Optical breathers in anisotropic media

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Anisotropic crystals are shown to have three different mechanisms of the formation of breathers depending on the direction of the wave propagation and on the symmetry of the medium. Explicit analytic expressions for the parameters of breathers and the effective nonlinear susceptibilities for extraordinary waves are obtained. All uniaxial crystals with quadratic susceptibilities can be divided into three different groups, according to the crystal classes. Each group is characterized by a universal structure of the breathers zones. The structure of the breathers zones of the media with cubic susceptibilities depends neither on the crystal systems (syngonies) nor on the crystal classes and coincides with the structure of the breathers zones of the crystals with quadratic susceptibilities and crystal classes 3, $3m$, 4, $4mm$, 6, $6mm$. The initial-value and boundary-value problems are considered separately.

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I. INTRODUCTION

The propagation of optical waves in a medium is accompanied by various changes of their shape. The main effects that change the shape of waves are nonlinearity and dispersion. The most interesting are those wave processes for which the effects distorting the shape of the wave compensate each other exactly and breathers are formed. The existence of breathers is one of the most interesting and important manifestations of nonlinearity in optical systems. They are of particular interest because they have many solitonlike properties, but unlike solitons, breathers can be excited also for relatively small areas (intensities) of pulses [1–3]. Determination of the mechanisms causing the formation of optical breathers and investigation of their properties in different media are among the principal problems of the physics of nonlinear waves.

In the propagation of a pulse in a nonlinear medium the effects of nonlinearity leads to a progressive deformation of the initial pulse profile. The basic sources of the optical nonlinearity in dielectrics and semiconductors may be the following.

(i) Nonresonance nonlinearity. Media possess nonlinear susceptibilities, the most important of which are nonlinearities of second (quadratic) and third (cubic) order. There is a great variety of dielectrics and semiconductors possessing nonresonance nonlinearity. For example, $LiNbO_3$, α-quartz (SiO_2), GaAs, InSb, etc., have quadratic nonresonance nonlinearity, but melted quartz, CS_2, etc., have cubic nonresonance nonlinearity [4–7]. Unlike the coefficient of nonlinear susceptibility of the second order d_{ijk}, which is nonzero only for noncentrosymmetric crystals, the coefficient of nonlinear susceptibility of the third order ρ_{ijkl} is nonzero in any media—in cubic crystals (Kerr media) and even in isotropic media.

(ii) Resonance nonlinearity. A medium which contains optically active impurities whose excitation frequency is in resonance with the frequency of a nonlinear optical wave. The experimental studies of optical resonance nonlinear effects in crystals $LaF_3:Pr^{3+}$, $YAlO_3:Pr^{3+}$, $Y_2O_3:Eu^{3+}$, $CaWO_4:Nd^{3+}$, on diphenyl with impurity molecules of piren, in semiconductors PbTe, InSb are described in [8,9].

On the other hand, in the propagation of an optical pulse in a dispersive medium, its shape will not remain unchanged: its width will spread [10–14]. Depending on the nature of the nonlinearity, the nonresonance or resonance mechanism of the formation of breathers (MFB) is realized. In the case of nonresonance nonlinearity, which is expressed by means of quadratic or cubic susceptibilities, its competition with dispersion leads to the formation of nonresonance optical breathers [1,15].

A resonance optical nonlinear wave can be formed with the help of the resonance (McCall-Hahn) mechanism of the formation of nonlinear waves—i.e., from a nonlinear coherent interaction of an optical pulse with resonance impurity atoms in solids, when the conditions of the self-induced transparency, $\omega T \gg 1$ and $T \ll T_{1,2}$, have to be fulfilled, where T and ω are the width and frequency of the pulse, and T_1 and T_2 are the longitudinal and transverse relaxation times of the impurity atoms [16–20]. When the area of the pulse, $\Theta > \pi$, the solitons are generated, but for $\Theta \ll 1$ resonance optical breathers are formed [1–3]. In the experiments of McCall and Hahn [16] in a crystal of ruby $Al_2O_3:Cr^{3+}$ the excitation of resonance soliton was reached when the pulse intensity exceeds some critical value about 100 W/cm^2. The necessary intensity for exciting resonance optical breathers of small area is significantly smaller than the intensity necessary for exciting a resonance soliton (2π pulse). Therefore, the breathers can be excited more easily. The resonance optical waves of small area $\Theta \ll 1$ are particularly interesting also because they can take part in a wide variety of nonlinear optical phenomena—for instance, in the processes of the formation of optical double breathers [21,22]. Resonance breathers of some equations of nonlinear optics are also highly stable. The breather can be also con-

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sidered as a "zero-area pulse" which is experimentally studied in Ref. [23] (see also [24]).

The coherent interaction of an optical pulse with impurity atoms is characterized by the coefficient of a photon-atom connection $K=2\pi\omega^2 n_0 d_0$, where n_0 and d_0 are the concentration and magnitude of the vector of the electrical dipole moment of the impurity atoms.

For small values of K, a linear coupling of an optical pulse with a medium takes place and nonresonance MFB is realized. In this case the interaction of a pulse with impurities does not contribute to the formation of the nonresonance breathers, except for a renormalization of their parameters [15].

For large values of K, a nonlinear interaction of the pulse with impurities is realized, for instance, under the condition of a self-induced transparency. In this case resonance MFB is realized and resonance breathers are formed [2,3].

Besides these two basic mechanisms, in some situations another "blended" MFB can also take place when resonance and nonresonance mechanisms are acting effectively simultaneously. The condition for the realization of these mechanisms of the formation of optical breathers depends on the quantities d_{ijk} (or ρ_{ijkl}) and K and is realized when these quantities are equal to each other.

The numerical values of the quantities d_{ijk}, ρ_{ijkl} and K can vary very strongly depending on the medium. Indeed d_{ijk} is of order 10^{-20}–10^{-24} (mks units) and can change two to three orders of magnitude. ρ_{ijkl} is of order 10^{-31}–10^{-34} (mks units) and also changes in a wide region [6,8,25]. For most noncentrosymmetric crystals $d_{ijk} \gg \rho_{ijkl}$ and usually third-order nonlinearity can be neglected. The quantity n_0 can vary in an interval of order 10^{14}–10^{19} cm^{-3}, while the quantity d_0 is of order 5×10^{-21} (cgs units) in a crystal of a ruby, but in some semiconductors it is of order of 10^{-17} (cgs units) [18]. Because the numerical values of these quantities can vary very strongly in different media, different solids will realize different mechanisms of the formation of optical breathers. But even more interesting for the study and comparison of different mechanisms is the investigation of these processes in one and the same crystal. Such a possibility is given if we consider anisotropic uniaxial crystals and investigate processes of the formation of optical breathers for optical extraordinary waves (Fig. 1). It is well known that the properties of extraordinary waves depend on the direction of their propagation and therefore for different directions of the propagation of the waves different relations between the quantities d_{ijk}, ρ_{ijkl}, and K are realized. Hence, if we change the direction of the propagation of nonlinear waves, different mechanisms of the formation of optical breathers—nonresonance MFB ($K \ll L$), resonance MFB ($K \gg L$), and "blended" MFB ($K=L$, where $L=d_{ijk}$ in noncentrosymmetric media and $L=\rho_{ijkl}$ in Kerr media)—will be realized.

Consequently, in uniaxial media we expect the existence of certain propagation directions (and zones around them) along which one of the above-mentioned MFB will be realized or not.

Investigation of the breather formation processes and specific peculiarities of the propagation of nonlinear waves in anisotropic media are also of interest because many laser crystals are anisotropic [6,7] and isotropic solids can become optically anisotropic ones in the presence of a constant electric field or under the influence of a deformation [26]. It is also very important to note that the anisotropic uniaxial crystals are used in many modern optical devices. Consequently, the considered problem has rather general character.

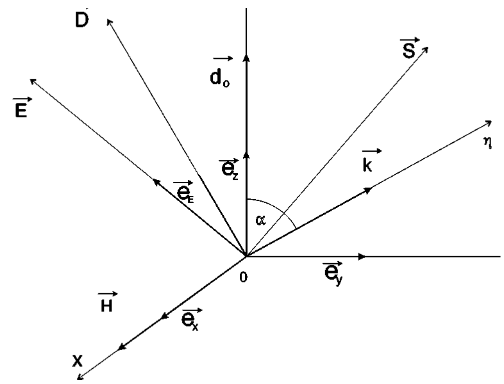

FIG. 1. The direction of the propagation of the extraordinary wave along the η axis making an angle α with the principal optical O axis of the uniaxial crystal. The vectors $\vec{E}$, $\vec{D}$, $\vec{S}$, and $\vec{k}$ lie in the yz plane. The optical O axis and the vector of electrical dipole moment of the impurity atoms $\vec{d}_0$ are directed along the z axis.

