# Robust measure for characterizing generalized synchronization

Xiao Hu and Valeriy Nenov
*Brain Monitoring and Modeling Laboratory, Division of Neurosurgery, School of Medicine, University of California, Los Angeles, California 90095, USA*

(Received 29 August 2003; revised manuscript received 18 November 2003; published 26 February 2004)

Generalized synchronization between two coupled systems can be characterized by recently proposed interdependency measures calculated from two simultaneously observed time series from them. However, numerical tests have shown that these measures cannot consistently indicate the direction of the coupling for strongly coupled systems or in situations with a large phase space neighbor size. An interdependency measure is proposed here quantifying how close a conditional neighbor is to a true neighbor in terms of the degree of alignment of their principal axes. Numerical tests are carried out on time series generated from a coupled Hénon map and a Lorenz model driven by a Rossler model. Given that a driving system is more dependent on a response system, the results show that the direction of the coupling is consistently detected by using the proposed measure even in those unfavorable cases for the measures mentioned above.

DOI: 10.1103/PhysRevE.69.026206 PACS number(s): 05.45.Xt, 05.45.Tp

## I. INTRODUCTION

Generalized synchrony is a state where the temporal evolutions of dynamic systems are synchronized but not necessarily identical. The technique of time delay embedding [1] has made it possible to investigate this phenomenon by completely depending on discrete observations of the system states without any knowledge of the internal system equations. Following this line has led to the recent invention of several dynamical interdependency measures between two simultaneously recorded time series [2–5]. In addition to being able to characterize the nonlinear correlations between two signals, these measures are also inherently asymmetrical such that the calculated degree of dependency of one system on another is different from that calculated vice versa. This last property makes them very promising in discovering the direction of the coupling for situations where one system is driven by another without having significant feedback on the driving system.

The core of the computation of the aforementioned interdependency measures is the formation of conditional neighbors as originally proposed in Ref. [3]. It was reasoned that if two systems $A$ and $B$ are in a synchronized state, the contemporary states of $B$ corresponding to the neighbors of the state of $A$ at time $t$ should also be close to the state of $B$ at time $t$. Suppose that two trajectories $\mathbf{x}_i$ and $\mathbf{y}_i$, $i=1,\ldots,N$, are reconstructed from two simultaneously recorded time series $x$ and $y$. Furthermore, let $d_{n,j}$ denote the time index of the $j$th nearest neighbor point of $\mathbf{x}_n$ on the trajectory $\mathbf{x}$ and $r_{n,j}$ denote the time index of the $j$th nearest neighbor point of $\mathbf{y}_n$ on the trajectory $\mathbf{y}$. With these notations, a $k$ nearest conditional neighbor of $\mathbf{x}_n$ on $\mathbf{y}_n$ is $\mathbf{x}_{r_{n,j}}$, $j=1,\ldots,k$ and thus $\mathbf{y}_{d_{n,j}}$, $j=1,\ldots,k$ designates the $k$ nearest conditional neighbor of $\mathbf{y}_n$ on $\mathbf{x}_n$. Next, we proceed to briefly summarize some of the publicized interdependency measures.

The definition of the interdependency measure in Ref. [3] is the average of the following ratio of a set of reference points $\mathbf{x}_n$ and $\mathbf{y}_n$:

$$P(n)=\frac{|\mathbf{x}_n-\mathbf{x}_{r_{n,1}}|}{|\mathbf{x}_n-\mathbf{x}_{d_{n,1}}|}\frac{|\mathbf{y}_n-\mathbf{y}_{d_{n,1}}|}{|\mathbf{y}_n-\mathbf{y}_{r_{n,1}}|} \tag{1}$$

where $|\mathbf{x}|$ represents the squared length of the vector $\mathbf{x}$. Theoretically, the above ratio should be close to 1 when $\mathbf{x}$ and $\mathbf{y}$ are synchronized; otherwise it should be much greater than 1. As pointed out in Ref. [5], this measure can be easily contaminated by noise; thus it is only suitable for data generated from theoretical models. Furthermore, it is not asymmetrical.

A prediction error calculated using a zero-order nonlinear cross predicator was adopted in Ref. [4] for defining an interdependency measure as

$$\delta(\mathbf{x}|\mathbf{y})=\frac{\sum_{n=1}^{N}\left|\mathbf{x}_n-(1/k)\sum_{j=1}^{k}\mathbf{x}_{r_{n,j}}\right|}{\sum_{n=1}^{N}|\mathbf{x}_n-\overline{\mathbf{x}}|}, \tag{2}$$

where $k$ is the number of neighbor points for constructing the predictor, $N$ is the total number of points on the trajectory, and $\overline{\mathbf{x}}$ is the arithmetic average of $\mathbf{x}_n$. $\delta(\mathbf{y}|\mathbf{x})$ can be defined in the same manner as above. Ideally, when two systems are completely independent, $\delta(\mathbf{x}|\mathbf{y})$ will be 1, while small $\delta(\mathbf{x}|\mathbf{y})$ indicates a strong dependency of $\mathbf{x}$ on $\mathbf{y}$.

