Soliton of modified nonlinear Schrödinger equation with random perturbations

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Soliton of the modified Schrödinger equation which describes the propagation of femtosecond optical pulses in nonlinear optical fibers is studied in the presence of random perturbations. Two cases are considered—an initial random perturbation and a multiplicative noise. A perturbation theory based on the inverse scattering transform is used. Spectral distribution of the emitted radiation and statistical characteristics of the soliton parameters are obtained.

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I. INTRODUCTION

The propagation of ultrashort pulses in nonlinear media has become a topic of intense research, stimulated by the rapid progress of femtosecond laser sources. As is well known [1,2], the classical, mathematical model for nonlinear pulse propagation in the picosecond time scale in an isotropic, homogeneous, lossless, nonamplifying, single mode optical fiber is the nonlinear Schrödinger equation (NLSE), which is integrable by an inverse scattering transform (IST) [3]. This equation models the effects of group velocity dispersion and Kerr nonlinearity. As a result of the balance between them, a special class of pulses (solitons) can propagate in optical fibers without changing their forms. Optical solitons are therefore expected to be suitable information carriers in optical communication systems. However, experiments and theories on the propagation of ultrashort pulses in the long monomode fibres have shown that dynamics of femtosecond pulses ($\leqslant$100 fs) is not well governed by the NLSE. The spectral width of the pulses becomes comparable with the carrier frequency and additional effects should be taken into account: the Kerr nonlinearity dispersion (self-steepening), the Raman stimulated scattering (soliton self-frequency shift), third-order linear dispersion, etc. The account for only the nonlinearity dispersion leads to the so called perturbed modified NLSE (MNLSE) [4]

$$i\partial_t u+\frac{1}{2}\partial_x^2 u+i\alpha\partial_x(|u|^2 u)+\beta|u|^2 u+p[u,u^*]=0, \quad (1)$$

where $u(x,t)$ is the normalized slowly varying amplitude of the complex field envelope, t is the normalized propagation distance along the fiber, x is the normalized time measured in a frame of reference moving with the pulse at the group velocity (the retarded time), real parameters β and α govern the effects of the Kerr nonlinearity and Kerr nonlinearity dispersion, respectively, and $p[u,u^*]$ accounts for effects which we will consider as a perturbation. Equation (1) with $p[u,u^*]=0$ is still integrable by the IST [5–8] though the associated spectral problem is different from the Zakharov-Shabat one. Namely, the initial value problem for the MNLSE can be solved within the framework of the Wadati-Konno-Ichikawa problem [9].

The aim of this paper is twofold. First, for the unperturbed MNLSE we investigate evolution of the soliton parameters when initial (launching, in the optical fiber literature) soliton pulse contains a small additional random part. Several forms of the noise spectrum are considered. We also determine a spectral distribution of radiation accompanying the soliton into the fiber. The importance of this problem is evident since practical lasers cannot be designed to excite only the pure soliton mode, but also excite an entire continuum of linearlike dispersive (radiative) waves, so that the input pulse always contains a small noise additive part. Second, on the basis of the "classical" formulation of the IST we develop a perturbation theory for Eq. (1) with $p[u,u^*]\neq 0$ and then consider a perturbation in the form of a multiplicative noise $p=\varepsilon(x,t)u$, where $\varepsilon(x,t)$ is a random Gaussian field. This case corresponds to fluctuating part of the refractive index. We obtain the averaged power spectral density of radiation emitted by the moving soliton. Note that a variant of the perturbation theory for the MNLSE, based on the Riemann-Hilbert formulation of the IST, had been suggested in Ref. [10], but explicit expressions for the nonsoliton (radiative) part of the scattering data had not been written. On the basis of the results obtained in Ref. [10], influence of the additive δ-correlated Gaussian noise on the MNLSE soliton was studied in Ref. [11]. A direct, independent of the IST, perturbation theory for the MNLSE was developed in Ref. [12], where some results of Ref. [10] were confirmed, and, in addition, perturbation-induced radiation in the physical space was obtained. We obtain the radiative part in the spectral (natural for the IST) form that can be more preferable from the practical point of view.

The paper is organized as follows. Section II begins with a review of the theory of the scattering transform for the corresponding linear eigenvalue problem. A simple method for finding N-soliton solutions and corresponding Jost functions is also suggested. Evolution of randomly perturbed initial soliton pulse is considered in Sec. III. Then, in Sec. IV we present the perturbation theory for the MNLSE and in Sec. V study the influence of an external multiplicative noise on the MNLSE soliton.

Regarding notations, we will use stars for complex conjugation, and matrices will be written with bold letters, except for the Pauli matrices

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$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

II. INVERSE SCATTERING THEORY FOR THE MNLSE

A. Scattering data

When $p[u,u^*]=0$, Eq. (1) can be represented as the compatibility condition

$$\mathbf{U}_t-\mathbf{V}_x+[\mathbf{U},\mathbf{V}]=0 \tag{2}$$

of two linear matrix equations

$$\partial_x\mathbf{M}=\Lambda(\lambda)\sigma_3\mathbf{M}+2i\lambda\mathbf{QM}\equiv\mathbf{UM}, \tag{3}$$

$$\partial_t\mathbf{M}=\Omega(\lambda)\sigma_3\mathbf{M}+\lambda(2i\lambda\sigma_3\mathbf{Q}^2-2\Lambda(\lambda)\mathbf{Q}-2i\alpha\mathbf{Q}^3+\sigma_3\partial_x\mathbf{Q})\mathbf{M}\equiv\mathbf{VM}, \tag{4}$$

where λ is a spectral parameter, the 2×2 matrix $\mathbf{Q}=\begin{pmatrix}0&u\\u^*&0\end{pmatrix}$ corresponds to the potential of the linear spectral problem Eq. (3), and

$$\Lambda(\lambda)=-(2i/\alpha)(\lambda^2-\beta/4), \tag{5}$$

$$\Omega(\lambda)=-(4i/\alpha^2)(\lambda^2-\beta/4)^2. \tag{6}$$

This means [13], in particular, that Eq. (1) with $p[u,u^*]=0$ is integrable in the sense of the IST.

Further in this section we assume, generally speaking, $p[u,u^*]\neq0$. In this case Eq. (1) is not integrable (and there is no any *compatibility* condition), but it can be cast in matrix form

$$\mathbf{U}_t-\mathbf{V}_x+[\mathbf{U},\mathbf{V}]+\mathbf{P}=0, \tag{7}$$

where $\mathbf{P}=\begin{pmatrix}0&\lambda p2\\-2\lambda p^*&0\end{pmatrix}$, and matrices $\mathbf{U}$ and $\mathbf{V}$ are the same as in Eqs. (3) and (4). It is assumed that $u(x)$ and $p[u,u^*]$ for some fixed t are vanishing fast enough as $x\to\pm\infty$. Let us consider Eq. (3). For $\lambda^2\in\mathbb{R}$ denote by $\mathbf{M}^{\pm}(x,t,\lambda)$ the 2×2 matrix Jost solutions of Eq. (3), satisfying the boundary conditions

$$\mathbf{M}^{\pm}\to\mathbf{E}(x,\lambda)\equiv\exp(\Lambda(\lambda)\sigma_3x) \tag{8}$$

as $x\to\pm\infty$. Since $\mathrm{Tr}\mathbf{U}=0$, these boundary conditions guarantee that $\det\mathbf{M}^{\pm}=1$ for all x. The symmetry properties of $\mathbf{M}^{\pm}$ follow from Eq. (3),

$$\mathbf{M}^{\pm *}(x,t,\lambda)=\sigma_2\mathbf{M}^{\pm}(x,t,\lambda)\,\sigma_2, \tag{9}$$

$$\mathbf{M}^{\pm}(x,t,\lambda)=\sigma_3\mathbf{M}^{\pm}(x,t,-\lambda)\,\sigma_3. \tag{10}$$