The main goal of this work is to investigate the structure of breathers zones (SBZ's) and the conditions for realization of the resonance, nonresonance, and "blended" MFB in different anisotropic media and to determine the explicit analytic expressions for the parameters of breathers and effective nonlinear susceptibilities for the extraordinary waves.

II. BASIC EQUATIONS

We consider the mechanisms of the formation of the optical breathers in the (quadratic or cubic) nonlinear and second-order (space and/or time) dispersive optically uniaxial media containing impurity atoms in the case where an optical pulse of width $T \ll T_{1,2}$ and frequency $\omega \gg T^{-1}$, propagating in the positive direction along the η axis. We shall consider the optically uniaxial media—trigonal, tetragonal, and hexagonal crystals with components of the permittivity tensor $\varepsilon_{xx}=\varepsilon_{yy}\neq\varepsilon_{zz}$. In these crystals, one of the principal axes of the permittivity tensor ε_{ij} coincides with the axis of the symmetry of third, fourth, and sixth order, respectively. This axis is called the optical axis of the uniaxial crystal and we assume that this axis O is pointing along the z axis (Fig 1). The corresponding principal value of the tensor ε_{ij} is $\varepsilon_{zz}=\varepsilon_{\|}$. The directions of the two other principal axes (in the plane perpendicular with the z axis) are arbitrary and we determine them as $\varepsilon_{xx}=\varepsilon_{yy}=\varepsilon_{\perp}$. Without specifying the physical nature of the dispersive process, we describe the dependence of the permittivity tensor ε_{ij} on two variables—the wave vector $\vec{k}$ and frequency ω of wave (spatially and/or temporally dispersion) [10–12]. There are two different cases: the regime of normal dispersion when the group-velocity dispersion $\partial^2\vec{k}/\partial\omega^2>0$ and the regime of anomalous dispersion when this quantity is negative.

In uniaxial media, the electric displacement vector $\vec{D}$ and the vector of the strength of the electric field $\vec{E}$ of the pulse are parallel only if $\vec{k}$ points in the direction of one of the principal optical axis, but not in general. There are two systems of orthogonal vector triplets: ($\vec{D}$, $\vec{H}$, $\vec{k}$) and ($\vec{E}$, $\vec{H}$, $\vec{S}$). We assume without any loss of generality that the vectors $\vec{E}$, $\vec{D}$, and $\vec{k}$ and the Poynting vector $\vec{S}=(c/4\pi)[\vec{E},\vec{H}]$ lie in the single yz plane perpendicular to the strength of the magnetic field $\vec{H}$, where c is the light velocity in vacuum (Fig. 1). Then $\vec{k}\vec{r}=yk_y+zk_z=k\eta$, where $\eta=z\cos\alpha+y\sin\alpha$.

The wave equation for the strength of the electrical field $\vec{E}(\eta,t)$ of the optical pulse in uniaxial media has the form

$$\frac{\partial^2\vec{E}}{\partial t^2}-c^2\frac{\partial^2\vec{E}}{\partial\eta^2}=-4\pi\left(\frac{\partial^2\vec{P}}{\partial t^2}-c^2\,\mathrm{graddiv}\,\vec{P}\right), \tag{1}$$

where the polarization of the medium

$$\vec{P}=\int\hat{\chi}^{(1)}(x_1,t_1)\vec{E}(\eta-x_1,t-t_1)dt_1dx_1+\vec{P^{(2)}}+\vec{P^{(3)}}+\vec{P}'. \tag{2}$$

The first-order susceptibility tensor

$$\chi_{ij}^{(1)}=\frac{\varepsilon_{ij}-1}{4\pi}$$

has two independent nonzero components: $\chi_\perp^{(1)}=\chi_{xx}^{(1)}=\chi_{yy}^{(1)}$ and $\chi_{\|}^{(1)}=\chi_{zz}^{(1)}$.

The components of the second- and third-order nonresonance nonlinear polarizations have the forms

$$P_j^{(2)}=\int\chi_{2,j}(\eta_1,\eta_2,t_1,t_2;\alpha)E_z(\eta-\eta_1,t-t_1)E_z(\eta-\eta_1-\eta_2,t-t_1-t_2)dt_1dt_2d\eta_1d\eta_2,$$

$$\begin{aligned}P_j^{(3)}=&\int\rho_{3,j}(\eta_1,\eta_2,\eta_3,t_1,t_2,t_3;\alpha)E_z(\eta-\eta_1,t-t_1)E_z(\eta-\eta_1-\eta_2,t-t_1-t_2)\\&\times E_z(\eta-\eta_1-\eta_2-\eta_3,t-t_1-t_2-t_3)d\eta_1d\eta_2d\eta_3dt_1dt_2dt_3,\end{aligned} \tag{3}$$

where

$$\chi_{2,j}(\eta_1,\eta_2,t_1,t_2;\alpha)=\chi_{jmn}(\eta_1,\eta_2,t_1,t_2)\frac{e_{E,m}e_{E,n}}{e_{E,z}^2},$$

$$\rho_{3,j}(\eta_1,\eta_2,\eta_3,t_1,t_2,t_3;\alpha)=\rho_{jmnr}(\eta_1,\eta_2,\eta_3,t_1,t_2,t_3)\frac{e_{E,m}e_{E,n}e_{E,r}}{e_{E,z}^3},\quad j,n,m,r=y,z;$$

χ_{jmn} and ρ_{jmnr} are the components of the quadratic and cubic susceptibility tensors [4,6,8,25]. $e_{n,m}=\vec{e}_n\cdot\vec{e}_m$, $\vec{e}_m$ are unit vectors directed along the vector $\vec{E}$ and x,y,z coordinate axes; $n,m=E,x,y,z$. $\vec{E}=\vec{e}_EE$, $E_z=e_{E,z}E$. The unit vector $\vec{e}_E$, the direction of polarization of a linearly polarized optical wave, is determined (Fig. 1). Although for convenience in the equations we are keeping both quantities χ_{jmn} and ρ_{jmnr}, in fact only one of them is not zero, and depending on this we consider it a noncentrosymmetric or Kerr medium.

The quantity $\vec{P}'$ is the resonance nonlinear polarization describing the effects of the one-photon resonance interaction of the optical pulse with the optically active impurity atoms. We shall assume, as is true of a large class of laser crystals (see, for example, Ref. [7]), that the vector of the electric dipole moment $\vec{d}_0$ of impurity atoms and the optical axis O of the uniaxial matrix coincide. In such a system the theory of self-induced transparency has been constructed in Ref. [27]. Another situation of self-induced transparency in anisotropic media when the vector $\vec{d}_0$ does not coincide with the optical axis of the crystal was considered in Ref. [28]. In the present work, we assume that the vector $\vec{d}_0$ and the optical axis O of the matrix coincide and are directed along the z axis. In such a case the vector $\vec{E}$ and the vector of polarization of the impurity atoms, $\vec{P}'$, are coupling to each other through their z components [27]. Consequently, we have to consider the nonlinear wave equation for the z component of $\vec{E}(\eta,t)$.

Substituting the expressions (2) and (3) into the wave equation (1) and employing the condition that the field $\vec{D}$ be transverse, $\mathrm{div}\,\vec{D}=0$, we obtain a nonlinear wave equation for E_z in the form

$$-c^2\frac{\partial^2}{\partial\eta^2}\int\left(\delta(\eta_1)\delta(t_1)+4\pi\cos\alpha\left[\cos\alpha\chi^{(1)}{}_{||}(\eta_1,t_1)+\sin\alpha\chi_{\perp}^{(1)}(\eta_1,t_1)\frac{e_{E,y}}{e_{E,z}}\right]\right)E_z(\eta-\eta_1,t-t_1)d\eta_1 dt_1$$

$$+\frac{\partial^2}{\partial t^2}\int[\delta(\eta_1)\delta(t_1)+4\pi\chi^{(1)}{}_{||}(\eta_1,t_1)]E_z(\eta-\eta_1,t-t_1)d\eta_1 dt_1$$

$$=-4\pi\frac{\partial^2}{\partial t^2}\int\chi_{2,z}(\eta_1,\eta_2,t_1,t_2;\alpha)E_z(\eta-\eta_1,t-t_1)E_z(\eta-\eta_1-\eta_2,t-t_1-t_2)dt_1dt_2d\eta_1d\eta_2$$

$$-4\pi\frac{\partial^2}{\partial t^2}\int\rho_{3,z}(\eta_1,\eta_2,\eta_3,t_1,t_2,t_3;\alpha)E_z(\eta-\eta_1,t-t_1)E_z(\eta-\eta_1-\eta_2,t-t_1-t_2)$$

$$\times E_z(\eta-\eta_1-\eta_2-\eta_3,t-t_1-t_2-t_3)d\eta_1d\eta_2d\eta_3dt_1dt_2dt_3-4\pi\frac{\partial^2P_z'}{\partial t^2}$$