Two interdependence measures were proposed in Ref. [5], the first one being

$$S^k(\mathbf{x}|\mathbf{y})=\frac{1}{N}\sum_{n=1}^{N}\frac{R_n^k(\mathbf{x})}{R_n^k(\mathbf{x}|\mathbf{y})} \tag{3}$$

and the second one being

$$H^k(\mathbf{x}|\mathbf{y})=\frac{1}{N}\sum_{n=1}^{N}\log_2\frac{R_n^{N-1}(\mathbf{x})}{R_n^k(\mathbf{x}|\mathbf{y})}, \tag{4}$$

where $R_n^k(\mathbf{x})$ is computed as

 

$$R_n^k(\mathbf{x}) = \frac{1}{k} \sum_{i=1}^{k} (\mathbf{x}_n - \mathbf{x}_{d_{n,i}})^2 \tag{5}$$

and $R_n^k(\mathbf{x}|\mathbf{y})$ is

$$R_n^k(\mathbf{x}|\mathbf{y}) = \frac{1}{k} \sum_{i=1}^{k} (\mathbf{x}_n - \mathbf{x}_{r_{n,i}})^2. \tag{6}$$

By its construction, $S^k(\mathbf{x}|\mathbf{y})$ is in (0,1]. A low value of $S^k(\mathbf{x}|\mathbf{y})$ indicates weak dependencies between $\mathbf{x}$ and $\mathbf{y}$. As for the $H$ measure, ideally, when $\mathbf{x}$ and $\mathbf{y}$ are from two completely independent systems, $H^k(\mathbf{x}|\mathbf{y})$ will be zero. A large value of $H$ indicates synchronization between them.

In addition to characterizing the coupling strength, it has been proposed that one can infer the direction of the coupling by using the asymmetrical measures reviewed above. However, it is this most desired application that has been problematic, as discussed in comprehensive numerical experiments performed in Ref. [6]. Specifically, the $S$ and $H$ measures as defined in Eq. (3) and Eq. (4), respectively, were studied using bivariate time series generated from a unidirectionally coupled discrete-time map and a unidirectionally coupled continuous-time system. It was demonstrated that the $H$ measure differentiated more consistently between a driving and a response system based on the criterion that the time series observed for a driving system is more dependent on that from the corresponding response system. However, this criterion holds only in a certain range of neighborhood size $k$, outside which the inverse of the criterion becomes true. The dependency of driver-response direction as detected using either the $S$ or the $H$ measure on $k$ is not desirable for processing real data where no prior knowledge regarding how to choose an appropriate neighbor size is available.

It is the purpose of the present work to propose an asymmetrical measure of interdependency that is more robust to neighbor size as well as more sensitive to the direction of coupling. It was realized that the interpoint distance, which unfortunately is directly affected by the size of the neighbor; is the only information that has been explored by existing interdependency measures. To capture a more complete geometric picture of the neighborhoods, the proposed measure is based on the degree of the alignment between the matched principal axes of a conditional neighbor and a true neighbor.

More recently, several publications [7–9] discussed the intrinsic limitation of detecting generalized synchronization for two coupled systems posing a nondifferentiable synchronization function. Results in these publications have shown that any measure of dependency based on the assumption of a continuous synchronization function will probably fail to correctly characterize the degree of the synchronization if a conventional neighborhood formulation is assumed. A possible corrective procedure as discussed in [8] was to use two nontrivial neighborhood formulation methods. Their usefulness was demonstrated using the $\varepsilon_{max} - \delta$ test [8] for detecting synchronization, and later in the work by Rulkov and Afraimovich from a more theoretical perspective [9]. Nevertheless, it remains unknown whether or not the adoption of these two nontraditional neighborhood formulations can benefit the existing measures of coupling strength in characterizing generalized synchronization. To shed light on this important issue, two coupled maps with a nondifferentiable synchronization function were used for calibrating the proposed measure and the existing ones in their abilities to harvest two neighborhood formulation algorithms for an appropriate characterization of generalized synchronization.