For each λ^2 there can only be two linearly independent columns of $\mathbf{M}^{\pm}(x,t,\lambda)$; therefore there is a matrix $\mathbf{S}(t,\lambda)$, $\lambda^2\in\mathbb{R}$, the *scattering matrix*, such that

$$\mathbf{M}^{-}(x,t,\lambda)=\mathbf{M}^{+}(x,t,\lambda)\mathbf{S}(t,\lambda) \tag{11}$$

with the symmetry properties

$$\mathbf{S}^*(t,\lambda)=\sigma_2\mathbf{S}(t,\lambda)\,\sigma_2, \tag{12}$$

$$\mathbf{S}(t,\lambda)=\sigma_3\mathbf{S}(t,-\lambda)\,\sigma_3. \tag{13}$$

It follows from Eq. (11) that the coefficients S_{11} and S_{12} are

$$S_{11}(t,\lambda)=\det(M_1^-(x,t,\lambda),M_2^+(x,t,\lambda)), \tag{14a}$$

$$S_{12}(t,\lambda)=\det(M_2^-(x,t,\lambda),M_2^+(x,t,\lambda)), \tag{14b}$$

where $M_j^{\pm}$ means the jth column of $\mathbf{M}^{\pm}$. The corresponding integral equations for $\mathbf{M}^{\pm}$ can be obtained from Eqs. (3) and (8)

$$\mathbf{M}^{\pm}(x,t,\lambda)=\mathbf{E}(x,\lambda)\mp2i\lambda\int_x^{\pm\infty}\mathbf{E}(x-y,\lambda)\mathbf{Q}(y,t)\times\mathbf{M}^{\pm}(y,t,\lambda)\,dy. \tag{15}$$

The standard analysis of these Volterra-type integral equations yields the expressions for the Jost solutions at $\lambda=0$,

$$\mathbf{M}^{+}(x,t,0)=\mathbf{M}^{-}(x,t,0)=\mathbf{E}(x,0), \tag{16}$$

and the asymptotics at $\lambda\to\infty$,

$$\mathbf{M}^{\pm}(x,t,\lambda)=\mathbf{E}(x,\lambda)\exp\{\mp i\sigma_3\theta^{\pm}(x,t)\}\left[\mathbf{I}+\frac{1}{4\lambda}\mathbf{Q}(x,t)\right]\times\left[1+O\left(\frac{1}{\lambda^2}\right)\right], \tag{17}$$

where we have introduced the notations

$$\theta^{+}(x,t)=\alpha\int_x^{\infty}|u(y,t)|^2\,dy, \tag{18}$$

$$\theta^{-}(x,t)=\alpha\int_{-\infty}^{x}|u(y,t)|^2\,dy. \tag{19}$$

The vector functions $M_1^-(x,t,\lambda)$, $M_2^+(x,t,\lambda)$ turn out to be analytically continuable to $\mathrm{sgn}\alpha\,\mathrm{Im}\,\lambda^2>0$, while M_2^-, M_1^+ are analytically continuable to $\mathrm{sgn}\alpha\,\mathrm{Im}\,\lambda^2<0$. It then follows from Eq. (14a) that the coefficient $S_{11}(\lambda)$ as a function of λ is analytically continuable to $\mathrm{sgn}\alpha\,\mathrm{Im}\,\lambda^2>0$ with the asymptotic at $\lambda\to\infty$,

$$S_{11}=\exp(-i\theta_0)\left[1+O\left(\frac{1}{\lambda^2}\right)\right], \tag{20}$$

where $\theta_0=\theta^-+\theta^+$. Likewise $S_{22}(\lambda)$ is analytically continuable to $\mathrm{sgn}\alpha\,\mathrm{Im}\,\lambda^2<0$. It follows from Eqs. (12) and (13) that

$$S_{22}(\lambda)=S_{11}^*(\lambda^*),\quad S_{21}(\lambda)=-S_{12}^*(\lambda^*) \tag{21}$$

and

$$S_{11}(\lambda)=S_{11}(-\lambda),\quad S_{12}(\lambda)=-S_{12}(-\lambda). \tag{22}$$

Also, for $\lambda^2\in\mathbb{R}$ the fact that $\det\mathbf{S}=1$ implies the normalization condition $|S_{11}(\lambda)|^2+\mathrm{sgn}\lambda^2|S_{12}(\lambda)|^2=1$.

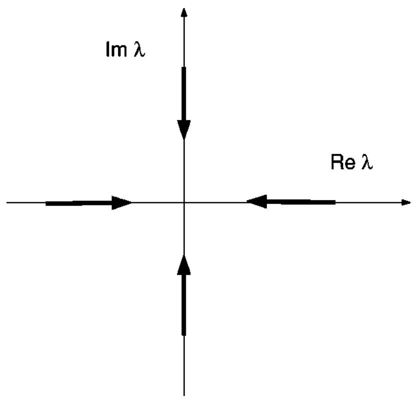

FIG. 1. Integration contour Γ for $\alpha>0$ (reverse direction is implied for $\alpha<0$).

The analytic function $S_{11}(t,\lambda)$ may have zeros $\lambda_1(t),\ldots,\lambda_N(t)$ in the region of its analyticity $\mathrm{sgn}\alpha\,\mathrm{Im}\,\lambda^2>0$. The determinant equation (14a) then shows that the columns M_2^+ and M_1^- are linearly dependent and there exist complex numbers $\gamma_1(t),\ldots,\gamma_N(t)$ such that

$$M_2^+(x,t,\lambda_k(t))=\gamma_k(t)M_1^-(x,t,\lambda_k(t)) \tag{23}$$

for $k=1,\ldots,N$. At the complex conjugate points $\lambda_k^*(t)$ we have

$$M_1^+(x,t,\lambda_k^*(t))=-\gamma_k^*(t)M_2^-(x,t,\lambda_k^*(t)) \tag{24}$$

for $k=1,\ldots,N$.

Since $S_{11}\exp(i\theta_0)\to1$ as $\lambda\to\infty$ with $\mathrm{sgn}\alpha\,\mathrm{Im}\,\lambda^2>0$ and $S_{11}(\lambda)=S_{11}(-\lambda)$, standard methods of the Hilbert transform theory [13] can be used in conjunction with the normalization condition to express $S_{11}(t,\lambda)$ for $\mathrm{sgn}\alpha\,\mathrm{Im}\,\lambda^2>0$ in terms of its zeros and the values of $|S_{12}(t,\lambda)|$ on the contour $\Gamma=\{\lambda;\mathrm{Im}(\lambda^2)=0\}$ (oriented as in Fig. 1),

$$S_{11}(t,\lambda)=\prod_{k=1}^{N}\frac{\lambda^2-\lambda_k^2(t)}{\lambda^2-\lambda_k^{*2}(t)}\exp\Biggl\{-i\theta_0+\frac{1}{2\pi i}\times\int_\Gamma\frac{\mu\ln(1-\mathrm{sgn}(\mu^2)|S_{12}(t,\mu)|^2)}{\mu^2-\lambda^2}d\mu\Biggr\}. \tag{25}$$

From Eqs. (16) and (14a) we have $S_{11}(t,0)=1$, then, setting $\lambda=0$ in Eq. (25), one can find $\theta_0=\alpha\int_{-\infty}^{\infty}|u(x,t)|^2\,dx$ in terms of the scattering data

$$\theta_0=4\sum_{k=1}^{N}\arg(\lambda_k)-\frac{1}{2\pi}\int_\Gamma\frac{\ln(1-\mathrm{sgn}(\lambda^2)|\,S_{12}(\lambda)|^2)}{\lambda}d\lambda, \tag{26}$$

where $0<\arg(\lambda_k)<\pi/2$ if $\alpha>0$, and $-\pi/2<\arg(\lambda_k)<0$ if $\alpha<0$.