$$+4\pi c^2\frac{\partial^2}{\partial\eta^2}\left\langle\int[\cos^2\alpha\chi_{2,z}(\eta_1,\eta_2,t_1,t_2;\alpha)+\cos\alpha\sin\alpha\chi_{2,y}(\eta_1,\eta_2,t_1,t_2;\alpha)]\right.$$

$$\left.\times E_z(\eta-\eta_1,t-t_1)E_z(\eta-\eta_1-\eta_2,t-t_1-t_2)dt_1dt_2d\eta_1d\eta_2\right\rangle$$

$$+4\pi c^2\frac{\partial^2}{\partial\eta^2}\left\langle\int[\cos^2\alpha\rho_{3,z}(\eta_1,\eta_2,\eta_3,t_1,t_2,t_3;\alpha)+\cos\alpha\sin\alpha\rho_{3,y}(\eta_1,\eta_2,\eta_3,t_1,t_2,t_3;\alpha)]\right.$$

$$\left.\times E_z(\eta-\eta_1,t-t_1)E_z(\eta-\eta_1-\eta_2,t-t_1-t_2)E_z(\eta-\eta_1-\eta_2-\eta_3,t-t_1-t_2-t_3)d\eta_1d\eta_2d\eta_3dt_1dt_2dt_3\right\rangle$$

$$+4\pi c^2\cos^2\alpha\frac{\partial^2}{\partial\eta^2}\langle P_z'\rangle, \tag{4}$$

where we have used the notation $\langle g\rangle=\int\varepsilon_{\perp}^{-1}(\eta_1,t_1)g(\eta-\eta_1,t-t_1)d\eta_1dt_1$ for any function g.

The dependence of the quantity $P_z'=n_0d_0s_1$ on the strength of the electrical field E_z is governed by the optical Bloch equations which are based on the representation of the resonance impurity atoms by an ensemble of two-level atoms whose evolution is caused by processes of interaction with optical extraordinary waves [16–22]:

$$\frac{\partial s_1}{\partial t}=-\omega_0s_2,\quad\frac{\partial s_2}{\partial t}=\omega_0s_1+\kappa_0E_zs_3,\quad\frac{\partial s_3}{\partial t}=-\kappa_0E_zs_2, \tag{5}$$

where $\kappa_0=2d_0/\hbar$, $s_i(t)=\langle\hat{\sigma}_i(t)\rangle$ $(i=1,2,3)$; here, $\langle\hat{\sigma}_i\rangle$ is the average value of the Pauli operator $\hat{\sigma}_i$ and ω_0 is the frequency of excitation of the two-level impurity atoms. To take into account that we consider coherent interactions of pulses with two-level atoms—i.e., $T\ll T_{1,2}$—in the system of equations (5) the relaxation effects are neglected. Since we investigate the situation of a small concentration of impurity atoms (as is true of a large class of crystals with impurities), the interaction of one impurity atom with another as usual is ignored in the Bloch equations [16–22,27,28].

We can simplify equations (4) and (5) using the method of slowly changing profiles. For this purpose, we represent the functions E_z and P_z' in the form

$$E_z=\sum_l\hat{E}_lZ_l,\quad P_z'=\frac{n_0d_0}{2}\sum_l Z_ld_{-l}(\delta_{l,1}+\delta_{l,-1}), \tag{6}$$

where $\hat{E}$ and $d_l=d_{l,1}+ild_{l,2}$ are the slowly varying complex amplitudes of the optical wave and polarization of the impurities, and l runs through the values $\pm1,\pm2,\ldots,s_i=\Sigma_l d_{l,i}Z_l$, $Z_l=e^{il(k\eta-\omega t)}$ $(i=1,2)$. To guarantee the reality of the quantities E_z and P_z', we set $\hat{E}_l=\hat{E}^*_{-l}$ and $d_l=(d_{-l})^*$. We note that such a representation of the solution of a nonlinear wave equation has been widely used in the theory of nonlinear waves [4,16–22]. This approximation is based on the consideration that the envelopes $\hat{E}_l$ and $d_{l,i}$ vary sufficiently slowly in space and time as compared with the carrier wave parts—i.e.,

$$\left|\frac{\partial \hat{E}_l}{\partial t}\right| \ll \omega |\hat{E}_l|, \quad \left|\frac{\partial \hat{E}_l}{\partial \eta}\right| \ll k|\hat{E}_l|, \quad \left|\frac{\partial d_{l,i}}{\partial t}\right| \ll \omega |d_{l,i}|,$$

$$\left|\frac{\partial d_{l,i}}{\partial \eta}\right| \ll k|d_{l,i}|,$$

and is called the slowly varying envelope approximation.

On substituting the expansions (6) in the system of nonlinear equations (4) and (5) we obtain for the envelopes the nonlinear wave equation

$$\sum_l Z_l \Bigg\{ \Bigg[W_l(\alpha)\hat{E}_l - i\alpha_l(\alpha)\frac{\partial \hat{E}_l}{\partial \eta} + i\beta_l(\alpha)\frac{\partial \hat{E}_l}{\partial t} - \mu_l(\alpha)\frac{\partial^2 \hat{E}_l}{\partial \eta^2} - \gamma_l(\alpha)\frac{\partial^2 \hat{E}_l}{\partial \eta \partial t} - \delta_l(\alpha)\frac{\partial^2 \hat{E}_l}{\partial t^2}\Bigg] - \sum_{l'} \chi^{(2)}_{l,l'}(\alpha)\hat{E}_{l-l'}\hat{E}_{l'} - \sum_{l',l''} \rho^{(3)}_{l,l',l''}(\alpha)\hat{E}_{l-l'-l''}\hat{E}_{l'}\hat{E}_{l''} - r_l(\alpha)(\delta_{l,1}+\delta_{l,-1}) \Bigg\} = 0 \tag{7}$$

and connect with them the system of Bloch equations

$$\frac{\partial d_l}{\partial t} = il(\omega_0 - \omega)d_l + il\kappa_0 \hat{E}_{-l}s_3,$$

$$\frac{\partial s_3}{\partial t} = \frac{il\kappa_0}{2}(d_l\hat{E}_l - d_{-l}\hat{E}_{-l}), \tag{8}$$

where

$$W_l = l^2(c^2k^2\kappa_l^{(2)} - \omega^2\kappa_l^{(1)}),$$

$$\alpha_l = l(c^2lk^2A_l^{(2)} + 2kc^2\kappa_l^{(2)} - l\omega^2A_l^{(1)}),$$

$$\beta_l = l(c^2lk^2B_l^{(2)} - 2\omega\kappa_l^{(1)} - l\omega^2B_l^{(1)}),$$

$$\gamma_l = l(2\omega A_l^{(1)} + l\omega^2T_l^{(1)} - 2c^2kB_l^{(2)} - c^2lk^2T_l^{(2)}),$$

$$\delta_l = c^2l^2k^2D_l^{(2)} - l^2\omega^2D_l^{(1)} - 2l\omega B_l^{(1)} - \kappa_l^{(1)},$$

$$\mu_l = c^2l^2k^2C_l^{(2)} + c^2\kappa_l^{(2)} + 2c^2lkA_l^{(2)} - l^2\omega^2C_l^{(1)},$$

$$A_l^{(j)} = \frac{\partial \kappa_l^{(j)}}{\partial (lk)}, \quad B_l^{(j)} = \frac{\partial \kappa_l^{(j)}}{\partial (l\omega)}, \quad C_l^{(j)} = \frac{1}{2}\frac{\partial^2 \kappa_l^{(j)}}{\partial (lk)^2},$$

$$D_l^{(j)} = \frac{1}{2}\frac{\partial^2 \kappa_l^{(j)}}{\partial (l\omega)^2}, \quad T_l^{(j)} = \frac{\partial^2 \kappa_l^{(j)}}{\partial (lk)\partial (l\omega)} \quad j = 1,2,$$

$$r_l(\alpha) = K\psi_l(\alpha)d_{-l}, \quad \psi_l(\alpha) = 1 - \frac{k^2c^2}{\omega^2}\cos^2\alpha\,\varepsilon_\perp^{-1}(l\omega, lk),$$

$$\kappa_l^{(1)} = \varepsilon_{||}(lk, l\omega),$$

$$\kappa_l^{(2)} = \int \Bigg\{ \delta(\eta)\delta(t) + 4\pi\cos^2\alpha\Bigg[\chi^{(1)}{}_{||}(\eta,t) + \tan\alpha\,\chi_\perp^{(1)}(\eta,t)\frac{e_{E,y}}{e_{E,z}}\Bigg]\Bigg\} e^{il(\omega t - k\eta)}dt\,d\eta.$$