## II. METHOD

### A. Coupling strength based on degree of principal axis alignment

To proceed, singular value decomposition (SVD) of two true neighbor matrices and two conditional neighbor matrices is to be obtained as the following

$$[\mathbf{x}_{d_{n,1}}, \ldots, \mathbf{x}_{d_{n,k}}] = U_x \Sigma_x V_x^T, \tag{7}$$

$$[\mathbf{x}_{r_{n,1}}, \ldots, \mathbf{x}_{r_{n,k}}] = U_{x|y} \Sigma_{x|y} V_{x|y}^T, \tag{8}$$

$$[\mathbf{y}_{r_{n,1}}, \ldots, \mathbf{y}_{r_{n,k}}] = U_y \Sigma_y V_y^T, \tag{9}$$

$$[\mathbf{y}_{d_{n,1}}, \ldots, \mathbf{y}_{d_{n,k}}] = U_{y|x} \Sigma_{y|x} V_{y|x}^T. \tag{10}$$

SVD was introduced to the field of nonlinear dynamic analysis in Ref. [10]. Here, only a brief description is provided for the convenience of later discussion. A detailed treatment of SVD can be found in the textbook [11]. SVD of a $M \times N$ matrix $A$ results in two orthonormal matrices $U_{M\times M}$, $V_{N\times N}$, and a $M \times N$ singular value matrix $\Sigma$ whose off-diagonal entries are all zero and whose diagonal entries are termed singular value. Their relationship is

$$A = U \Sigma V^T. \tag{11}$$

As a convention, the columns of $U$ are organized with regard to their corresponding singular values in a descending fashion. What is relevant in motivating the proposed measure is a geometric interpretation of the SVD. Each column of $A$ can be looked upon as a point in an $M$-dimensional space. Hence, $A$ is a ''cloud'' of $N$ such points whose geometric structure is revealed via SVD. Each column of $U$ represents an axis of this $M$-dimensional space while its corresponding singular value gives the variability of the cloud in the direction of that axis. In a synchronized state, the two clouds represented by a conditional neighbor matrix and a true neighbor matrix will be similar in their ''shape.'' Thus, quantifying the degree of alignment of these two clouds will result in a measure of coupling strength. Since $U$ is an orthonormal matrix, a measure of $x$'s dependency on $y$ can hence be defined succinctly as

$$C(x|y) = \frac{\mathrm{tr}(|U_x^T U_{x|y}|, d)}{d}, \tag{12}$$

and similarly $y$'s dependency on $x$ as

$$C(y|x)=\frac{\mathrm{tr}(|U_y^T U_{y|x}|,d)}{d}, \tag{13}$$

where $\mathrm{tr}(|A|,d)$ is the summation of the absolute values of the first $d$ diagonal entries of the matrix $A$. Here, the axes of $U_x$ and $U_{x|y}$ are paired with regard to their singular value order. $C(x|y)$ and $C(y|x)$ are in the range of $[0,1]$. A greater $C(x|y)$ indicates a stronger dependency of $x$ on $y$.

### B. Formulation of neighborhoods

Formation of neighborhoods itself is a straightforward procedure. The only concern was the computational issue which has been largely alleviated by using a K-dimensional tree searching algorithm [12]. This algorithm scales up well with the dimensionality of the searching space as well as the number of points available for searching. However, in a recent paper [7], So *et al.* pointed out the inherent limits on the detectability of the generalized synchronization that are caused by the complicated nondifferentiable geometric structure of the synchronization function that associates the driving and response state variables. Existing numerical methods for quantifying interdependencies including the one proposed here are all based on the assumption that this function is continuous; therefore a straightforward way of formulating neighborhoods as conventionally employed will probably lead to a failure of detection of generalized synchronization using these interdependency measures. Facing these limitations, the authors of [8] have proposed two remedies, both of which utilize some nontrivial neighborhood formulating algorithms. The essence of the proposed algorithms is to impose additional constraints on the neighborhood candidates. Those constraints were chosen to be the conditions that preimages of the current neighborhood should satisfy. Specifically, the first remedy was the "$\delta^p$ neighbor" approach. In addition to the condition that a conventional neighborhood $\mathbf{x}_{d_{n,j}}, j=1,\ldots,k$, should satisfy, i.e., $|\mathbf{x}_{d_{n,j}}-\mathbf{x}_n|<\delta$, neighbor points in a $\delta^p$ neighborhood should also satisfy

$$|\mathbf{x}_{d_{n,j}-i}-\mathbf{x}_{n-i}|<\delta, \quad i=1,\ldots,p. \tag{14}$$

This means that the $p$ preimages of a conventional neighborhood should also be a neighborhood. It is clear that $\delta^0$ can be used to represent a conventional neighborhood formulation, which is just a special case with $p=0$. Their second remedy was an even stronger one, which was termed $\delta^{p,q}$ neighbor searching. To be qualified as being a $\delta^{p,q}$ neighbor point of $\mathbf{x}_n$, the following criterion should be satisfied in addition to Eq. (14):

$$|\mathbf{y}_{d_{n,j}-i}-\mathbf{y}_{n-i}|<\delta, \quad i=1,\ldots,q. \tag{15}$$

Therefore, a $\delta^{p,q}$ neighborhood requires that the $q$ preimages of simultaneous states of the response system $y$, corresponding to those of a $\delta^p$ neighbor, should be a neighborhood as well.