Dynamics of the scattering data turns out to be trivial when $p=0$

$$S_{12}(t)=S_{12}(0)\exp(2\Omega(\lambda)t), \tag{27}$$

$$\lambda_k(t)=\lambda_k(0), \tag{28}$$

$$\gamma_k(t)=\gamma_k(0)\exp(2\Omega(\lambda_k)t). \tag{29}$$

B. The Jost functions and the potential in the reflectionless case

When $p=0$, the Jost functions $\mathbf{M}^\pm(x,t,\lambda)$ and the potential $u(x,t)$ can be recovered for each fixed t from the scattering data, namely, the reflection coefficient $S_{12}(t,\lambda)$ for $\lambda\in\Gamma$, the eigenvalues $\{\lambda_k(t)\}$ with $\mathrm{sgn}\alpha\mathrm{Im}\lambda^2>0$, and the proportionality constants $\{\gamma_k(t)\}$. An important particular case is that of the reflectionless (solitonic) potentials $u(x)$ when $S_{12}(t,\lambda)\equiv0$ as a function of λ for some fixed t. It then follows from Eq. (25) that

$$S_{11}(t,\lambda)=\prod_{k=1}^{N}\frac{\lambda_k^{*2}(\lambda^2-\lambda_k^2)}{\lambda_k^2(\lambda^2-\lambda_k^{*2})}, \tag{30}$$

which extends to $\mathrm{sgn}\alpha\,\mathrm{Im}\,\lambda^2<0$ as a meromorphic function. One also sees that $S_{22}(t,\lambda)=1/S_{11}(t,\lambda)$ and that $S_{21}(t,\lambda)\equiv0$. Since $\mathbf{S}(t,\lambda)$ is diagonal in this case, it can be factorized in such a way $\mathbf{S}^-(t,\lambda)=\mathbf{S}^+(t,\lambda)\mathbf{S}(t,\lambda)$ that the Jost solution matrices $\mathbf{M}^\pm$ is expressed through a common solution matrix $\mathbf{A}(x,t,\lambda)$,

$$\mathbf{M}^\pm(x,t,\lambda)=\mathbf{A}(x,t,\lambda)\mathbf{S}^\pm(t,\lambda), \tag{31}$$

where

$$\mathbf{S}^+=\mathrm{diag}\left(\prod_{k=1}^{N}\frac{\lambda_k}{\lambda_k^*(\lambda^2-\lambda_k^2)},\prod_{k=1}^{N}\frac{\lambda_k^*}{\lambda_k(\lambda^2-\lambda_k^{*2})}\right) \tag{32}$$

and

$$\mathbf{S}^-=\sigma_1\mathbf{S}^+\sigma_1. \tag{33}$$

The columns of $\mathbf{A}(x,t,\lambda)$ necessarily satisfy the relations

$$A_2(x,t,\lambda_k(t))=\gamma_k(t)A_1(x,t,\lambda_k(t)), \tag{34a}$$

$$A_1(x,t,\lambda_k^*(t))=-\gamma_k^*(t)A_2(x,t,\lambda_k^*(t)) \tag{34b}$$

for all $k=1,\ldots,N$. Since $\mathbf{A}(\lambda)$ is analytical in the λ plane, it follows from Eqs. (17) and (31) that diagonal and off-diagonal elements of the matrix $\mathbf{A}(\lambda)\mathbf{E}^{-1}(\lambda)$ are polynomials in λ of degrees $2N$ and $2N-1$, respectively. In addition, from Eqs. (10) and (31) one sees that the diagonal elements are even in λ, while the off-diagonal ones are odd. This means that

$$\mathbf{A}(x,t,\lambda)\mathbf{E}^{-1}(x,\lambda)=\begin{pmatrix}A_{11}^{(0)} & 0\\ 0 & A_{22}^{(0)}\end{pmatrix}+\sum_{p=1}^{N}\lambda^{2p-1}\begin{pmatrix}\lambda A_{11}^{(p)} & A_{12}^{(p)}\\ A_{21}^{(p)} & \lambda A_{22}^{(p)}\end{pmatrix}. \tag{35}$$

Setting here $\lambda=0$, we readily get from Eqs. (16) and (31)—(33) the expressions for the functions $A_{11}^{(0)}(x,t)$ and $A_{22}^{(0)}(x,t)$,

$$A_{11}^{(0)}=A_{22}^{(0)}=-\prod_{k=1}^{N}|\lambda_k(t)|^2. \quad (36)$$

The remaining unknown matrix coefficients $\mathbf{A}^{(p)}(x,t)$ with $p=1,\ldots,N$ are determined uniquely from Eqs. (34). Indeed, the first row of Eqs. (34) is a linear algebraic system of $2N$ equations in $2N$ unknowns, the coefficients $A_{12}^{(p)}$ and $A_{11}^{(p)}$ with $p=1,\ldots,N$. Likewise, the second row of Eqs. (34) is the system for determining $A_{21}^{(p)}$ and $A_{22}^{(p)}$ with $p=1,\ldots,N$. By direct substitution one can check that Eq. (35) is compatible with Eqs. (3) and (31) if and only if

$$u(x,t)=\frac{2A_{12}^{(N)}(x,t)}{\alpha A_{22}^{(N)}(x,t)}. \quad (37)$$

This formula reconstructs $u(x,t)$ from the discrete scattering data $\{\lambda_k(t)\}$, $\{\gamma_k(t)\}$ in the case when $S_{12}(t,\lambda)\equiv 0$ and it gives N-soliton solution of Eq. (1). An explicit form of the solution can be easily written in terms of the determinants of corresponding matrices. Equations (31) and (35) determine the N-soliton Jost functions. This treatment of the soliton solutions follows the ideas of Refs. [14–16].

In conclusion of this section we note that elements $A_{ij}(x,t,\lambda)$ of the matrix $\mathbf{A}$ and the potential $u(x,t)$ satisfy the important relations

$$2i\lambda(uA_{21}A_{22}+u^*A_{11}A_{12})=\partial_x(A_{12}A_{21}), \quad (38)$$

$$2i\lambda(uA_{22}^2+u^*A_{12}^2)=\partial_x(A_{12}A_{22}), \quad (39)$$

which we will use below. These relations can be easily verified by taking the derivative and using $\partial_x\mathbf{A}=\mathbf{UA}$.

C. Changing the scattering data under the variation of the potential

The variations of the reflection δS_{12} and transmission δS_{11} coefficients under the small variations of the potentials $\delta u(x,t)$, $\delta u^*(x,t)$ for some fixed t are

$$\delta S_{11}(\lambda)=\int_{-\infty}^{\infty}\left\{\frac{\delta S_{11}}{\delta u}\delta u(x)+\frac{\delta S_{11}}{\delta u^*}\delta u^*(x)\right\}dx, \quad (40)$$

$$\delta S_{12}(\lambda)=\int_{-\infty}^{\infty}\left\{\frac{\delta S_{12}}{\delta u}\delta u(x)+\frac{\delta S_{12}}{\delta u^*}\delta u^*(x)\right\}dx. \quad (41)$$

From Eqs. (14) we have

$$\frac{\delta S_{11}}{\delta u}=\frac{\delta}{\delta u}(M_{11}^-M_{22}^+-M_{12}^+M_{21}^-), \quad (42)$$

$$\frac{\delta S_{12}}{\delta u}=\frac{\delta}{\delta u}(M_{12}^-M_{22}^+-M_{12}^+M_{22}^-), \quad (43)$$

and similar expressions for $\delta S_{11}/\delta u^*$, $\delta S_{12}/\delta u^*$. The variational derivatives in the right-hand side (rhs) of Eqs. (42) and (43) can be found from the integral equations (15). As a result we obtain (λ dependence in $\mathbf{M}^{\pm}$ is omitted)

$$\delta S_{11}(\lambda)=2i\lambda\int_{-\infty}^{\infty}\{M_{22}^+(x)M_{21}^-(x)\delta u(x)+M_{12}^+(x)M_{11}^-(x)\delta u^*(x)\}dx, \quad (44)$$

$$\delta S_{12}(\lambda)=2i\lambda\int_{-\infty}^{\infty}\{M_{22}^+(x)M_{22}^-(x)\delta u(x)+M_{12}^+(x)M_{12}^-(x,\lambda)\delta u^*(x)\}dx. \quad (45)$$

III. ONE-SOLITON PULSE WITH A RANDOM INITIAL PERTURBATION

The reflectionless scattering data with the single ($N=1$) zero $\lambda_1^2\equiv\varkappa=\xi+i\eta$ of the function $S_{11}(\lambda)$ correspond to one-soliton solution. It can be found from Eqs. (34)–(37)

$$u_s(x,t)=\frac{ik_0}{|\lambda_1|}\frac{\cosh(k_0z-i\varphi)}{\cosh^2(k_0z+i\varphi)}e^{i\psi}, \quad (46)$$

where we have introduced the notations

$$z=x_0-x+vt, \quad \psi=\psi_0+vx+\frac{1}{2}(k_0^2-v^2)t, \quad (47)$$

$$k_0=\frac{4\eta}{\alpha}, \quad v=\frac{\beta-4\xi}{\alpha}, \quad (48)$$

$$\varphi=\arg(\lambda_1)=\frac{1}{2}\arctan(\eta/\xi). \quad (49)$$

Parameters x_0 and ψ_0 determine initial position and initial phase of the soliton. The parameter η is, up to the multiplier $4/\alpha$, the soliton inverse width k_0 (note that $\eta/\alpha>0$), and ξ is, up to constant multiplier and shift, the soliton velocity v. An explicit expression for u_s in terms of the soliton amplitude and phase is

$$u_s=\frac{ik_0}{|\lambda_1|}\frac{\exp\{i\psi-3i\arctan[\tanh(k_0z)\tan\varphi]\}}{\sqrt{\cosh^2(k_0z)-\sin^2\varphi}}. \quad (50)$$

Some of the properties of the MNLS soliton differ from those of the NLS soliton. First, the MNLS soliton has nonzero phase difference at its limits

$$\arg(u_s(x\to\infty))-\arg(u_s(x\to-\infty))=6\varphi. \quad (51)$$

Second, the number of particles or the optical energy of the soliton is

$$E_s=\int_{-\infty}^{\infty}|u_s(x,t)|^2dx=4\alpha\varphi, \quad 0<\varphi\,\mathrm{sgn}\alpha<\frac{\pi}{2}, \quad (52)$$

and, therefore, has an upper limit $2\pi/|\alpha|$. These properties of the MNLSE soliton resemble those of the NLSE soliton with condensate (nonvanishing at the infinity) boundary conditions (dark NLSE soliton).

