{\int_0^1 g(\alpha + f'/\alpha + \mu\phi)du}{\int_0^1 g\,du} \leqslant \sup_{u\in[0,1]} \left[\alpha + \mu\phi + \frac{1}{\alpha} f' \right]. \tag{13}$$

The above inequality holds for any positive α, hence

$$c_* \leqslant \inf_\alpha \sup_{u\in[0,1]} \left[\alpha + \frac{1}{\alpha} f'(u) + \mu\,\phi(u) \right]. \tag{14}$$

The bound (14) in the case $\mu=0$ differs from the classical AW [3] result for fronts of the parabolic reaction diffusion equation $c \leqslant c_{AW} = 2\,\sup_{u\in[0,1]}\sqrt{f(u)/u}$. The bound (3) obtained in Ref. [18] on the other hand, reduces, when $\mu=0$ to the classical AW result. Here we show that this last bound can be derived from the variational principle as well. Using the inequality (12), now with $a=\sqrt{fg/u}$ and $b=\sqrt{uh}$, we have that

$$2\int_0^1 \sqrt{fgh}\,du \leqslant \int_0^1 \left[\alpha\frac{fg}{u} + \frac{1}{\alpha} hu \right] du = \int_0^1 \left[\alpha\frac{fg}{u} + \frac{1}{\alpha} g \right] du,$$

where the last expression is obtained after integrating the second term by parts. We have then

$$c_* \leqslant \sup_g \frac{\int_0^1 g\left(\alpha\frac{f}{u} + \frac{1}{\alpha} + \mu\phi \right) du}{\int_0^1 g\,du}$$

$$\leqslant \sup_{u\in[0,1]} \left[\alpha\frac{f}{u} + \frac{1}{\alpha} + \mu\phi \right].$$

Choosing $\alpha = 1/\sup\sqrt{f/u}$ we obtain

$$c_* \leqslant \sup_{u\in]0,1]} \left[2\sqrt{\frac{f(u)}{u}} \right] + \mu \max_{u\in[0,1]} \phi(u).$$

In the classical AW case $\mu=0$, we know that when the reaction term is concave then $\sup 2\sqrt{f(u)/u} = 2\sqrt{f'(0)}$. In this case the upper bound coincides with the linear lower bound c_L and the minimal speed is univocally determined. A similar criterion can be obtained in the present problem. The minimal speed for the existence of a front is known unambiguously to be the linear value whenever the upper bound (14) coincides with the linear lower bound c_L. A sufficient condition for this to occur is that the supremum of the function $K(u) = \alpha + f'(u)/\alpha + \mu\phi(u)$ in (0,1] does not exceed the value of K at the origin. Effectively, if $\sup_u K(u) = K(0) = \alpha + f'(0)/\alpha$; minimizing with respect to α we obtain $\alpha = \sqrt{f'(0)}$ and the upper bound is precisely the linear value. A sufficient condition that guarantees that the maximum (supremum) of K occurs at zero is that $K(u)$ be decreasing. With $\alpha=\sqrt{f'(0)}$ this condition is

$$\frac{f''(u)}{\sqrt{f'(0)}} + \mu\phi'(u) < 0.$$

Whenever this condition is fulfilled, for all u, we know that the minimal speed of a monotonic front is the linear value $c_L = 2\sqrt{f'(0)}$. Again as it occurs in the standard case $\mu=0$, this condition is sufficient but not necessary.

III. AN EXACTLY SOLVABLE CASE

Here we illustrate the above results by applying them to the exactly solvable case discussed in Ref. [16]: a Fisher type reaction term $f(u)=u(1-u)$ and the simplest convective term $\phi(u)=u$. By means of a phase space analysis Murray [16] found that the minimal speed for the existence of a monotonic decaying front is

$$c_* = \begin{cases} \dfrac{2}{\mu} + \dfrac{\mu}{2} & \text{if } \mu > 2 \\ 2 & \text{if } \mu \leqslant 2. \end{cases} \tag{15}$$

Here we show that the results of the preceding section allow for the exact determination of the speed.