## III. NUMERICAL EXPERIMENTS

For comparison of the $C$ measures with existing ones, we first conducted numerical experiments as in Ref. [6]. Two coupled systems were studied. In the first system, the Rossler system is the driving system

$$\dot{x}_1=-\alpha(x_2+x_3),$$
$$\dot{x}_2=\alpha(x_1+0.2x_2),$$
$$\dot{x}_3=\alpha[0.2+x_3(x_1-5.7)], \tag{16}$$

and the response system is the Lorenz system with an additional input from the above Rossler system as

$$\dot{y}_1=-10(y_2-y_1),$$
$$\dot{y}_2=28y_1-y_2-y_1y_3+\gamma x_2^2,$$
$$\dot{y}_3=y_1y_2-\frac{8}{3}y_3. \tag{17}$$

The parameter $\gamma$ thus controls the coupling strength. $\alpha=6$ and $\alpha=10$ were tested. The above equations were integrated using the ODE45 solver implemented in MATLAB™ 6.0 and sampled at $t_s=0.003$. The same embedding dimensions as selected in Ref. [6] were used here with $m_x=4$ and $m_y=5$. The delay time for embedding was chosen to be 0.3 as used in Ref. [6].

The second system is composed of two coupled Hénon maps with driving equations

$$x_1(n+1)=1.4-x_1^2(n)+b_1x_2(n),$$
$$x_2(n+1)=x_1(n) \tag{18}$$

and the equations

$$y_1(n+1)=1.4-\gamma x_1(n)y_1(n)+(1-\gamma)y_1^2(n)+b_2y_2(n),$$
$$y_2(n+1)=y_1(n) \tag{19}$$

for the response. Again, $\gamma$ controls the coupling strength. $b_1=0.1$, $b_2=0.3$ and $b_1=0.3$, $b_2=0.1$ were tested. The embedding dimension for this coupled system was set as 4 and the delay time was chosen to be 1.

In addition to the above two systems, two coupled maps with a nondifferentiable synchronization function (denoted as a "wrinkling" case in Ref. [7]) were also used for testing the two neighborhood formulation algorithms. This system has a two-dimensional driving map consisting of

$$u_{n+1}=\begin{cases}\lambda u_n, & v_n<\alpha,\\ \lambda+(1-\lambda)u_n, & v_n\geqslant\alpha,\end{cases} \tag{20}$$

and

$$v_{n+1}=\begin{cases}v_n/\alpha, & v_n<\alpha,\\ (v_n-\alpha)/(1-\alpha), & v_n\geqslant\alpha,\end{cases} \tag{21}$$

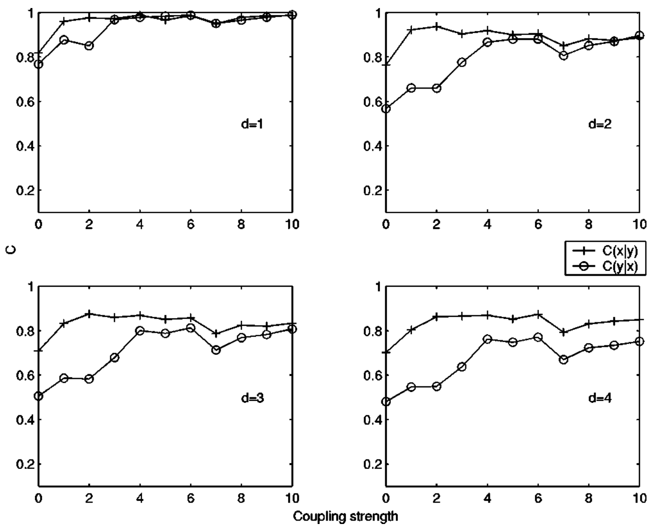


FIG. 1. $C(x|y)$ and $C(y|x)$ at various numbers of directions compared for calculating interdependencies. This number $d$ is marked in each panel. The bivariate time series generated from the Lorenz system uniquely driven by a Rossler system with parameter $\alpha=6$ is used.

and its response map is

$$y_{n+1}=\gamma y_n+\cos(2\pi u_{n+1}) \tag{22}$$

with $0<\alpha<1$ and $0<\lambda<1$. The constant $\alpha$ was chosen as 0.7 with $\lambda$ as 0.8 in the numerical test. When the controlling parameter $\gamma$ satisfies $|\gamma|<1$, the response is asymptotically stable for all $u$ and thus the system should always be in a generalized synchronization state. Consequently, any measure of coupling strength should in principle approach 1. However, as $|\gamma|$ increases from 0 to 1, the degree of "wrinkling" also increases, leading to a decreasing apparent coupling strength if conventional neighborhoods are formulated. This phenomenon is actually caused by the increasing number of nondifferentiable regions as $|\gamma|$ increases.