The one-soliton Jost functions and the corresponding scattering data are given in the Appendix.

Suppose now that the soliton input is randomly perturbed so that a pulse $u(x)=u_s(x)[1+\varepsilon(x)]$ is injected into the fiber. We assume that the noise $\varepsilon(x)$ is a zero mean, real Gaussian process with correlation function

$$\langle \varepsilon(x)\varepsilon(x')\rangle = D(x-x'), \tag{53}$$

where $\langle\cdots\rangle$ means statistical averaging. It is assumed that the intensity of the noise is small, $D(x)\ll 1$. The presence of $\varepsilon(x)$ will modify the soliton eigenparameter λ_1 in a random way, and, aside from this, will result in a continuum (radiative) contribution δu_c accompanying the modified soliton into the fiber. As was shown in Ref. [17], the only asymptotic ($t\to\infty$) effect of the radiation on the soliton solution (with the modified eigenparameter) of the MNLSE is, as usual, a shift (random in our case) in phase ψ_0 and position x_0 of the soliton.

The corresponding variation of the parameter $\varkappa$ can be written as

$$\delta\varkappa = \left(\frac{\partial S_{11}(\lambda^2)}{\partial \lambda^2}\bigg|_{\lambda=\lambda_1} \right)^{-1} \delta S_{11}(\lambda_1), \tag{54}$$

where δS_{11}, given by Eq. (44), is the variation of the transmission coefficient $S_{11}(\lambda)$ induced by the given realization of δu. Using the perturbative approach, we substitute the unperturbed soliton eigenparameter λ_1 and one-soliton Jost functions into the rhs of Eq. (54). Thus, we have

$$\delta\varkappa = \frac{2i\lambda_1}{\varkappa-\varkappa^*}\int_{-\infty}^{\infty}\varepsilon(x)(u_s A_{21}A_{22}+u_s^* A_{11}A_{12})\,dx, \tag{55}$$

where A_{11}, A_{12}, A_{21}, and A_{22} are defined in Eqs. (A4)–(A7) and evaluated at $\lambda=\lambda_1$. One can see that $\delta\varkappa$ is Gaussian random value with $\langle\delta\varkappa\rangle=0$. Therefore, in this approximation the averaged soliton velocity v and the inverse width k_0 remain unchanged. Using the relation (38), one can simplify Eq. (55). Then, writing down expression for $|\delta\varkappa|^2$, performing averaging over $\varepsilon(x)$ with the aid of Eq. (53), and introducing Fourier transform of $D(x)$ in the form $D(x)=\int_{-\infty}^{\infty}\tilde{D}(q)\exp(-iqx)\,dq$, we get

$$\langle|\delta\varkappa|^2\rangle = \frac{1}{4\eta^2}\int_{-\infty}^{\infty}\tilde{D}(q)|I(q)|^2\,dq, \tag{56}$$

where $I(q)=\int_{-\infty}^{\infty}\exp(-iqx)\partial_x(A_{12}A_{21})dx$. Integrating by parts gives for the averaged $|\delta\varkappa|^2$,

$$\langle|\delta\varkappa|^2\rangle = \pi^2\alpha^2(1+1/\mu^2)k_0^2 G(\xi,\eta), \tag{57}$$

where $\mu=\eta/\xi$ and we have introduced the function

$$G(\xi,\eta) = k_0\int_{-\infty}^{\infty} y^2\tilde{D}(k_0 y)\frac{\sinh^2(\arctan(\mu)y/2)}{\sinh^2(\pi y/2)}\,dy, \tag{58}$$

which depends on the specific form of the noise correlation function. Note that the integral in Eq. (58) is always convergent, since $|\varphi|=|\arctan(\mu)/2|<\pi/2$. Similarly, for $\langle\delta\varkappa\,\delta\varkappa\rangle$ one can find

$$\langle\delta\varkappa\,\delta\varkappa\rangle = \pi^2\alpha^2(1/\mu^2-1-2i/\mu)k_0^2 G(\xi,\eta). \tag{59}$$

Splitting real and imaginary parts, we obtain variances of the (normalized) soliton velocity ξ and inverse width η,

$$\langle\delta\xi^2\rangle = (1/\mu^2)\langle\delta\eta^2\rangle, \quad \langle\delta\eta^2\rangle = \pi^2\alpha^2 k_0^2 G(\xi,\eta), \tag{60}$$

and the cross correlation

$$\langle\delta\xi\,\delta\eta\rangle = \langle\delta\eta^2\rangle/\mu. \tag{61}$$

The function $G(\xi,\eta)$ can be explicitly calculated in two limiting cases—zero and infinite noise correlation time.

If the noise is δ correlated in time (zero correlation time), so that

$$D(x) = D_0\delta(x), \tag{62}$$

the function G takes the form

$$G(\xi,\eta) = \frac{8\pi^2 D_0 k_0\mu^2}{1+\mu^2}F(\varphi), \tag{63}$$

where

$$F(\varphi) = \frac{2\sin\phi\cos^2\phi(\cos\phi-2)-\cos\phi(5\sin\phi-12\phi)-8\sin\phi+3\phi}{24\sin^5\phi(\cos\phi-1)} \tag{64}$$

and we have introduced the notation $\phi=2\varphi$. The function $F(\varphi)$ increases monotonically with φ and $F(0)=1/35$.

Consider now the case when the random function $\varepsilon(x)$ has the form

$$\varepsilon(x) = \varepsilon_0\cos(q_0 x+\vartheta), \tag{65}$$

where the random amplitude ε_0 is a zero mean, normally distributed value with variance σ^2, and the random phase ϑ is uniformly distributed between 0 and 2π. The correlation function of such a process is $D(x)=(\sigma^2/2)\cos(q_0 x)$ or, in the frequency domain

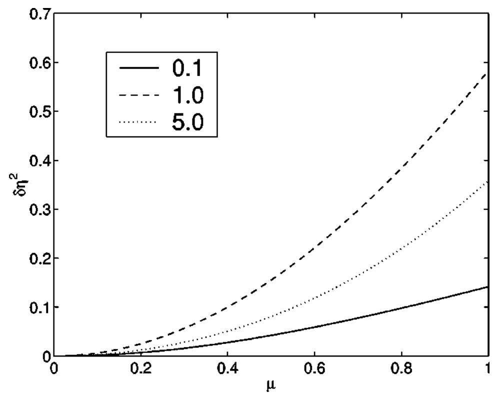

FIG. 2. The dependence of the variance of η on the parameter $\mu=\eta/\xi$ for different values of a.

$$\tilde{D}(q)=\frac{\sigma^2}{4}[\delta(q-q_0)+\delta(q+q_0)]. \tag{66}$$

In this case the noise has an infinite correlation time and is concentrated at the frequency q_0. For G we have

$$G(\xi,\eta)=\frac{\sigma^2 b^2 \sinh^2(\arctan(\mu)b/2)}{2\sinh^2(\pi b/2)}, \tag{67}$$

where $b=q_0/k_0$.