In this example the linear marginal stability value is given by $c_L = 2$. We first use the variational principle to obtain a lower bound. Take the trial function

$$g(u) = \left(\frac{1-u}{u} \right)^\lambda \quad \text{with } 0<\lambda<1.$$

A straightforward integration of Eq. (10) leads to

$$c \geqslant 2\sqrt{\lambda} + \frac{\mu}{2}(1-\lambda) \equiv c(\lambda).$$

If $\mu>2$ the maximum of $c(\lambda)$ occurs for $\lambda = 4/\mu^2$ and it is given by $2/\mu + \mu/2$. For $\mu\leqslant 2$, however, the supremum of $c(\lambda)$ occurs as $\lambda\to 1$. We have then

$$c_* \geqslant \sup c(\lambda) = \frac{2}{\mu} + \frac{\mu}{2} \quad \text{for } \mu>2,$$

and

$$c_* \geqslant \sup c(\lambda) = 2 \quad \text{for } \mu \leqslant 2.$$

To obtain an upper bound we use Eq. (13), that is,

$$c_* \leqslant \sup_{u\in[0,1]}\left[\alpha+\frac{1}{\alpha}+u\left(\mu-\frac{2}{\alpha}\right)\right] \quad \forall\alpha>0.$$

We will separate the two cases $\mu\leqslant 2$ and $\mu>2$.

If $\mu\leqslant 2$, choose $\alpha=1$, then

$$c_* \leqslant \sup_{u\in[0,1]}[2+u(\mu-2)]=2.$$

If $\mu>2$ choose $\alpha=2/\mu$, then

$$c_* \leqslant \sup_{u\in[0,1]}\left(\frac{2}{\mu}+\frac{\mu}{2}\right)=\frac{2}{\mu}+\frac{\mu}{2}.$$

The lower bound obtained from the variational expression coincides with the upper bound obtained from Eq. (14), therefore we know with certainty that the minimal speed is indeed Eq. (15), and had been previously demonstrated by phase space methods.

Note that Eq. (3) constitutes a poorer bound in this case. Effectively, from Eq. (3) it follows that $c_*\leqslant 2+\mu$.

IV. DENSITY DEPENDENT DIFFUSION

The effect of the convective term on the minimal speed of fronts of the reaction diffusion equation for nonconstant diffusion follows in a simple way from the previous results. Consider traveling fronts of the equation

$$u_t+\mu\phi(u)u_x=[D(u)u_x]_x+f(u),$$

where $f(u)$ and $\phi(u)$ satisfy the properties spelled in the previous sections. The diffusion coefficient $D(u)$ is continuous and $D(u)>0$ in (0,1]. $D(0)$ is either positive or zero. By a suitable change of variables [20,21] the equation for the fronts is reduced to the usual reaction diffusion equation with a reaction term $\tilde{f}(u)=D(u)f(u)$. This reaction term satisfies $\tilde{f}>0$, and $\tilde{f}'(0)=D(0)f'(0)$. We must distinguish two cases. If $D(0)\neq 0$, then $\tilde{f}$ satisfies the same properties as f and the results of the preceding sections can be applied directly. If $D(0)=0$, then $\tilde{f}'(0)=0$ and we expect a sharp wave front. In this case it has been shown [3] that the front approaches $u=0$ as cu. A variational principle exists also in this case [22]. We have then that, in both cases, the minimal speed of the wave fronts is given by

$$c_*=\sup_g\left[\frac{\int_0^1\{2\sqrt{D(u)f(u)g(u)[-g'(u)]}+\mu\phi(u)g(u)\}du}{\int_0^1 g(u)\,du}\right],$$

where the supremum is taken over all positive monotonic decreasing functions $g(u)$ for which the integrals exist and $g(1)=0$. Upper and lower bounds can be obtained following the methods of the preceding sections. We do not spell them out here.

V. SUMMARY

We have studied the effect of a convective term on the speed of monotonic reaction-diffusion fronts. The minimal speed for the existence of fronts has been shown to derive from a variational principle from which the exact speed can be determined in principle. The existence of this variational characterization permits the obtention of upper and lower bounds. The classical result that establishes that for concave reaction terms, the minimal speed of the fronts is the linear value is extended to the case where convective terms are present. The extension to the case of density dependent diffusion has been given for positive diffusion terms.

We have found that a convective term increases the minimal speed of the traveling front only if it is sufficiently strong, if not, the minimal speed is determined by the reaction term alone.

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APPENDIX

In this appendix we prove that

$$\lim_{\alpha\to 0}\mathcal{E}(g_\alpha)=c_L=2\sqrt{f'(0)},$$

where $g_\alpha=u^{\alpha-1}-1$ with $0<\alpha<1$.