## IV. RESULTS

The parameter $d$ in Eqs. (12) and (13) controls the number of axial directions for checking the degree of alignment. The effect of this parameter on the final calculated $C$ measures is shown in Fig. 1 and Fig. 2. Due to the fact that system under examination is a low dimensional system, a small $d$ is enough to provide adequate sensitivity for telling the direction of coupling when the coupling strength is weak, as in the case of $d=1$. On the other hand, to detect the direction of coupling in a highly synchronized situation, $d$ should be increased. However, an excessively large $d$ might be harmful because a comparison of those directions corresponding to very small singular values might not be relevant for comparing two spaces. This is due to the fact that very little variation is manifested in these directions. Nevertheless, in these two experiments, the direction of the coupling was always correctly detected even with a high $d$. Further investigation of this issue is necessary, especially for a noisy data set. The proposed $C$ measure is capable of telling the coupling direction even in a strongly coupled situation. In contrast, the existing measures cannot differentiate the coupling direction in such cases.

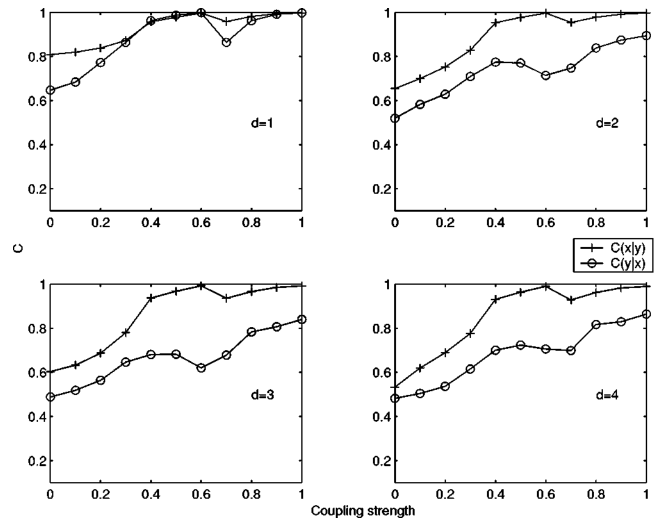


FIG. 2. Same as Fig. 1, but the coupled Hénon map is studied.

Next, we will investigate the performance of the $C$ measure at different neighbor sizes. Shown in Figs. 3–6 are the $C$ measures calculated at $k=10$, 20, 30, 40, 50, and 60 for the four bivariate time series generated as described above. Clearly, the proposed $C$ measure is robust with regard to the neighborhood size in detecting the direction of coupling. As reported in Ref. [6], the $H$ measure was the most robust one among the existing measures tested, but at a neighbor size of $k=30$ and for strong coupling ($\gamma>2$), the $H$ measure became inconsistent in detecting the direction of the coupling between the two systems.

Three measures of interdependency including the pro-

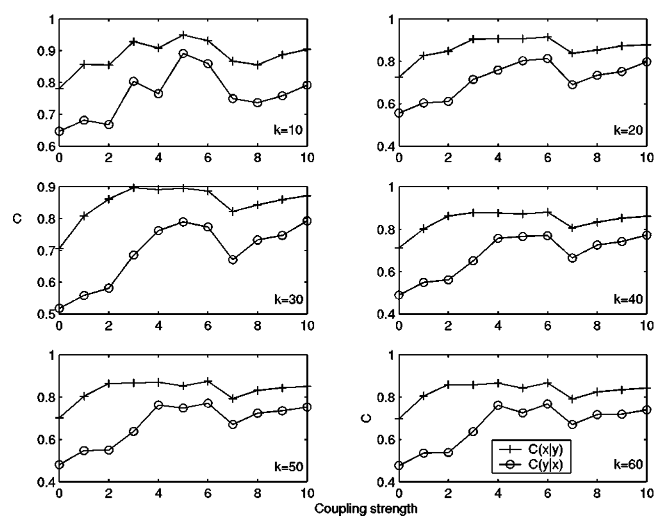


FIG. 3. $C(x|y)$ and $C(y|x)$ at various coupling strengths ($\gamma$) and at various neighborhood sizes ($k$) as calculated from the bivariate time series from a Lorenz system uniquely driven by a Rossler system. The parameter $\alpha$ in the driving equations is 6. $k$ is marked in each panel.