To take into account a finite correlation time we consider an important particular case, when the noise spectrum has a Lorenzian shape

$$\tilde{D}(q)=\frac{D_0}{\pi\tau_c[q^2+(1/\tau_c)^2]}, \tag{68}$$

where D_0 is the integral intensity of the noise. In x space this corresponds to the correlation function $D(x)=D_0\exp(-|x|/\tau_c)$, where τ_c is a correlation time. In this case, the function $G(\xi,\eta)$ takes the form

$$G(\xi,\eta,a)=\frac{D_0 a}{\pi}\int_{-\infty}^{\infty}\frac{y^2\sinh^2(\arctan(\mu)y/2)}{(y^2+a^2)\sinh^2(\pi y/2)}dy, \tag{69}$$

where the parameter $a=1/(k_0\tau_c)$ is the ratio of the soliton width to the correlation time. In Fig. 2 the dependence of variance of the inverse width η on the parameter μ is shown for different values of a at $\eta=1$, $D_0=0.025$, $\alpha=1$. The dependence of the variance of η on a for different μ at the same η, D_0, α is presented in Fig. 3. One can see that the influence of the noise (under the fixed integral intensity) on the soliton parameters becomes larger at $a\sim1$, that is, when the soliton width is comparable with the noise correlation time.

Consider now the radiative contribution Eq. (1) with $p=0$ conserves the "optical energy,"

$$E=\int_{-\infty}^{\infty}|u(x,t)|^2\,dx, \tag{70}$$

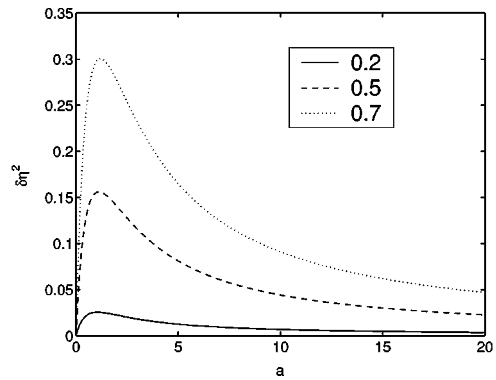

FIG. 3. The dependence of the variance of η on the parameter a for different values of μ.

which, as it follows from Eq. (26), can be explicitly expressed in terms of the continuum ($\lambda^2\in\mathbb{R}$) and discrete scattering data

$$E=\int_{-\infty}^{\infty}\mathcal{N}_{rad}(\lambda^2)\,d\lambda^2+\frac{4}{\alpha}\varphi, \tag{71}$$

where

$$\mathcal{N}_{rad}(\lambda^2)=-\frac{\ln(1-\mathrm{sgn}(\lambda^2)|S_{12}(\lambda)|^2)}{2\pi|\alpha|\lambda^2}. \tag{72}$$

In Eq. (71) the soliton contribution is separated from that of the radiative component ($\int d\lambda^2$) of the wave field described by the continuous-spectrum scattering data. The dispersion relation corresponding linearized version of Eq. (1) is $p=-q^2/2$. If we consider the radiative component as a superposition of free waves governed by the linear Schrödinger equation, the spectral parameter λ^2, as it follows from Eq. (27), is connected with the frequency of the emitted linear waves q by

$$q=\frac{4}{\alpha}\left(\lambda^2-\frac{\beta}{4}\right). \tag{73}$$

The quantity $\mathcal{N}_{rad}(\lambda^2)$ can be regarded as spectral density of the optical energy carried by the radiation.

The reflection coefficient is no longer zero and, for a given realization of $\varepsilon(x)$, we have $|S_{12}(\lambda)|=|\delta S_{12}(\lambda)|\ll1$. It then follows from Eq. (72) that the averaged spectral density $n_{rad}(\lambda)=\langle\mathcal{N}_{rad}(\lambda)\rangle$ is

$$n_{rad}(\lambda)=\frac{\langle|\delta S_{12}(\lambda)|^2\rangle}{2\pi|\alpha\lambda^2|}. \tag{74}$$

Inserting the unperturbed one-soliton Jost functions into Eq. (45), one obtains

$$\delta S_{12}(\lambda)=\frac{2i\lambda}{(\lambda^2-\varkappa)(\lambda^2-\varkappa^*)}\int_{-\infty}^{\infty}\varepsilon(x)[u_s A_{22}^2(\lambda)+u_s^* A_{12}^2(\lambda)]dx. \tag{75}$$

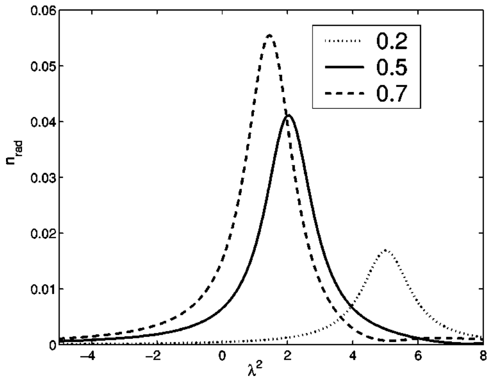

FIG. 4. The spectral distribution of radiation for different values of the parameter a.

Writing down expression for $|\delta S_{12}(\lambda)|^2$, performing averaging over $\varepsilon(x)$, calculating integrals with the aid of Eq. (39), and substituting the result into Eq. (74) one can find for the spectral distribution of radiation

$$n_{rad}(\lambda)=\frac{8\pi\eta^3}{\alpha^2|\lambda_1|^2[(\lambda^2-\xi)^2+\eta^2]^2}R(\lambda), \tag{76}$$

where we have introduced the function

$$R(\lambda)=\int_{-\infty}^{\infty}[y+c(\lambda)]^2\tilde{D}(k_0y+k_0c(\lambda)) \times\frac{(\lambda^2e^{-\varphi y}-|\lambda_1|^2e^{\varphi y})^2}{\cosh^2(\pi y/2)}dy, \tag{77}$$

with the notation $c(\lambda)=v/k_0+(\lambda^2-\beta/4)/\eta$. For the noise correlator of the form Eq. (68) we have

$$R(\lambda)=\frac{D_0a}{\pi k_0}\int_{-\infty}^{\infty}\frac{(y+c)^2(\lambda^2e^{-\varphi y}-|\lambda_1|^2e^{\varphi y})^2}{[a^2+(y+c)^2]\cosh^2(\pi y/2)}dy. \tag{78}$$

The spectral distribution of radiation in this case is shown in Fig. 4 for different values of the parameter a. The distribution has one peak and exponentially decaying tails. The maximum of the distribution shifts to higher frequencies with decreasing the ratio of the soliton width $1/k_0$ to the correlation time τ_c.

If $\varepsilon(x)$ is a white noise in time, with the aid of Eq. (62) one can find an explicit expression for $\langle|S_{12}(\lambda)|^2\rangle$ and the spectral density is

$$n_{rad}(\lambda)=\frac{16D_0\eta^3[(\lambda^4+\xi^2+\eta^2)F_1-\lambda^2|\lambda_1|^2F_2]}{\pi\alpha^2|\lambda_1|^2[(\lambda^2-\xi)^2+\eta^2]^2}, \tag{79}$$

where

$$F_1(\varphi)=\frac{\phi-\sin(2\phi)+\phi\cos^2(\phi)}{\sin^3(\phi)},\quad F_2(\varphi)=\frac{\cos(2\phi)}{6}. \tag{80}$$

In this case the spectral density decays algebraically ($\sim1/\lambda^4$) with increasing of $|\lambda^2|$. This slow decay is connected with the fact that the spectrum of the δ-correlated noise contains Fourier harmonics of all frequencies with equal amplitudes. Having determined the reflection coefficient $S_{12}(\lambda)$ in a closed form, we can evaluate the averaged position Δx_0 and phase $\Delta\psi_0$ shifts of the soliton due to interaction with the radiation. As was shown in Ref. [17], these shifts (in the deterministic case) are expressed through the coefficient S_{12} in the following way (the $\pm$ signs correspond to $x,t\to\pm\infty$):

$$\Delta x_0\equiv x_0^+-x_0^-=-\frac{\alpha}{4}\ln\left(\frac{|\delta^+(\lambda_1;\lambda_0)|}{|\delta^-(\lambda_1;\lambda_0)|}\right), \tag{81}$$

$$\Delta\psi_0\equiv\psi_0^+-\psi_0^-=\frac{\alpha}{4}\arg\left(\frac{\delta^+(\lambda_1;\lambda_0)}{\delta^-(\lambda_1;\lambda_0)}\right), \tag{82}$$

where

$$\delta^+(\lambda;y)=\exp\left\{\frac{1}{2\pi i}\int_0^y\frac{\ln(1+|S_{12}(\mu)|^2)d\mu^2}{(\mu^2-\lambda^2)} -\frac{1}{2\pi i}\int_0^\infty\frac{\ln(1-|S_{12}(i\mu)|^2)d\mu^2}{(\mu^2+\lambda^2)}\right\}, \tag{83}$$

$$\delta^-(\lambda;y)=\exp\left\{\frac{1}{2\pi i}\int_y^\infty\frac{\ln(1+|S_{12}(\mu)|^2)d\mu^2}{(\mu^2-\lambda^2)}\right\}, \tag{84}$$

and $\lambda_0=x/t-\beta/\alpha$. Taking into account $\langle|S_{12}|^2\rangle\ll1$, integrals in Eqs. (83) and (84) can be calculated analytically for the case of a δ-correlated noise, but appearing formulas for the averaged shifts (81) and (82) turn out to be too cumbersome and are omitted here.