The effective susceptibilities of the second and third order in uniaxial media for E_z have the form

$$\chi^{(2)}_{l,l'}(\alpha) = 4\pi l^2\omega^2[\psi_l(\alpha)(\chi_{zyy}\lambda_l^2\cot^2\alpha + \chi_{zzz}) + \lambda_l\cos^2\alpha(\chi_{yyz} + \chi_{yzy}) - \chi_{yyy}\cot\alpha\cos^2\alpha\lambda_l^2], \tag{9}$$

$$\rho^{(3)}_{l,l',l''}(\alpha) = 4\pi l^2\omega^2[\psi_l(\alpha)\lambda_l^2\cot^2\alpha(\rho_{zzyy} + \rho_{zyyz} + \rho_{zyzy}) + \psi_l(\alpha)\rho_{zzzz} - \psi_l(\alpha)\lambda_l^3\rho_{zyyy}\cot^3\alpha + \lambda_l\cos^2\alpha(\rho_{yyzz} + \rho_{yzyz} + \rho_{yzzy}) + \cot^2\alpha\cos^2\alpha\lambda_l^3\rho_{yyyy} - \cos^2\alpha\cot\alpha(\rho_{yyyz} + \rho_{yyzy} + \rho_{yzyy})\lambda_l^2], \tag{10}$$

where

$$\lambda_l = \frac{\varepsilon_{||}(lk, l\omega)}{\varepsilon_\perp(lk, l\omega)}, \quad \chi_{ijn} = \chi_{ijn}(lk, l\omega, l'k, l'\omega),$$

$$\rho_{ijnm} = \rho_{ijnm}(lk, l\omega, l'k, l'\omega, l''k, l''\omega), \quad i,j,n,m = y,z.$$

It is easy to make sure that all the quantities in Eqs. (7)–(10) depend on the direction of wave propagation—i.e., from the quantity α. The system of equations (7) and (8) is for the slowly varying variables in a sufficiently general form and can describe various processes of the formation and propagation of the nonlinear waves in the anisotropic, nonlinear, and dispersive media containing small concentrations of the optical active impurity atoms. A lot of effects that were considered earlier can be investigated, as special cases, by these equations. For instance, under the condition when $\chi^{(2)}_{l,l'}$ (or $\rho^{(3)}_{l,l',l''}) \ll r_l$ we obtain a situation of self-induced transparency in anisotropic media which was considered in Ref. [27], but for the situation when condition $\chi^{(2)}_{l,l'}$ (or $\rho^{(3)}_{l,l',l''}) \gg r_l$ is fulfilled we obtain nonresonance solitons and breathers which were considered in Refs. [4,15].

In the media with quadratic susceptibility and first-order dispersion $\partial^2\vec{k}/\partial\omega^2 = \rho^{(3)}_{l,l',l''} = r_l = 0$, for the direction of synchronism the requirement of phase matching is satisfied and second-harmonic generation is effective [12–14], if we take into account the wave equation for the E_x component too.

Unlike second-harmonic generation, for other relations between phases of the waves it is possible to realize another type of interaction: namely, the reactive interaction when the interacting waves do not exchange any energy and under this condition can form "bright" (for fundamental mode) and "dark" (for second-harmonic mode) solitons [12] and also many other nonlinear effects can be investigated by Eqs. (7) and (8).

III. OPTICAL BREATHERS OF THE EXTRAORDINARY WAVES

To further analyze Eqs. (7) and (8), we make use of the perturbative reduction method, developed in Refs. [29,30], under the condition

$$|\Theta_l| \ll 1,$$

where

$$\Theta_l(\eta,t)=\kappa_0\int_{-\infty}^{t}\hat{E}_l(\eta,t')dt'$$

is the area of the optical pulse envelope.

The solution of Eqs. (7) and (8) can be carried out by two different methods, depending on whether we investigate the problem of the evolution of the initial perturbation (initial-value problem) or we consider the propagation in the medium of a pulse, which is specified on the boundary of the medium (boundary-value problem). Although the corresponding equations appear different, we must note that in some sense they are identical to each other.

In the case of an initial-value problem, $\hat{E}_l$ can be represented as [29,30]

$$\hat{E}_l(\eta,t)=\sum_{\alpha=1}^{\infty}\sum_{n=-\infty}^{+\infty}\varepsilon^{(\alpha)}Y_n\varphi_{l,n}^{(\alpha)}(\zeta,\tau), \tag{11}$$

where $Y_n=e^{in(Q\eta-\Omega t)}$, $\zeta=\varepsilon Q(\eta-vt)$, $\tau=\varepsilon^2 t$, $v=d\Omega/dQ$, and ε is a small parameter.

In the case of a boundary-value problem, we can represent the quantity $\hat{E}_l$ as

$$\hat{E}_l(\eta,t)=\sum_{\alpha=1}^{+\infty}\sum_{n=-\infty}^{+\infty}\varepsilon^{(\alpha)}X_n f_{l,n}^{(\alpha)}(\xi,\nu), \tag{12}$$

where $X_n=e^{in(\tilde{Q}\eta-\tilde{\Omega}t)}$, $\xi=\varepsilon(t-\eta/U)$, $\nu=\varepsilon^2\eta$, and $U=(d\tilde{Q}/d\tilde{\Omega})^{-1}$. Such a representation allows us to separate from $\hat{E}_l$ the still more slowly changing quantities $\varphi_{l,n}^{(\alpha)}$ and $f_{l,n}^{(\alpha)}$. Consequently, it is assumed that the quantities Ω, Q, $\tilde{Q}$, $\tilde{Q}$, $\varphi_{l,n}^{(\alpha)}$, and $f_{l,n}^{(\alpha)}$ satisfy the inequalities

$$\omega\gg\Omega,\quad k\gg Q,\quad \omega\gg\tilde{\Omega},\quad k\gg\tilde{Q},$$

$$\left|\frac{\partial\varphi_{l,n}^{(\alpha)}}{\partial t}\right|\ll\Omega|\varphi_{l,n}^{(\alpha)}|,\quad \left|\frac{\partial\varphi_{l,n}^{(\alpha)}}{\partial\eta}\right|\ll Q|\varphi_{l,n}^{(\alpha)}|,$$

$$\left|\frac{\partial f_{l,n}^{(\alpha)}}{\partial t}\right|\ll\tilde{\Omega}|f_{l,n}^{(\alpha)}|,\quad \left|\frac{\partial f_{l,n}^{(\alpha)}}{\partial\eta}\right|\ll\tilde{Q}|f_{l,n}^{(\alpha)}|.$$

In the interaction of an optical pulse with a resonantly absorbing medium, the most significant effects are usually observed at exact resonance. Therefore, for simplicity, we consider the system of Bloch equations (8) at exact resonance—i.e., with $\omega=\omega_0$.

For the determination of the explicit form of the quantity P'_z, we expand d_l and s_3 in a perturbation-theory series in the small nonlinearity parameter ϵ:

$$d_l=\sum_{\alpha=1}\varepsilon^{(\alpha)}b_l^{(\alpha)},\quad s_3=\sum_{\alpha=0}\varepsilon^{(\alpha)}N^{(\alpha)}.$$

Substituting these expansions and expression (11) into Eqs. (8), we obtain

$$r_l(\alpha)=\sum_n Y_n\frac{l}{n}\left\{R_l(\alpha)\sum_{\alpha=1}\varepsilon^{\alpha}\varphi_{l,n}^{(\alpha)}-\varepsilon^3R_{l,0}(\alpha)\varphi_{l,n-n'-m}^{(1)}\varphi_{l,n'}^{(1)}\varphi_{-l,m}^{(1)}\right\}+O(\varepsilon^4). \tag{13}$$

Analogously we can obtain an expression for P'_z in the case of the boundary-value problem using Eq. (12):

$$r_l(\alpha)=\sum_n X_n\frac{l}{n}\left\{R_l(\alpha)\sum_{\alpha=1}\varepsilon^{\alpha}f_{l,n}^{(\alpha)}-\varepsilon^3R_{l,0}(\alpha)f_{l,n-n'-m}^{(1)}f_{l,n'}^{(1)}f_{-l,m}^{(1)}\right\}+O(\varepsilon^4), \tag{14}$$

where

$$R_l(\alpha)=\frac{4\pi n_0 d_0^2\omega^2\tau_0}{\hbar\Omega}\psi_l(\alpha),$$

$$R_{l,0}(\alpha)=\frac{8\pi n_0 d_0^4\omega^2\tau_0}{(\hbar\Omega)^3}\psi_l(\alpha). \tag{15}$$

The plus sign of the quantity τ_0 corresponds to the initial condition in which the impurity atoms are initially in the ground state—i.e., at $t\to\infty, s_3=-1$ (attenuating medium). The minus sign corresponds to the case where, at $t\to\infty, s_3=+1$; i.e., all the impurity atoms are initially in the excited state (amplifying medium). From Eqs. (13) and (14) we can see that the resonance nonlinear polarization P'_z or r_l for one-photon processes contains not only nonlinear but linear parts too.