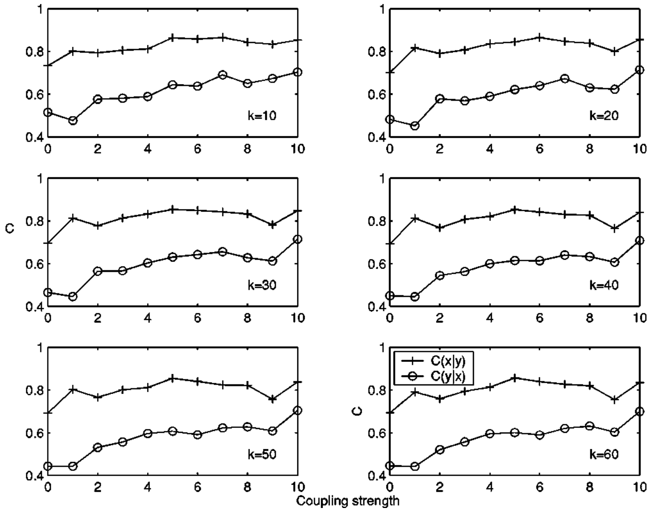


FIG. 4. Same as Fig. 3, with $\alpha=10$.

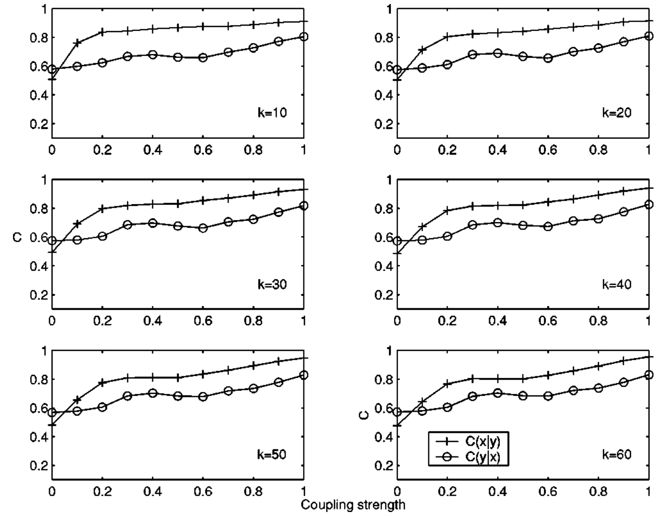


FIG. 6. Same as Fig. 5, with $b_1=0.3$, $b_2=0.1$.

posed $C$ measure, the $H$ measure, and the $S$ measure for coupled systems with a non differentiable synchronization function were calculated using different neighborhood formulation algorithms with an embedding dimension chosen as 3, delay time as 1, and $d=3$. These results are shown in Figs. 7–9. Although theoretical analysis indicates that the system is always in a general synchronization state as long as $|\gamma|<1$, as can be seen from those figures, the nondifferentiability of the synchronization function will affect the detection of generalized synchronization if a conventional neighborhood formulation procedure is adopted because all the tested independency measures start to decrease around $\gamma=0.5$. This happens because the ''wrinkling'' effect becomes more widespread at this point. While the adoption of the $\delta^6$ and $\delta^{3,3}$ neighborhood formulation this effect lessens for all the measures; the $C$ measure seems to be able to achieve the most significant improvement by recognizing that there is almost no decrease of the $C$ measure with increasing $\gamma$ on using a $\delta^{3,3}$ neighborhood formulation. It is also expected that the $\delta^{3,3}$ should help more than the $\delta^6$ method for handling a ''wrinkling'' synchronization function [8] like the one produced in this numerical experiment.

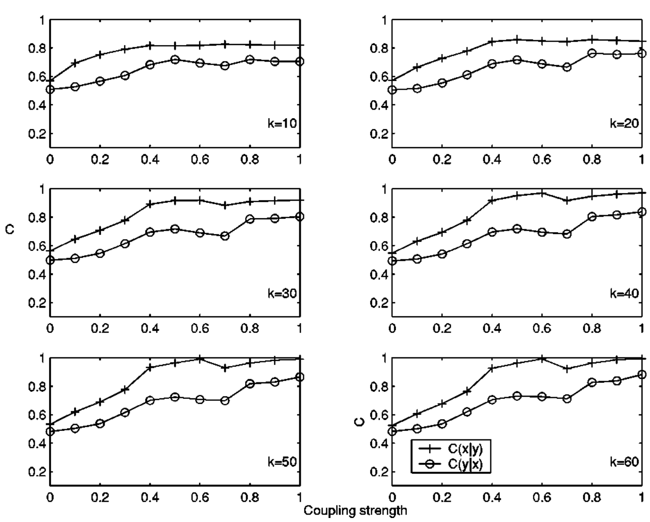


FIG. 5. Same as Fig. 3 except the bivariate time series are generated from two coupled Hénon maps with $b_1=0.3$ and $b_2=0.1$. Coupling strength $\gamma$ increases from 0 to 1 with a step size of 0.1.