IV. PERTURBATION THEORY

In this section we derive evolution equations for the scattering data of the MNLSE in the presence of a perturbation. These equations replace the well-known corresponding equations [2,16,18–21] of the NLSE.

Let us consider dynamics of the scattering data when $p\neq0$. From Eq. (7) and the fact that $\mathbf{M}^\pm$ satisfies Eq. (3) one can get

$$(\partial_x-\mathbf{U})(\partial_t-\mathbf{V})\mathbf{M}^\pm+\mathbf{P}\mathbf{M}^\pm=\mathbf{0}. \tag{85}$$

Introducing a new unknown $\mathbf{J}^\pm(x,t,\lambda)$ defined through the relation $(\partial_t-\mathbf{V})\mathbf{M}^\pm=\mathbf{M}^\pm\mathbf{J}^\pm$, one finds that $\mathbf{J}^\pm$ satisfies $\partial_x\mathbf{J}^\pm=-\mathbf{M}^{\pm-1}\mathbf{P}\mathbf{M}^\pm$. Integrating and taking into account the boundary conditions (8) and the fact that $\mathbf{V}\to\Omega(\lambda)\sigma_3$ as $|x|\to\infty$, one can obtain $\mathbf{J}^\pm=-\Omega(\lambda)\sigma_3+\int_x^{\pm\infty}\mathbf{M}^{\pm-1}\mathbf{P}\mathbf{M}^\pm dx'$ and, hence, the following equations of motion for $\mathbf{M}^\pm$:

$$(\partial_t - \mathbf{V})\mathbf{M}^{\pm} = \mathbf{M}^{\pm}\left(-\Omega(\lambda)\sigma_3 + \int_x^{\pm\infty} \mathbf{M}^{\pm\,-1}\mathbf{P}\mathbf{M}^{\pm}dx'\right). \tag{86}$$

Equation (86) makes sense only for $\operatorname{Im}\lambda^2=0$. For $\lambda^2\in\mathbb{R}$, the columns M_1^- and M_2^+ are the boundary values of functions analytic for $\operatorname{sgn}\alpha \operatorname{Im}\lambda^2>0$, and we also need equations for them that hold for $\operatorname{sgn}\alpha\operatorname{Im}\lambda^2>0$. Introducing the matrix $\mathbf{M}(x,t,\lambda)=(M_1^-,M_2^+)$, and as before defining the new unknown $\mathbf{J}(x,t,\lambda)=(J_1,J_2)$ through the relation $(\partial_t-\mathbf{V})\mathbf{M}=\mathbf{M}\mathbf{J}$, and then integrating, we obtain

$$J_1=\begin{pmatrix}-\Omega(\lambda)\\0\end{pmatrix}-\int_{-\infty}^{x}\mathbf{M}^{-1}\mathbf{P}M_1^-dx', \tag{87}$$

$$J_2=\begin{pmatrix}0\\\Omega(\lambda)\end{pmatrix}+\int_{x}^{\infty}\mathbf{M}^{-1}\mathbf{P}M_2^+dx'. \tag{88}$$

As before, these expressions are used in $(\partial_t-\mathbf{V})\mathbf{M}=\mathbf{M}\mathbf{J}$ to yield the equation of motion for $\mathbf{M}$, valid for $\operatorname{sgn}\alpha\operatorname{Im}\lambda^2>0$ except at λ_k, where $\mathbf{M}$ fails to be invertible. Making the natural assumption that the zeros $\lambda=\lambda_k$ are simple, one can show [13,16], however, that each singularity is removable since $\det\mathbf{M}=S_{11}$. Hence, the evolution equation for $\mathbf{M}$ makes sense as $\lambda\to\lambda_k(t)$, and one can introduce

$$\mathbf{H}_k(x,x',t)=\lim_{\lambda\to\lambda_k(t)}\mathbf{M}(x,t,\lambda)\mathbf{M}(x',t,\lambda)^{-1}, \tag{89}$$

where the limit can be calculated by using Hôpital rule.

The equation of motion for $\mathbf{M}^{\pm}$ and $\mathbf{M}$ determine the evolution of the scattering data. Differentiating Eq. (11) with respect to t and using Eq. (86) yields for real λ^2,

$$\partial_t\mathbf{S}(t,\lambda)-\Omega(\lambda)[\sigma_3,\mathbf{S}(t,\lambda)] = -\int_{-\infty}^{\infty}\mathbf{M}^{+}(x',t,\lambda)^{-1}\mathbf{P}\mathbf{M}^{-}(x',t,\lambda)dx'. \tag{90}$$

Note that since $\mathbf{P}$ is off diagonal, the equation for $S_{11}(\lambda,t)$ only involves quantities analytic for $\operatorname{sgn}\alpha\operatorname{Im}\lambda^2>0$.

The equation of motion for the reflection coefficient $S_{12}(\lambda,t)$ is contained in that for $\mathbf{S}$,

$$\partial_t S_{12}-2\Omega(\lambda)S_{12}=-2\lambda\int_{-\infty}^{\infty}(pM_{22}^+M_{22}^-+p^*M_{12}^+M_{12}^-)dx'. \tag{91}$$

The expression defining the zeros $\lambda_k^2(t)$ of $S_{11}(t,\lambda)$ is $S_{11}(t,\lambda_k(t))=0$. Differentiating with respect to t gives

$$\partial_t S_{11}(t,\lambda_k(t))+\frac{d\lambda_k^2}{dt}\partial_{\lambda^2}S_{11}(t,\lambda_k(t))=0. \tag{92}$$

Using the equation of motion for $\mathbf{S}$, one therefore finds

$$\frac{d\lambda_k^2}{dt}=\frac{2\lambda_k}{\partial_{\lambda^2}S_{11}(\lambda_k,t)}\int_{-\infty}^{\infty}(pM_{22}^+M_{21}^-+p^*M_{12}^+M_{11}^-)dx'. \tag{93}$$

The integrand here is evaluated at x', t, and $\lambda=\lambda_k(t)$. Differentiating Eq. (23) with respect to t and using the evolution equation for $\mathbf{M}$ taken in the limit $\lambda\to\lambda_k(t)$ yields the equation for $\gamma_k(t)$

$$\begin{aligned}\left[\frac{d\gamma_k}{dt}-2\Omega(\lambda)\gamma_k\right]M_1^-(x,t,\lambda) &= -\gamma_k\int_{-\infty}^{\infty}\mathbf{H}_k(x,x',t)\mathbf{P}M_1^-(x',t,\lambda)dx'\\ &=[\partial_\lambda M_2^+(x,t,\lambda)-\gamma_k\partial_\lambda M_1^-(x,t,\lambda)]\frac{d\lambda_k}{dt}\end{aligned} \tag{94}$$

with $\lambda=\lambda_k(t)$. Equations (91), (93), and (94) describe the evolution of the scattering data, but are coupled to the equations for $\mathbf{M}$ and $\mathbf{M}^{\pm}$. The coupling disappears for $\mathbf{P}=0$, as a result we have Eqs. (27)–(29).

If $p[u,u^*]$ is a small perturbation, one can substitute the unperturbed u, u^*, and Jost functions $\mathbf{M}^{\pm}$ into the right-hand side of Eqs. (91), (93), and (94). This yields evolution equations for the scattering data in the lowest approximation of perturbation theory. This procedure can be iterated to yield higher orders of perturbation theory. The appearing hierarchy of Eqs. (91), (93), and (94) are applied to arbitrary number of solitons and, in particular, describe nontrivial many-soliton effects in the presence of perturbations. Here we restrict ourselves to the case of one-soliton pulse.