A. Initial-value problem

We begin by considering the solution of Eqs. (7) and (8) in case of an initial-value problem. In this analysis we use the expansion (11). On substituting Eqs. (11) and (13) into Eq. (7), we obtain the nonlinear wave equation

$$\sum_{\alpha,l,n} \varepsilon^{\alpha} Z_l Y_n \Bigg\{ \Bigg[W_{l,n} + \varepsilon J_{l,n} \frac{\partial}{\partial \zeta} + \varepsilon^2 H_{l,n} \frac{\partial^2}{\partial \zeta^2} + \varepsilon^2 h_{l,n} \frac{\partial}{\partial \tau} \Bigg] \varphi_{l,n}^{(\alpha)}$$

$$- \sum_{\alpha',l',n' \alpha'',l'',n''} \varepsilon^{\alpha'} [F_{l,l'} \varphi_{l-l',n-n'}^{(\alpha)} \varphi_{l',n'}^{(\alpha')}$$

$$- \varepsilon^{\alpha''} \lambda_{l,l',l''} \varphi_{l-l'-l'',n-n'-n''}^{(\alpha)} \varphi_{l',n'}^{(\alpha')} \varphi_{l'',n''}^{(\alpha'')}] \Bigg\}$$

$$+ \sum_{l,n} Z_l Y_n \frac{l}{n} (\delta_{l,1} + \delta_{l,-1}) \Bigg[R_l \sum_{\alpha=1} \varepsilon^{\alpha} \varphi_{l,n}^{(\alpha)}$$

$$- \varepsilon^3 R_{l,0} \varphi_{l,n-n'-m}^{(1)} \varphi_{l,n'}^{(1)} \varphi_{-l,m}^{(1)} + O(\varepsilon^4) \Bigg] = 0, \tag{16}$$

where the coefficients

$$W_{l,n} = W_l + \alpha_l n Q + \beta_l n \Omega - \gamma_l n^2 Q \Omega + \mu_l n^2 Q^2 + \delta_l n^2 \Omega^2,$$

$$J_{l,n} = -iQ[\alpha_l + \beta_l v_g + 2nQ\mu_l + 2n\delta_l \Omega v_g - \gamma_l n(\Omega + Q v_g)],$$

$$H_{l,n} = Q^2(\gamma_l v_g - \mu_l - \delta_l v_g{}^2),$$

$$h_{l,n} = i(\beta_l + 2n\Omega\delta_l - nQ\gamma_l),$$

$$F_{l,l'} = 4\pi l^2 \omega^2 \chi_{l,l'}^{(2)},$$

$$\lambda_{l,l',l''} = 4\pi l^2 \omega^2 \rho_{l,l',l''}^{(3)}, \tag{17}$$

depending on α. To determine the values of $\varphi_{l,n}^{(\alpha)}$ we equate to zero the terms corresponding to like powers of ε. As a result, we obtain a chain of equations: to first order in ε,

$$\Bigg[W_{l,n}(\alpha) + \frac{l}{n} R_l(\alpha) \Bigg] \varphi_{l,n}^{(1)} = 0. \tag{18}$$

In dispersive media $W_0 = W_{\pm 1} = 0$ and $W_{|l|>1} \neq 0$. The equation $W_{l=\pm 1} = 0$ provides the dispersion law for extraordinary waves. In what follows, we shall also be interested in a breather which vanishes at $t \to \pm\infty$. Consequently, according to Eq. (18), only the following terms of all the quantities $\varphi_{l,n}^{(1)}$ differ from zero: $\varphi_{\pm 1,\pm 1}^{(1)}$ or $\varphi_{\pm 1,\mp 1}^{(1)}$. The relation between the quantities Ω and Q, for fixed values of l and $n = \pm 1$, is determined from Eq. (18):

$$\alpha_l n Q + \beta_l n \Omega - \gamma_l Q \Omega + \mu_l Q^2 + \delta_l \Omega^2 + R_l \frac{l}{n} = 0. \tag{19}$$

Since all the coefficients in this equation are functions of α, the relation between Ω and Q will depend on the angle α, too.

We have to consider the situation when $\varphi_{\pm 1,\pm 1}^{(1)} = 0$ and $\varphi_{\pm 1,\mp 1}^{(1)} \neq 0$ separately from the situation when $\varphi_{\pm 1,\pm 1}^{(1)} \neq 0$ and $\varphi_{\pm 1,\mp 1}^{(1)} = 0$. First we consider the situation when $\varphi_{\pm 1,\pm 1}^{(1)} = 0$ and $\varphi_{\pm 1,\mp 1}^{(1)} \neq 0$. Then the relation between Ω and Q is determined from Eq. (20) at $l = -n = \pm 1$.

Substituting Eq. (19) into Eq. (17), we easily see that the following relation holds: $J_{\pm 1,\mp 1} = 0$.

To second order in ε, from Eq. (16) we obtain the relation between $\varphi_{\pm 2,\mp 2}^{(2)}$ and $\varphi_{\pm 1,\mp 1}^{(1)}$:

$$\varphi_{\pm 2,\mp 2}^{(2)} = \frac{F_{\pm 2,\pm 1}}{w_{\pm 2,\mp 2}} (\varphi_{\pm 1,\mp 1}^{(1)})^2. \tag{20}$$

Substituting Eqs. (19) and (20) into Eq. (16), we obtain the well-known nonlinear Schrödinger equation (NSE) for the quantity $\Psi_{l,-l} = \varepsilon \sqrt{q_l} \varphi_{l,-l}^{(1)}$ (for $l = \pm 1$):

$$il \frac{\partial \psi_{l,-l}}{\partial t} + \frac{\partial^2 \psi_{l,-l}}{\partial y_l^2} + |\psi_{l,-l}|^2 \psi_{l,-l} = 0, \tag{21}$$

where

$$y_l = \frac{\eta - v_g t}{\sqrt{p_l}}, \quad p_l = \frac{ilH_{l,-l}}{h_{l,-l} Q^2} = -\frac{1}{2} \frac{\partial^2 \Omega}{\partial Q^2},$$

$$q_l = \frac{m_l - R_{l,0}}{2\Omega\delta_l - l\beta_l - Q\gamma_l}.$$

The quantity q_l contains terms coming from the resonance $R_{l,0}$ and nonresonance m_l nonlinear terms, where the quantity

$$m_l = M_l = \frac{16\pi^2 \omega^4 (\chi_{l,-l}^{(2)} + \chi_{l,2l}^{(2)}) \chi_{2l,l}^{(2)}}{W_{2l} - 2l\alpha_l Q - 2l\beta_l \Omega - 4\gamma_l Q \Omega + 4\delta_l \Omega^2 + 4\mu_l Q^2 - \frac{1}{2}(\hbar\Omega/d_0)^2 R_{2l,0}}$$

for quadratic nonlinearity crystals and

$$m_l = L_l = 4\pi\omega^2 (\rho_{l,l,-l}^{(3)} + \rho_{l,-l,l}^{(3)})$$

for crystals with cubic nonlinearity.

The NSE (21) under the condition $p_l q_l > 0$ has the soliton solution

$$\psi_{l,-l} = 2il\eta_0 \frac{e^{-il\varphi_{1,l}}}{\cosh 2\eta_0 \varphi_{2,l}}, \tag{22}$$

where

$$\varphi_{1,l}=\frac{2\xi_0\eta}{\sqrt{p_l}}+2\left[2(\xi_0^2-\eta_0^2)-\frac{\xi_0 v_g}{\sqrt{p_l}}\right]t-\varphi_0,$$

$$\varphi_{2,l}=\frac{\eta}{\sqrt{p_l}}+\left(4\xi_0-\frac{v_g}{\sqrt{p_l}}\right)t-y_0. \quad (23)$$

The quantities ξ_0,η_0,φ_0, and y_0 denote the scattering data of the inverse scattering transform (IST) [1,30,31] when applied to the nonlinear equation. Substituting the soliton solution for the superenvelope, Eq. (22), into Eq. (11), we obtain for the envelope $\hat{E}_l$ the breather solution [1–3,15,30–32]

$$\hat{E}_l=\frac{2il\eta_0}{\sqrt{q_l}}\frac{e^{-il(\varphi_{1,l}+\Omega t-Q\eta)}}{\cosh 2\eta_0\varphi_{2,l}}. \quad (24)$$

Using the IST, we can obtain the breather solution (24) for any initial value $\hat{E}(t=0,\eta)$. The appearance in expression (24) of the factor $e^{il(Q\eta-\Omega t)}$ indicates the formation of periodic beats (slow in comparison with coordinates and time, with characteristic parameters Ω and Q), as a result of which the soliton solution (22) for $\psi_{l,-l}$ is transformed into the solution (24) for complex envelope $\hat{E}_l$. This is exact regular time (and/or space) periodic solution of the nonlinear wave equation (7) called a breather (pulsing soliton) which loses no energy in the process of propagation through the medium [1–3,30–32]. Equation (24) is a breather under the condition of phase modulation $(\varphi_{1,l}\neq 0)$. From Eqs. (9), (10), (15), (19), (23), and (24) we can see that all parameters of the breather depend on the direction of the wave propagation.