## V. DISCUSSION

Although the proposed $C$ measure and the existing interdependency measures share the concept of the conditional neighbor as a common theoretical basis, they are fundamentally different in the way the information regarding the interdependency is extracted from these neighborhoods. First, the $C$ measure relies on the difference between a conditional and a true neighbor for quantization of the coupling, thus always being normalized in each individual system. In this way, it

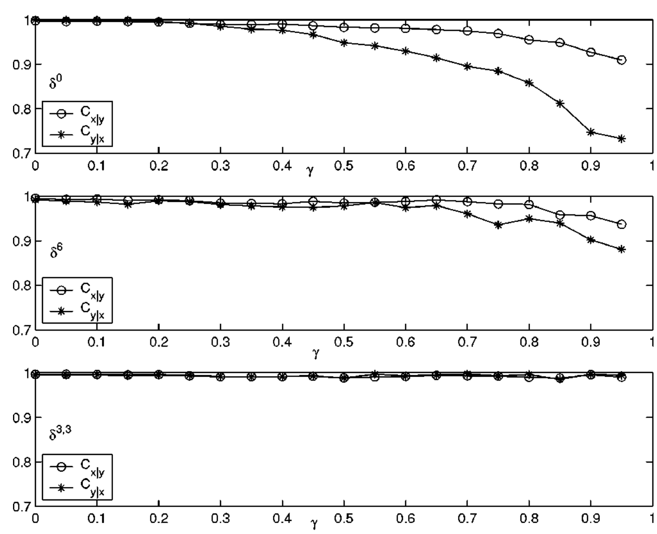


FIG. 7. $C$ measure of interdependency calculated using three different neighborhood formulation algorithms for two coupled systems with a ''wrinkling'' synchronization function. The top panel shows the result obtained using the conventional $\delta^0$ neighborhood while the results using a $\delta^6$ and a $\delta^{3,3}$ neighborhood are displayed in the middle and bottom panels, respectively.

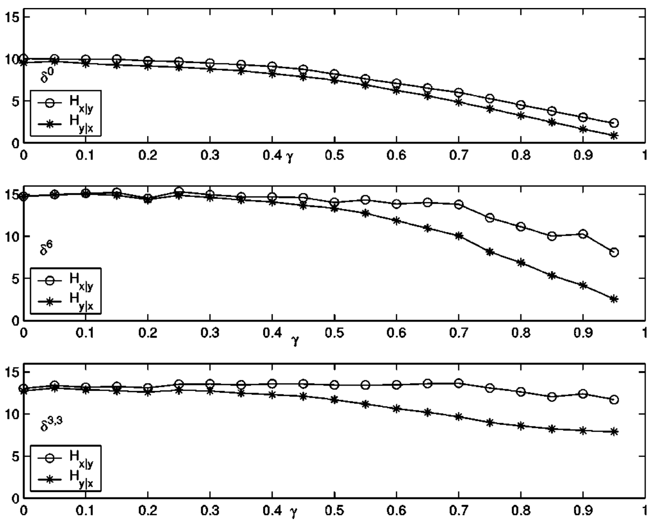


FIG. 8. Same as Fig. 7 except interdependency was characterized using $H$ measure.

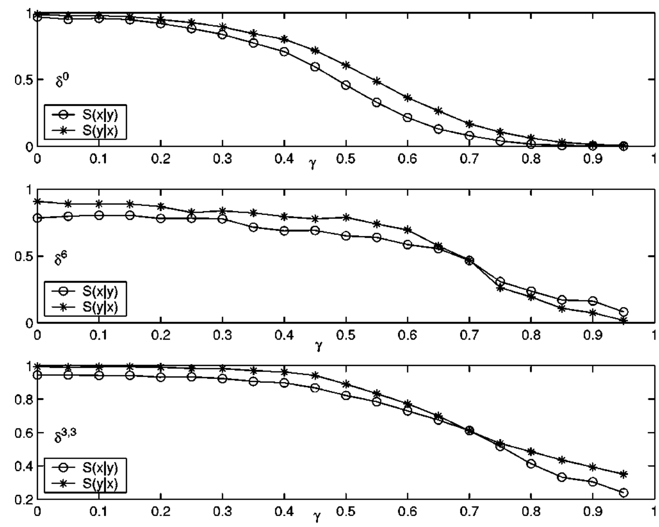


FIG. 9. Same as Fig. 7 except interdependency was characterized using $S$ measure.

avoids the problem that the existing measures might reflect the complexity of the individual system instead of the coupling strength [6,13]. Second, it does not depend on the size of the neighborhood for measuring the difference between two spaces but on the degree of alignment of their principal axes. In this way, the interdependency measure is not directly affected by the neighbor size and thus, as the numerical results indicate, it is more robust than the existing measures with regard to neighbor size.