Evolution equations for the parameters $\varkappa$ and γ_1 can be derived from the general equations (93) and (94) for the discrete-spectrum scattering data with the use of the one-soliton Jost functions given in the Appendix. After substituting these Jost functions into Eqs. (93) and (94) we obtain

$$\frac{d\varkappa}{dt}=\frac{2\lambda_1}{(\varkappa-\varkappa^*)}\int_{-\infty}^{\infty}(pA_{21}A_{22}+p^*A_{11}A_{12})\,dx \tag{95}$$

and

$$\begin{aligned}\left[\frac{d\gamma_1}{dt}-2\Omega(\lambda_1)\gamma_1\right]A_{11} &= -2\gamma_1\lambda_1\int_{-\infty}^{\infty}(pH_{11}A'_{21}-p^*H_{12}A'_{11})\,dx'\\ &=(S_{11}^-)^{-1}[\partial_\lambda(S_{11}^-A_{12})-\gamma_1\partial_\lambda(S_{11}^-A_{11})]\frac{d\lambda_1}{dt},\end{aligned} \tag{96}$$

where [writing A'_{ij} for $A_{ij}(x',t,\lambda)$]

$$H_{11}=\frac{i}{2\eta}\partial_\lambda(A_{11}A'_{22}-A_{12}A'_{21}), \tag{97}$$

$$H_{12}=\frac{i}{2\eta}\partial_\lambda(A_{12}A'_{11}-A_{11}A'_{12}) \tag{98}$$

$$S^-_{11}=\frac{\lambda_1^*}{\lambda_1(\lambda^2-\lambda_1^{*2})}, \tag{99}$$

where A_{11}, A_{12}, A_{21}, and A_{22} are defined in Eqs. (A4)–(A7) and evaluated at $\lambda=\lambda_1(t)$, $\gamma_1=\gamma_1(t)$, and p, p^* are evaluated at the soliton solution Eq. (46) with time dependent parameters. Similarly, for the reflection coefficient $S_{12}(\lambda,t)$ we find from Eq. (91),

$$\frac{dS_{12}(\lambda)}{dt}+ik(\lambda)S_{12}(\lambda)=\frac{2\lambda}{(\lambda^2-\varkappa)(\varkappa^*-\lambda^2)}\times\int_{-\infty}^{\infty}(pA_{22}^2(\lambda)+p^*A_{12}^2(\lambda))dx, \tag{100}$$

where we have introduced the notation $k(\lambda)=(8/\alpha^2)(\lambda^2-\beta/4)^2$.

V. INFLUENCE OF MULTIPLICATIVE NOISE ON THE SOLITON

In this section we consider the MNLSE equation with an external multiplicative noise. We take a perturbation term in the form

$$p=\varepsilon(x,t)u, \tag{101}$$

where $\varepsilon(x,t)$ is a zero mean real Gaussian field with covariance

$$\langle\varepsilon(x,t)\varepsilon(x',t')\rangle=D(x-x')B(t-t'). \tag{102}$$

This form of perturbation corresponds to a random additional part of the refractive index.

Substituting the perturbation term of Eq. (101) into Eq. (95) we find after some calculations the following evolution equations:

$$\frac{d\varphi}{dt}=0, \tag{103}$$

$$\frac{d\eta}{dt}=8\eta\sin(2\varphi)\int_{-\infty}^{\infty}\frac{\varepsilon(x,t)\sinh(2y)\,dy}{[\cosh(2y)+\cos(2\varphi)]^2}, \tag{104}$$

where $x=x_0(t)+v(t)t-y/k_0$. As is seen from Eq. (103), the noise term in the first approximation does not affect the quantity φ. This can be easily understood from the expression for the integral of motion Eq. (71). The perturbation of the form (101) still exactly conserves the optical energy Eq. (70), which is, neglecting the radiative degrees of freedom, proportional to φ. Thus, $\varphi(t)=\varphi(0)\equiv\varphi_0$ is a deterministic constant which is determined by the deterministic initial conditions. Note that if ε is a function of t only (space irregularities) we have also $d\eta/dt=0$. In the general case, as it follows from Eq. (104), the parameter η will be a random function of t with a complicated strongly non-Gaussian statistics. We will assume therefore that the intensity of the noise is small enough, so that η can be regarded as a constant in t and will concern of the radiative effects.

According to Eq. (72), the spectral density of the radiation energy may be expanded as follows:

$$\mathcal{N}_{rad}(\lambda^2)=\frac{|S_{12}(\lambda)|^2}{2\pi|\alpha\lambda^2|}+O(|S_{12}(\lambda)|^4), \tag{105}$$

provided that $|S_{12}(\lambda)|^2\ll 1$. The emission intensity is characterized by its power, i.e., the energy emission rate. The emission power spectral density $w(\lambda)\equiv d\mathcal{N}_{rad}/dt$ is

$$w(\lambda)=\frac{1}{\pi|\alpha\lambda^2|}\mathrm{Re}\left\{S_{12}^*\frac{dS_{12}}{dt}\right\}. \tag{106}$$

Inserting the perturbation Eq. (101) into the general perturbation-induced evolution equation (100) for the reflection coefficient $S_{12}(\lambda,t)$, one can obtain for $s(\lambda,t)=S_{12}(\lambda,t)\exp[ik(\lambda)t]$

$$\frac{ds(\lambda)}{dt}=\frac{2\lambda\int_{-\infty}^{\infty}\varepsilon(x,t)[u_sA_{22}^2(\lambda)+u_s^*A_{12}^2(\lambda)]dx}{(\lambda^2-\varkappa)(\varkappa^*-\lambda^2)}e^{ik(\lambda)t}. \tag{107}$$

Let us integrate Eq. (107), the rhs of which should be multiplied by $\exp(\nu t)$ with an infinitely small $\nu>0$. As usual, this trick implies adiabatically turning on a perturbation that was absent at $t=-\infty$. Thus, we get

$$s(\lambda,t)=\frac{\int_{-\infty}^{t}\int_{-\infty}^{\infty}\varepsilon(x,\tau)F(x,\tau,\lambda)e^{ik(\lambda)\tau+\nu\tau}d\tau\,dx}{i(\lambda^2-\varkappa)(\varkappa^*-\lambda^2)}, \tag{108}$$

where

$$F(x,\tau,\lambda)=\partial_x(A_{12}(x,\tau,\lambda)A_{22}(x,\tau,\lambda)), \tag{109}$$

and we have used the relation (39). Multiplying Eq. (107) by the complex-conjugate expression (108) and averaging yield

$$\left\langle s^* \frac{ds}{dt} \right\rangle = \frac{\int_{-\infty}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(t-\tau)D(x-x')F(x,t)F^*(x',\tau)e^{ik(\lambda)(t-\tau)+\nu\tau}d\tau dx dx'}{[(\lambda^2-\xi)^2+\eta^2]^2}. \tag{110}$$

Introducing Fourier transforms of $B(t)$ and $D(x)$ through $B(t)=\int_{-\infty}^{\infty}\widetilde{B}(p)\exp(-ipt)\,dp$, $D(x)=\int_{-\infty}^{\infty}\widetilde{D}(q)\exp(-iqx)\,dq$, making the change of variables $y=k_0(vt-x)$, $y'=k_0(v\tau-x')$, calculating integrals over y, y', we can perform then the integration over $t-\tau$ and obtain

$$\left\langle s^*(\lambda)\frac{ds(\lambda)}{dt}\right\rangle = \frac{1}{[(\lambda^2-\xi)^2+\eta^2]^2} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\widetilde{B}(p)\widetilde{D}(q)q^2 I(\lambda,q)\,dp\,dq}{i\nu+K(\lambda)-p-qv}, \tag{111}$$

where $K(\lambda)=k(\lambda)+\omega(\lambda)v+(k_0^2+v^2)/2$, $\omega(\lambda)=(4\lambda^2-\beta)/\alpha$ and

$$I(\lambda,q)=\frac{\pi^2\alpha^2\lambda^2[\lambda^2 e^{-\varphi h(\lambda)}-|\lambda_1|^2 e^{\varphi h(\lambda)}]^2}{4|\lambda_1|^2\cosh^2[\pi h(\lambda)/2]} \tag{112}$$

with $h(\lambda)=(q-\omega(\lambda)-v)/k_0$. Then, making use of the relation $\lim_{\nu\to 0}(y-i\nu)^{-1}=P(1/y)+i\pi\delta(y)$, where P is the symbol of the principal value, one can find

$$\left\langle \mathrm{Re}\left\{s^*\frac{ds}{dt}\right\}\right\rangle = \frac{\pi\int_{-\infty}^{\infty}\widetilde{B}(K(\lambda)-qv)\widetilde{D}(q)q^2 I(\lambda)\,dq}{[(\lambda^2-\xi)^2+\eta^2]^2}. \tag{113}$$

Substituting Eq. (113) into Eq. (106) gives the averaged power spectral density $\langle w(\lambda)\rangle$ emitted by the soliton.