A situation similar to Eq. (7) occurs with the sine-Gordon equation; namely, the breather for a small amplitude of the sine-Gordon equation corresponds to the soliton of the NSE (see, for example, [1–3,32]). Indeed, if we consider the case when there is no phase modulation $(\varphi_{1,l}=0)$, then the complex quantity $\hat{E}_l$ reduces to the real quantity Re $(\hat{E}_l)$ and expression (24) goes over to the more usual form of a small-amplitude breather which is proportional to $\sin(\Omega t-Q\eta)\operatorname{sech}2\eta_0\varphi_{2,l}$ [1–3,32]. At the same time, we have to note that Eq. (24) is a breather solution of Eq. (7), but is not a breather solution of the NSE [unlike Eq. (24), because breather solutions of the NSE are unstable [30,31]].

In the case when $\varphi^{(1)}_{\pm1,\pm1}\neq 0$ and $\varphi^{(1)}_{\pm1,\mp1}=0$, the relation between Ω and Q is determined from Eq. (19) at $l=n=\pm1$. Expressions (21)–(24) are valid in this case also if we make the changes

$$\psi_{l,-l}\to\varepsilon\sqrt{Q_{\pm1}}\varphi^{(1)}_{\pm1,\pm1},\quad p_l\to\frac{ilH_{l,l}}{h_{l,l}Q^2},$$

$$q_l\to\frac{\tilde{M}_l+L_l+R_{l,0}}{ilh_{l,l}},\quad \tilde{M}_l=(F_{l,-l}+F_{l,2l})\frac{F_{2l,l}}{w_{2l,2l}}.$$

B. Boundary-value problem

Here we consider the same problem in case of a boundary-value problem. In this case, we use expansion in Eq. (12) for the solution of Eqs. (7) and (8). On substituting Eqs. (12) and (14) into Eq. (7), as was done in the preceding section, we obtain the NSE in the form

$$il\frac{\partial\chi_{l,-l}}{\partial\eta}+\frac{\partial^2\chi_{l,-l}}{\partial T_l^2}+|\chi_{l,-l}|^2\chi_{l,-l}=0,\quad l=\pm1, \quad (25)$$

where

$$\chi_{l,-l}(\eta,T_l)=\varepsilon\sqrt{\tilde{q}_l}f^{(1)}_{l,-l},\quad l=-n=\pm1,\quad T_l=\frac{t-\dfrac{\eta}{U}}{\sqrt{\tilde{p}_l}},$$

$$\tilde{p}_l=\frac{ilH^{(0)}_{l,-l}}{h^{(0)}_{l,-l}},\quad \tilde{q}_l=\frac{M_{l,-l}+L_{l,-l}-R_{l,0}}{ilh^{(0)}_{l,-l}},$$

$$H^{(0)}_{l,n}=-\frac{1}{U^2}(\mu_l-\gamma_l U+\delta_l U^2),$$

$$h^{(0)}_{l,n}=-i(\alpha_l-\gamma_l n\tilde{\Omega}+2n\mu_l\tilde{Q}). \quad (26)$$

In this case, the relation between the quantities $\tilde{\Omega}$ and $\tilde{Q}$, at fixed values of l and n, has the form (at $l=\pm1$)

$$\alpha_l n\tilde{Q}+\beta_l n\tilde{\Omega}+\mu_l\tilde{Q}^2+\delta_l\tilde{\Omega}^2-\gamma_l\tilde{Q}\tilde{\Omega}+R_l\frac{l}{n}=0. \quad (27)$$

Substituting the solution of NSE (25) into Eq. (12), we can obtain the breather solution of Eq. (7) in the form

$$\hat{E}_l=\frac{2il\eta_0}{\sqrt{\tilde{q}_l}}\frac{e^{-il(\delta_{1_l}+\tilde{\Omega}t-\tilde{Q}z)}}{\cosh 2\eta_0\delta_{2_l}}, \quad (28)$$

where

$$\delta_{1_l}=\frac{2\xi_0}{\sqrt{\tilde{p}_l}}t+\left[4(\xi_0^2-\eta_0^2)-\frac{2\xi_0}{\sqrt{\tilde{p}_l}U}\right]\eta-\varphi_0,$$

$$\delta_{2_l}=\frac{t}{\sqrt{\tilde{p}_l}}+\left(4\xi_0-\frac{1}{U\sqrt{\tilde{p}_l}}\right)\eta-y_0. \quad (29)$$

Using the IST, we can obtain the breather solution (28) of Eqs. (7) and (8) for any boundary value of the quantity $\hat{E}(t,\eta=0)$.

Under the condition $l=n=\pm1$ expressions (26)–(29) are valid in this case also if we make the changes

$$\chi_{l,-l}\to\varepsilon\sqrt{\tilde{Q}_l}f^{(1)}_{l,l},\quad \tilde{p}_l\to\frac{ilH^{(0)}_{l,l}}{h^{(0)}_{l,l}},\quad \tilde{q}_l\to\frac{M_l+L_l-R_{l,0}}{ilh^{(0)}_{l,l}}.$$

IV. STRUCTURE OF THE BREATHER ZONE

In the present paper we have shown that in the propagation of intense optical radiation through (quadratic or cubic) nonlinear and second-order (spatially and/or temporally) dispersion ($\partial^2\vec{k}/\partial\omega^2\neq 0$) anisotropic uniaxial crystals containing small concentrations of optical resonance impurity atoms optical breathers can arise. The explicit form of the breather, when we consider the initial-value problem, is given by Eq. (24), and if we investigate the boundary-value problem, the form of the breather is given by expression (28). The dispersion equation and the relation between Ω and Q ($\tilde{\Omega}$ and $\tilde{Q}$) are given by $W_{\pm 1}=0$ and Eq. (19) [and Eq. (27)].

The physical interpretation of the formation of a breather is the following. In the propagation of the pulse in a dispersive medium, its shape will not remain unchanged. The width of the pulse will increase during propagation. This is due to the fact that waves of different wavelength propagate in a dispersive wave with different velocities. In the NSE, this effect is taken into account through the terms $\partial^2\psi_{l,-l}/\partial y_l^2$ and $\partial^2\chi_{l,-l}/\partial T_l^2$. On the other hand, the effects of nonlinearity lead to a progressive deformation of the profile of the pulse, which increases with increasing t. In the NSE, the nonlinear effects are taken into account by the terms $\psi_{l,-l}|\psi_{l,-l}|^2$ and $|\chi_{l,-l}|^2\chi_{l,-l}$. As a result of the competition between the (nonresonance and/or resonance) nonlinearity, which increases the curvature of the profile of the pulse, and the dispersion, which causes the profile to spread out, the shape of the nonlinear wave is stabilized—an optical breather state is formed.

It should be noted that these results and their interpretation are applicable to pulses with sufficiently smooth envelopes, under the condition that the size of the pulse be large in comparison with the wavelength—i.e., $k\Lambda\gg 1$, where Λ is the length of the breather. Moreover, the length of the breather should be significantly greater than the characteristic length of the periodic beats, $\Lambda Q\gg 1$ ($\Lambda\tilde{Q}\gg 1$).

We considered the case of exact resonance $\omega=\omega_0$ and homogeneous broadening of the spectral line. Extension to the case $\omega\neq\omega_0$ and inhomogeneous broadening of the spectral line do not present difficulties. It is obvious that in this case we should not expect qualitatively new results compared to those given above.

We note that the NSE contains not only one-soliton (23), but also N-soliton solutions with a more complicated behavior. In particular, for many-soliton solutions of the NSE there are characteristic oscillations of the envelope and strong compression of the pulse peaks already in the initial stage of propagation of the wave. Under these conditions, we cannot always use the slowly varying envelope approximation (6) and still less Eqs. (11) and (12) (the separation from $\hat{E}_l$ of the more slowly varyings $\varphi_{l,n}^{(\alpha)}$ and $f_{l,n}^{(\alpha)}$). Therefore, the scheme presented above is not valid for such solutions, and for that a completely different method is needed (see, for example, [5]).

The stability of the breathers solutions (24) and (28) is connected with the stability of the sech-soliton solutions of the NSE which have been investigated in detail. It is well known that the soliton solutions of the NSE are highly stable [1,30,31]. Taking into account that the effects of anisotropy do not stimulate any specific instability, special consideration of the stability of the breathers of Eqs. (7) and (8) in anisotropic media is not required.