As made clear by recent research, a complicated nondifferentiable synchronization function might destroy the ability to detect generalized synchronization by interdependency measures. Although this is true for all the interdependency measures with an underlying continuity assumption, our numerical results demonstrate that the $C$ measure responded most to the adoption of a $\delta^{p}$ as well as a $\delta^{p,q}$ neighborhood for handling complicated geometries of a synchronization function. This is a highly desirable feature of an interdependence measure for practical use.

There is still one more point that needs clarification before concluding the paper. Throughout, we have suggested that a driver system depends more on a response system, as indicated by $C(x|y)>C(y|x)$. This is a little bit counterintuitive considering that the driver is usually an autonomous system. It should be realized that calculation of the proposed interdependency measure and other types of measures referred to in the paper involves the formulation of conditional neighborhoods. This inevitably introduces a type of neighborhood for any point on the driving system's trajectory that is dependent on the way it drives the response system. Therefore, this naturally leads to the possibility of defining a dependency measure that has been termed the dependency of $x$ on $y$. In essence, this dependency of $x$ on $y$ like its counterpart, the dependency of $y$ on $x$, reflects only how $x$ drives $y$. Then why would one define such an "$x$ dependent on $y$" measure if it represents essentially the same thing as the more intuitive "$y$ dependent on $x$"? The answer is that it provides a potential way to differentiate between a driving and a response by virtue of the asymmetry between $C(x|y)$ and $C(y|x)$. However, for a unidirectionally coupled system, the difference between $C(x|y)$ and $C(y|x)$ does not mean that there exists any dependency difference. Instead, this difference mainly reflects the nontrivial geometries of the synchronization function associating $x$ to $y$. As discussed in Ref. [6], the observed relationship $C(x|y)>C(y|x)$ may be mainly due to the fact that a response system usually has a more complicated phase portrait than a driving system and is more active.

## VI. CONCLUSION

A different way of using the conditional neighbor concept for inferring the coupling strength and its direction has been proposed. The proposed $C$ measure performed well in the standard numerical experiments. The incorporation of nontrivial neighborhood formulation algorithms may even make the $C$ measure applicable to more general nondifferentiable situations often encountered in generalized synchronization studies. Its further calibration with real patients' intracranial pressure and cerebral blood flow velocity data is being conducted.

## ACKNOWLEDGMENTS

We wish to acknowledge financial support for the present research from the Division of Neurosurgery at University of California, Los Angeles and the UCLA Brain Injury Research Center.

[1] F. Takens, in *Dynamical Systems and Turbulence (Warwick 1980)*, edited by D. A. Rand and L.-S. Young (Springer-Verlag, Berlin, 1980), Vol. 898, pp.366–381.

[2] L. M. Pecora, T. L. Carroll, and J. F. Heagy, Phys. Rev. E **52**, 3420 (1995).

[3] N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995).

[4] S. J. Schiff, P. So, C. Taeun, R. E. Burke, and T. Sauer, Phys. Rev. E **54**, 6708 (1996).

[5] J. Arnhold, P. Grassberger, K. Lehnertz, and C. E. Elger, Physica D **134**, 419 (1999).

[6] R. Q. Quiroga, J. Arnhold, and P. Grassberger, Phys. Rev. E **61**, 5142 (2000).

[7] P. So, E. Barreto, K. Josic, E. Sander, and S. J. Schiff, Phys. Rev. E **65**, 046225/1 (2002).

[8] D. He, Z. Zheng, and L. Stone, Phys. Rev. E **67**, 026223 (2003).

[9] N. F. Rulkov and V. S. Afraimovich, Phys. Rev. E **67**, 066218 (2003).

[10] D. S. Broomhead and G. P. King, Physica D **20**, 217 (1986).

[11] G. H. Golub and C. F. VanLoan, *Matrix Computations*, 3rd ed., Johns Hopkins Studies in the Mathematical Sciences (Johns Hopkins University Press, Baltimore, 1996).

[12] S. Arya, D. M. Mount, N. Netanyahu, R. Silverman, and A. Y. Wu, J. ACM **45**, 891 (1998).

[13] E. Pereda, R. Rial, A. Gamundi, and J. Gonzalez, Physica D **148**, 147 (2001).