First we consider the case when ε is a random function of time only, so that for the space correlator we have $\widetilde{B}(p)=\delta(p)$. Then the averaged emission power spectral density is

$$\langle w(\lambda)\rangle = \frac{K^2(\lambda)\widetilde{D}(K(\lambda)/v)I(\lambda,K(\lambda)/v)}{\alpha\lambda^2[(\lambda^2-\xi)^2+\eta^2]^2 v^2} \tag{114}$$

and can be explicitly written for the arbitrary form of the frequency correlator $\widetilde{D}(q)$. In particular, in the case of a noise spectrum of Lorenzian shape we substitute Eq. (68) into Eq. (114). The spectral composition of emitted power is shown in Fig. 5 for different values of η/ξ (with $\alpha=0.2$, $\beta=1.$, $D_0=0.1$, $a=0.2$, $\xi=1.$) The emitted power decays exponentially with increasing of the frequency. For "light" soliton ($\eta/\xi\ll 1$) the distribution has two maxima and left of them (for $\alpha>0$) disappears with increasing η/ξ.

Second, we consider the case when the noise is concentrated at the frequency q_0, so that the frequency correlator has the form Eq. (66). Then, the spectral density of the emitted power is

$$\langle w(\lambda)\rangle = \frac{\sigma^2 q_0^2}{4\alpha\lambda^2[(\lambda^2-\xi)^2+\eta^2]^2}\{\widetilde{B}(K-q_0v)I(\lambda,q_0) + \widetilde{B}(K+q_0v)I(\lambda,-q_0)\} \tag{115}$$

and can be written in a closed form for the arbitrary space correlator $\widetilde{B}(p)$. In particular, if ε depends only on x, Eq. (115) becomes

$$\langle w(\lambda)\rangle = \frac{\sigma^2 q_0^2\,\delta(K(\lambda)-q_0v)I(\lambda,q_0)}{4\alpha\lambda^2[(\lambda^2-\xi)^2+\eta^2]^2}. \tag{116}$$

It follows from Eq. (116) that the emission is concentrated at two points of the spectrum

$$\lambda_\pm^2=[\beta+\alpha(\pm\sqrt{2q_0v-k_0^2})]/4 \tag{117}$$

and takes place provided that the soliton velocity satisfies the condition $v>k_0^2/2q_0$. The total quanta number emission rate is

$$W\equiv\int_{-\infty}^{\infty}\langle w(\lambda)\rangle\,d\lambda^2=W_+ + W_-, \tag{118}$$

where

$$W_\pm=\frac{\sigma^2 q_0^2 I(\lambda_\pm,q_0)}{16\lambda_\pm^2[(\lambda_\pm^2-\xi)^2+\eta^2]^2\sqrt{2q_0v-k_0^2}}. \tag{119}$$

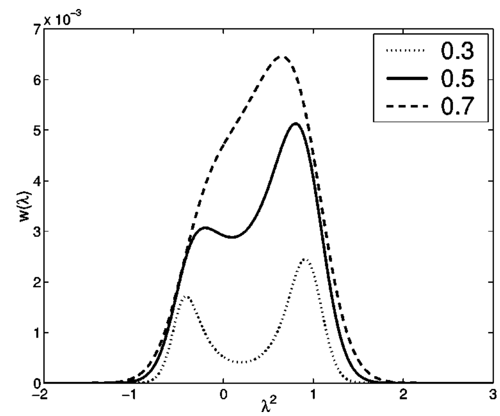

FIG. 5. Spectral density of radiation emitted by the soliton for different values of η/ξ.

Since the total optical energy $E=E_{rad}+E_s$ defined in Eq. (71) is conserved, the rate of soliton energy loss due to radiation can be evaluated from the balance equation

$$\frac{dE_s}{dt}=-W. \tag{120}$$

It then follows from Eq. (71) that the ratio η/ξ decreases with t.

VI. CONCLUSION

In conclusion we describe how the MNLSE soliton Eq. (46) with $\beta>0$ can be reduced [10] to a bright NLSE soliton at $\alpha\to 0$. To carry out this limit one should take into account that the Lax pair for the NLSE should be produced at $\alpha\to 0$ from the Lax pair (3) and (4) for the MNLSE. This condition implies that the spectral parameter λ depends on α and gives the following prescription: $2(\lambda_{MNLS}^2-\beta/4)/\alpha\to-\lambda_{NLS}$ at $\alpha\to 0$ or

$$\xi=\beta/4-\alpha\xi_0/2, \quad \eta=\alpha\eta_0/2, \tag{121}$$

where $\lambda_{MNLS}^2=\xi-i\eta$ and $\lambda_{NLS}=\xi_0+i\eta_0$. These formulas transform the MNLSE soliton Eq. (46) to the NLSE soliton $u_s=(2i\eta_0/\sqrt{\beta})\mathrm{sech}(z)\exp(i\psi)$, where $z=x_0-2\eta_0(x-2\xi_0 t)$, $\psi=\psi_0+2\xi_0 x-2(\xi_0^2-\eta_0^2)t$.

In this paper we have studied the influence of a random perturbation on the MNLSE soliton. First, we have shown that if the initial one-soliton pulse of the MNLSE has a small additional random component, the average asymptotic soliton width and velocity remain unchanged. Variances of the soliton inverse width and velocity, and the average shifts of the soliton position and phase have been determined analytically. The spectral distribution of radiation accompanying the soliton has also been obtained. Second, equations for the soliton eigenparameters and for the reflection coefficient describing nonsoliton (radiative) effects were derived for the perturbed MNLSE. The case when the perturbation has a form of multiplicative noise has been analyzed. This corresponds to additional random part of the refractive index. The averaged power spectral density emitted by the soliton was found for different forms of the noise correlator.

APPENDIX: ONE-SOLITON JOST FUNCTIONS

The scattering data corresponding to the one-soliton solution (46) are

$$S_{11}(\lambda)=\frac{(\xi+i\eta)(\lambda^2-\xi+i\eta)}{(\xi-i\eta)(\lambda^2-\xi-i\eta)}, \tag{A1}$$

$$S_{12}(\lambda,t)=0 \quad (\lambda^2 \text{ is real}), \tag{A2}$$

$$\lambda_1^2=\xi-i\eta, \quad \gamma_1(0)=\exp(k_0x_0+i\psi_0) \tag{A3}$$

and the corresponding Jost functions are given by Eq. (31) with

$$A_{11}(x,t,\lambda)=e^{\Lambda(\lambda)x}(\lambda^2A_{11}^{(1)}-|\lambda_1|^2), \tag{A4}$$

$$A_{12}(x,t,\lambda)=e^{-\Lambda(\lambda)x}\lambda A_{12}^{(1)}, \tag{A5}$$

$$A_{21}(x,t,\lambda)=e^{\Lambda(\lambda)x}\lambda A_{21}^{(1)}, \tag{A6}$$

$$A_{22}(x,t,\lambda)=e^{-\Lambda(\lambda)x}(\lambda^2A_{22}^{(1)}-|\lambda_1|^2), \tag{A7}$$

where

$$A_{12}^{(1)}(x,t)=-\frac{2i\eta}{|\lambda_1|}\frac{\exp(i\psi)}{\cosh(k_0z+i\varphi)}, \tag{A8}$$

$$A_{22}^{(1)}(x,t)=\frac{\cosh(k_0z+i\varphi)}{\cosh(k_0z-i\varphi)}, \tag{A9}$$

$$A_{11}^{(1)}=A_{22}^{(1)*}, \quad A_{21}^{(1)}=-A_{12}^{(1)*}. \tag{A10}$$

In Eqs. (A8) and (A9), z, ψ, k_0, and φ are the same as in Eqs. (47) and (48).

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