The NSE is the fundamental equation to describe solitary waves, which occur when dispersion is balanced by nonlinearity, when both group velocity dispersion and nonlinearity play an important role simultaneously. The condition for the existence of the soliton of the NSE is

$$T^2 q_l|\hat{E}_l(0,0)|^2=\frac{\partial^2 k}{\partial\omega^2},$$

where the quantity $|\hat{E}_l(0,0)|^2$ is proportional to an input power of the soliton. An analogous relation for the quantity $\tilde{q_l}$ is valid, too. The input power of the soliton necessary for the excitation of the soliton by means of the quantity, $\partial^2 k/\partial\omega^2$—i.e., from the dispersion properties $k(\omega)$ of the medium—is determined. The experimental study of solitons described by means of the NSE [33] in an AlGaAs sample is reported in Ref. [34]. In these experiments for the solitons, the maximum total power of the pulse inside the sample was maintained at 500–600 W.

Without phase-matching conditions no accumulation of nonresonance nonlinearity along the optical path occurs. In order to get the evident nonlinear effect, a very high power of radiation is necessary. When the phase-matching condition is not fulfilled exactly the efficiency of the cascaded nonlinearities is low and for the realization of the soliton regime of propagation a high-power pulse has to be used [35]. Such solitary waves have been observed experimentally in potassium-titaynl-posphate and in a $LiNbO_3$ uniaxial crystal [36].

We have to note that the input power (intensity) of the waves varies very strongly in different nonlinear optical experiments. For instance, for the excitation of the resonance solitons in ruby, the pulse intensity is of order 100 W/cm^2 [16], but for parametric amplification of the waves in the crystal $LiNbO_3$ an input intensity of 5×10^6 W/cm^2 is used [6]. Single-soliton formation in potassium-titanyl-phosphate has been observed in Ref. [37], with peak intensity thresholds of about 3 GW/cm^2. Multiple-soliton generation mediated by the amplification of asymmetries, with an input peak intensity 23 GW/cm^2 has been recently observed in Ref. [38].

For the breathers the situation is different. The input intensity for breather generation I_{br} is proportional to the quantity $\partial^2\Omega/\partial Q^2$ (or $\partial^2\tilde{\Omega}/\partial\tilde{Q}^2$) and it can be determined from the relation (19) for Ω and Q [or relation (27) for $\tilde{\Omega}$ and $\tilde{Q}$] The quantities Ω and Q ($\tilde{\Omega}$ and $\tilde{Q}$) characterize "internal properties" of the breather. Unlike solitons, the intensity of excitation of the breather is determined not only by the dispersive properties of the medium, but also by the internal parameters of the breather, the direction of wave propagation, and the symmetry of the medium.

We can make estimations of the optimal laser power P_{br} for nonresonance breather generation [breather zone (BZ) I]. For example, let us take typical numerical values for optical media and radiation parameters necessary for realization of the breather regime: $\omega=3\times 10^{14}$ Hz, $n_{||}(\omega)=1.462\,234$, $n_{\perp}(\omega)=1.498\,931$, $\alpha=\pi/12$, $T=3$ ns, and $\omega/\Omega=3\times 10^2$, where $P_{br}=I_{br}A$, $n_{||}$ and $n_{\perp}$ are components of the tensor of the refractive index at frequency ω, and A is the cross-sectional area of the medium, at this $\partial^2\Omega/\partial Q^2 = 10^{-17}$ s^2/cm. In the Kerr medium the total index of refraction has an additional term $n'I$ proportional to the laser intensity I. Using these estimations, for $n' \sim 10^{-10}$ m^2/W, we obtain the minimum power for nonresonance breather generation, $P_{br} \sim 520$ W, in the cubic medium. The power for breather generation in a noncentrosymmetric medium, $P_{br} \sim 350$ W, when the effective susceptibility is $\chi^2_{eff} \sim 5\times 10^{-23}$ (m/V)^2 and the values of the other parameters of the pulse and medium are the same as above. These estimations for the ''blended'' breathers (BZ II) are valid, too. All these estimations are based on the assumption that, for simplicity, the medium has temporal dispersion, cross section $A = 10^{-4}$ cm^2, and the direction of the wave propagation does not coincide with the direction of synchronism.

Because numerical values of the nonlinear susceptibilities can vary very strongly in different media, in different solids the minimum intensity for nonresonance breather generation will be different, too. It is clear that usually the minimum intensity for resonance breather generation is smaller than the minimum intensity for nonresonance or ''blended'' breather generation because in the BZ III always $R_{l,0} \gg m_l$.

In the general case, in a uniaxial nonlinear medium ordinary and extraordinary optical waves can propagate simultaneously, which are connected to each other by means of nonresonance susceptibililties when components $\chi_{xij} \neq 0$ and $\rho_{xijk} \neq 0$, where $i,j,k = x,y,z$. Independent of the kind of initial (or boundary) polarizations of the waves, during the process of propagation elliptical polarized waves will arise, because all x, y, and z components of the vector $\vec{E}$ will be excited. This statement is valid when the direction of the propagation of the waves coincides or is very close to the direction of syncronizm and phase-matching conditions are fulfilled. Under this condition the effect of second-harmonic generation is realized and an intensive exchange of energies between different modes takes place.

But the situation will be different when the soliton or breather regime of the propagation of the waves is satisfied. In the general case, their group and phase velocities are different and the phase-matching condition is not fulfilled. Under this condition the interaction between different modes has the character of the mechanism of cooperative self-action and a reactive interaction between waves takes place [12]. During propagation the breathers (solitons) do not exchange energies between different modes and their amplitudes are not changed. The polarization of the breather depends on the initial (or boundary) polarization of the wave. If the polarization was elliptical at the begining, then breathers appear for ordinary as well as for extraordinary waves. These waves will be propagating without energy exchange.

In the present work a special case is considered, when initially (or at the boundary) the waves are linearly polarized and lying in the yz plane. Consequently, during propagation of the waves only extraordinary waves, for which $\vec{E}$ has only two nonzero components E_y and E_z, will exist. An ordinary wave, for which the quantity E_x is not zero, will not be excited independently of the components of the nonlinear susceptibilities containing all x,y,z indexes that $\chi_{xij} \neq 0$ and $\rho_{xijk} \neq 0$, where $(i,j,k = x,y,z)$. It is because in the breather regime of propagation, different modes realize reactive interactions without a mutual exchange of energy [12]; i.e., the extraordinary waves do not exchange their energy with the ordinary waves. If the ordinary wave is not excited at the begining of a wave excitation, the breather of the extraordinary wave does not provide excitation energy for ordinary waves.

Of course we have neglected all transition processes and assumed that the medium is entered by the pulse already in the form of the breather. Hence we are considering breathers which are always linearly polarized, lying in the y,z plane, and do not become elliptically polarized waves.

Therefore Eq. (3) is valid under the condition when the extraordinary wave does not excite ordinary waves and the x component of $\vec{E}$ is equal to zero. Consequently, elliptically polarized waves do not arise and we consider the situation by means of Eq. (3) when vector $\vec{E}$ is always lying in the yz plane.

Problems of the rotation of the plane of the polarization and conditions for the generation of elliptically polarized states for solitary optical waves are considered in detail in Ref. [39].

The quantity q_l ($\tilde{q}_l$) contains terms coming both from the resonance $R_{l,0}$ and nonresonance m_l nonlinear terms. Depending on the values of these quantities, different mechanisms of the formation of optical breathers can take place.

(a) $m_l = R_{l,0}$ and $m_l R_{l,0} < 0$. This is the condition of the realization of the blended MFB when both the nonresonance and resonance nonlinearities are simultaneously effective and act together with the dispersion in the process of the formation of resonance optical breathers of the small area.

(b) $m_l \ll R_{l,0}$. The pulse interaction with optical impurities has nonlinear character and nonresonance interactions are ignored. This situation corresponds to the self-induced transparency and resonance optical breathers of the small area [2,3].

(c) $m_l \gg R_{l,0}$. The pulse interaction with optical impurities has linear character and does not contribute to the formation of the nonresonance breathers except for a renormalization of their parameters [15]. In particular, in Eqs. (20) and (28) which determine the connection between Ω and Q (or $\tilde{\Omega}$ and $\tilde{Q}$) we should substitute $R_{l,0} = 0$ (see [15]).

From expressions (9), (10), and (15) it is clear that the quantities m_l and $R_{l,0}$ depend not only differently on the direction of wave propagation, but also essentially on the symmetry of the medium. Hence the mechanisms of the formation of the optical breathers of extraordinary waves, which are determined by means of the quantities m_l and $R_{l,0}$, will depend both on the direction of propagation of the pulses and on the symmetry of the medium. Thus we expect that several wave propagation directions exist in uniaxial crystals at which different mechanisms [(a), (b), and (c)] of the formation of the optical breathers are effectively contributing. In order to find these directions we have to analyze the symmetry of the media.