Since $\int_0^1 g_\alpha(u)du=(1-\alpha)/\alpha$, and $h_\alpha=(1-\alpha)u^{\alpha-2}$, we may write

$$\mathcal{E}(g_\alpha)=J_1(\alpha)+J_2(\alpha),$$

where

$$J_1(\alpha)=\frac{2\alpha}{\sqrt{1-\alpha}}\int_0^1\sqrt{f(u)(u^{2\alpha-3}-u^{\alpha-2})}du$$

and

$$J_2(\alpha)=\frac{\mu\alpha}{1-\alpha}\int_0^1\phi(u)(u^{\alpha-1}-1)du.$$

Since $\phi(0)=0$ and ϕ is continuous the integral in J_2 has a finite value when $\alpha=0$. Then, due to the overall multiplicative factor of α, we see that

$$\lim_{\alpha\to 0} J_2(\alpha)=0.$$

To show that $\lim J_1(\alpha)=2\sqrt{f'(0)}$, as $\alpha\to 0$, write

$$J_1(\alpha)=\frac{2\alpha}{\sqrt{1-\alpha}}\int_0^1\sqrt{uf'(0)(u^{2\alpha-3}-u^{\alpha-2})}du+K(\alpha),$$

where

$$K(\alpha)=\frac{2\alpha}{\sqrt{1-\alpha}}\left[\int_0^1\sqrt{f(u)(u^{2\alpha-3}-u^{\alpha-2})}du - \int_0^1\sqrt{uf'(0)(u^{2\alpha-3}-u^{\alpha-2})}du\right].$$

The first integral is

$$\frac{2\alpha}{\sqrt{1-\alpha}}\int_0^1\sqrt{uf'(0)(u^{2\alpha-3}-u^{\alpha-2})}du = \frac{\sqrt{\pi f'(0)}}{\sqrt{1-\alpha}}\,\frac{\Gamma\left(\frac{1}{1-a}\right)}{\Gamma\left(\frac{-3+a}{2(a-1)}\right)}.$$

Now we prove that $\lim_{\alpha\to 0}K(\alpha)=0$.

$$|K(\alpha)|\leqslant\frac{2\alpha}{\sqrt{1-\alpha}}\int_0^1\left|\sqrt{f(u)(u^{2\alpha-3}-u^{\alpha-2})} - \sqrt{uf'(0)(u^{2\alpha-3}-u^{\alpha-2})}\right|du.$$

But $|\sqrt{a}-\sqrt{b}|\leqslant\sqrt{|b-a|}$, therefore

$$|K(\alpha)|\leqslant\frac{2\alpha}{\sqrt{1-\alpha}}\int_0^1\sqrt{|f(u)-uf'(0)|(u^{2\alpha-3}-u^{\alpha-2})}du.$$

Since $f(u)$ and its derivative are continuous, in $[0,1]$, there exist $d>0$, $q>0$ such that

$$\frac{|f(u)-uf'(0)|}{u}<du^q.$$

In particular, if $f(u)$ is analytic in a neighborhood of 0, $q=1$. Using this inequality in the expression above, we have that

$$|K(\alpha)|\leqslant\frac{2\alpha}{\sqrt{1-\alpha}}\int_0^1\sqrt{du^{q+1}(u^{2\alpha-3}-u^{\alpha-2})}du.$$

Finally, since $u^{2\alpha-3}-u^{\alpha-2}<u^{2\alpha-3}$,

$$|K(\alpha)|\leqslant\frac{2\alpha}{\sqrt{1-\alpha}}\sqrt{d}\int_0^1 u^{\alpha-1+q/2}du=\frac{2\alpha}{\sqrt{1-\alpha}}\,\frac{\sqrt{d}}{\alpha+q/2}.$$

Therefore, $\lim_{\alpha\to 0}|K(\alpha)|=0$.

To sum up,

$$\lim_{\alpha\to 0}\mathcal{E}(g_\alpha)=\lim_{\alpha\to 0}\frac{\sqrt{\pi f'(0)}}{\sqrt{1-\alpha}}\,\frac{\Gamma\left(\frac{1}{1-a}\right)}{\Gamma\left(\frac{-3+a}{2(a-1)}\right)}=2\sqrt{f'(0)}.$$

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