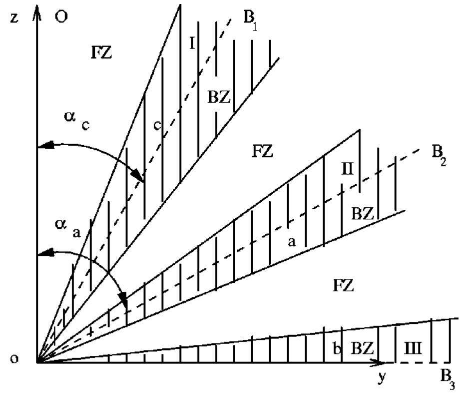

FIG. 2. Three different chosen directions B_1, B_2, and B_3 shown as dashed lines. The BZ correspond to the hatched regions. In the FZ the MFB is suppressed.

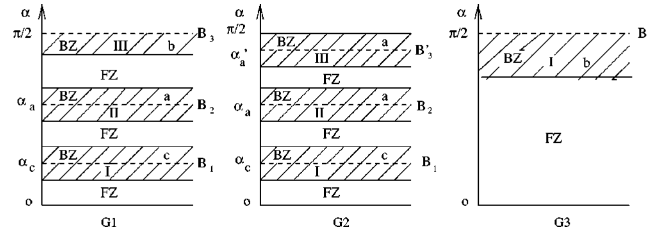

FIG. 3. Three different SBZ realizations for the G1, G2, and G3 of crystals with quadratic nonlinearity. For crystals with cubic nonlinearity the SBZ coincides with the SBZ of G2 of the quadratic nonlinearity crystals. The optical breathers of the small area are formed in the zones I, II, and III (BZ) by means of three different (a), (b), and (c) mechanisms as shown in the figures. These zones are hatched. The width of the zones depends on the nonlinearity parameters.

First we consider (noncentrosymmetric) media with quadratic nonlinearity in the angle interval $[0,\pi/2]$. For $\alpha=0$, the quantity E_z vanishes and hence for this direction expressions (9) and (10) are not determined [we can consider the wave equation (1) for the another y component of the vector $\vec{E}$ and instead of Eqs. (9) and (10) use analogous expressions for E_y]. For this direction $\psi_l(0)=R_{0,l}(0)=0$. Therefore along the z axis and for α close to zero E_z is very small and for these directions no breathers exist (Fig. 2). With increasing α the quantities E_z, $\psi_l(\alpha)$, and $R_{0,l}(\alpha)$ start also to increase. In this region the quantity $\chi^{(2)}_{l,l'}\neq 0$ for all classes of the trigonal, tetragonal, and hexagonal crystal systems except for the crystal classes (CC's) 32, 422, $\overline{4}2m$, and 622 for which all considered components of the quantity $\chi^{(2)}_{l,l'}$ equal zero [13]. When the quantity $R_{l,0}$ is very small, then the influence of the impurities on the wave processes is very small too and hence they do not contribute to the process of the formation of breathers. In this region we thus expect only nonresonance breathers to be excited, independent of the symmetry of the medium (except CC's 32, 422, $\overline{4}2m$, and 622). The corresponding direction is determined using the condition that the dispersion length equal the nonlinear length [13,15]. In this direction the (c) MFB is strongly enhanced.

For $\alpha=\pi/2$ the quantity $\psi_l(\pi/2)=1$, and hence the quantity $R_{l,0}(\pi/2)$ takes its maximum possible value. Under this condition the phenomenon of the self-induced transparency [MFB (b)] is the most effective one and resonance optical breathers of the small area are formed. When the quantity α is deflected from $\pi/2$ but still is very close to $\pi/2$, the quantity $R_{l,0}$ is also very close to its maximum value. Consequently the nonlinear interaction of the optical pulse with impurities is still dominant. From Eq. (9) we can see that depending on the symmetry of the crystals the quantity $\chi^{(2)}_{l,l'}(\alpha\to\pi/2)$ takes different values. For the CC's 4, 6, and $6m2$ of the tetragonal and hexagonal crystal systems $\chi^{(2)}_{l,l'}(\alpha\to\pi/2)=0$ and only MFB (b) will be effective, but for the CC's 3, $3m$, 4, $4mm$, 6, and $6mm$ of the trigonal, tetragonal, and hexagonal crystal systems the quantity $\chi^{(2)}_{l,l'}(\alpha\to\pi/2)\neq 0$ and depending on the ratio $m_l/R_{l,0}$ either (b) or (a) MFB will be realized.

The third important direction is α_{a} where the mechanism (a) of the formation of the resonance optical breathers of the small area is realized. It is obtained from the equation $M_l=R_{l,0}$ with the condition $M_lR_{l,0}<0$.

Thus in crystals with quadratic nonlinearity in the general case we have three chosen directions B_1, B_2, and B_3 (except crystals with CC's 32, 422, $\overline{4}2m$, and 622). While the directions B_1 and B_2 make angles α_c and α_a with the optical O axis, the B_3 direction coincides with the y axis. In Fig. 2 these directions are shown by dashed lines. Breather zones (BZ's) correspond to the hatched regions around these directions where one of the MFB will be most effective. There are forbidden zones (FZ's) located between the BZ's where the MFB are not or only weakly effective.

In the general case the number of chosen directions depends on the symmetry of the crystals. Analyzing expression (9), we can separate all quadratic uniaxial crystals into the three groups: The first group (G1) contains the crystals with the CC's 4, 6, and $6m2$ of the uniaxial tetragonal and hexagonal crystal systems; the second group (G2) contains the crystals with the CC's 3, $3m$, 4, $4mm$, 6, and $6mm$ of the trigonal, tetragonal, and hexagonal crystal systems; and the third group (G3) contains the crystals with the CC's 32, 422, $\overline{4}2m$, and 622 of the trigonal, tetragonal, and hexagonal crystal systems.

The situation considered in Fig. 2 corresponds to the G1 taking into account that for this group of crystals $\chi^{(2)}_{l,l'}(\alpha\to\pi/2)=0$. To investigate the dependence of mechanisms of the formation of breathers on the direction of the wave propagation it will be more convenient if we consider the single BZ (SBZ) in Fig 3.

Unlike the G1, the G2 of the crystals realizes another SBZ (see Fig 3, G2). In particular, for the G2 the quantity $\chi^{(2)}_{l,l'}(\alpha\to\pi/2)\neq 0$ and consequently in zone III the direction B_3' and the angle α_a' appear which are defined through the equation $M_l=R_{l,0}$ (but for a different value $\chi^{(2)}_{l,l'}\sim\chi_{zzz}$ than in zone II). In zones II and III resonance optical breathers will be formed by means of mechanism (a) but breather parameters in zones II and III will be different.

For the G3 of the crystals the situation is quite different as compared to G1 and G2 (see Fig. 3). In this case all components of the quantity $\chi^{(2)}_{l,l'}$ equal zero and consequently $M_l = 0$ everywhere and we have only one special direction B_3 which points along the y axis. The quantity $R_{l,0}(\pi/2)$ has a maximum value in this direction, meaning that in this single BZ the (b) MFB (i.e., self-induced transparency) will be realized. Note that we have ignored the influence of the weak cubic nonlinearity in order to study the SBZ in crystals with quadratic nonlinearity.

From Eq. (10) it follows that all uniaxial crystals with cubic nonlinearity (Kerr media) have the same chosen directions and SBZ as those of the crystals of G2 with quadratic nonlinearity and hence Fig. 2. G2 applies for crystals with cubic nonlinearity as well.

Note that mechanisms (b) and (c) do not act independently but also influence each other. They can support each other under the condition of (a) MFB but in other cases when $m_l - R_{l,0} = 0$—for example, in amplifier media at $\tau_0 < 0$—mechanism (a) is not realized and consequently the SBZ will be changed significantly: BZ II in Fig. 3, G1, and II and III in Fig. 3, G2, will be transformed to the FZ where no breathers exist.

Consequently in the anisotropic uniaxial media three mechanisms [(a), (b), and (c)] of the formation of optical breathers can be realized for different directions of the extraordinary wave propagation depending on the symmetry of the medium. The uniaxial crystals with quadratic nonlinearity can be divided into three different groups with each of them having its own SBZ. The SBZ within one of these groups does not depend on the crystal systems (syngonies or point groups) and is determined by means of the CC. Unlike quadratic media for uniaxial crystals with cubic nonlinearity the SBZ depends neither on the crystal systems nor on the CC and one single SBZ is realized which coincides with the SBZ of the G2 of crystals with quadratic nonlinearity. Hence the mechanisms of the formation of breathers depend on the direction of pulse propagation and this dependence is qualitatively different for media with quadratic and cubic susceptibilities.

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