Linear dispersive dielectrics as limits of Drude-Lorentz systems

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We investigate the limiting case of the Drude-Lorentz model for the complex electric permeability $\varepsilon(\omega)=1+\hat{\chi}(\omega)$ as the damping tends to zero. We find that $\varepsilon(\omega)$ becomes real except for a number of discrete frequencies. The Kramers-Kronig relations connecting the real and imaginary parts of $\hat{\chi}(\omega)$ remain valid.

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I. BACKGROUND

In general linear dielectrics the polarization $\boldsymbol{P}(\mathbf{x},t)$ and electric field $\boldsymbol{E}(\mathbf{x},t)$ are related according to

$$\boldsymbol{P}(\mathbf{x},t)=\int ds\,\chi(\mathbf{x},t-s)\boldsymbol{E}(\mathbf{x},s), \tag{1.1}$$

where $\chi(\mathbf{x},t)$ is the electric susceptibility. Since there is no polarization before the electric field is applied (causality) $\chi(\mathbf{x},t)$ vanishes for $t<0$ leading to the usual electric permeability (permittivity, dielectric function)

$$\varepsilon(\mathbf{x},\omega)=1+\int_0^\infty dt\,\exp[i\omega t]\chi(\mathbf{x},t)=1+\hat{\chi}(\mathbf{x},\omega). \tag{1.2}$$

Thus $\varepsilon(\mathbf{x},\omega)$, which is in general complex, can be continued analytically into the upper complex half-plane and the real and imaginary parts of $\hat{\chi}(\mathbf{x},\omega)$ are connected by a Kramers-Kronig relation (P indicates a principle value)

$$\mathrm{Re}\,\hat{\chi}(\mathbf{x},\omega)=\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{+\infty}d\omega'\frac{\mathrm{Im}\,\hat{\chi}(\mathbf{x},\omega')}{\omega'-\omega},$$

$$\mathrm{Im}\,\hat{\chi}(\mathbf{x},\omega)=-\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{+\infty}d\omega'\frac{\mathrm{Re}\,\hat{\chi}(\mathbf{x},\omega')}{\omega'-\omega}. \tag{1.3}$$

But it sometimes happens that in certain frequency interval absorption is quite small, in which case it makes sense to neglect absorption, while retaining dispersive behavior. This apparently violates the causality requirement, but, as we show below, this need not be the case.

An important example where absorption can be neglected is that of small silver spheres. The latter more or less behave as an absorptionless Drude metal (for Drude metals, see Ref. [1]) in part of the optical range, where they still show appreciable dispersion. Since in this regime

$$\varepsilon(\omega)=1-\frac{\Omega^2}{\omega^2},\quad \Omega>0, \tag{1.4}$$

can be small or even negative, there can be a large contrast relative to a background like vacuum. A photonic crystal made up from such spheres can possess (photonic) band gaps. This was verified numerically for the three-dimensional (3D) case in Refs. [2] and [3] for the 2D situation. But if absorption is present, the situation changes. In earlier work [4] we showed rigorously that band gaps do not occur in absorptive frequency regions. Then the band spectrum is no longer real but consists of islands in the lower complex half plane. This was confirmed numerically for the 2D absorptive Drude case [5], where we used

$$\varepsilon(\omega)=1-\frac{\Omega^2}{\omega(\omega+i\gamma)},\quad \Omega,\gamma>0. \tag{1.5}$$

In this note we address the question as to what happens with absorptive Drude-Lorentz dielectrics, characterized by

$$\varepsilon(\omega)=1+\hat{\chi}(\omega)=1-\sum_j\ \frac{\Omega_j^2}{\omega^2-\omega_j^2+i\gamma_j\omega},$$
$$\omega_j\geqslant 0,\gamma_j,\Omega_j>0 \tag{1.6}$$

in the limiting case $\gamma\downarrow 0$. In the literature derivations of Eq. (1.6), at various levels of sophistication, can be found [6,7]. The classical model for the simplest situation exists of the equation of motion

$$\partial_t^2\mathbf{x}(t)+\gamma\partial_t\mathbf{x}(t)+\omega_0^2\mathbf{x}(t)=\frac{e}{m}\boldsymbol{E}(t),\quad \gamma>0 \tag{1.7}$$

for a classical charged particle (mass m, charge e), subject to a driving electric field $\boldsymbol{E}(t)$ and experiencing friction through the term $\gamma\partial_t\mathbf{x}(t)$ and a harmonic restoring term $\omega_0^2\mathbf{x}(t)$. Taking the Fourier transform of the corresponding polarization then gives $\hat{\chi}(\omega)$ as in Eq. (1.6) with a single $\omega_j=\omega_0$. In Ref. [7], the standard derivation, based on a classical model, is given, together with an extension to a nonlinear situation.

We are in particular interested in energy conservation, the Kramers-Kronig relations and quantization in the limiting case above. Here we note in passing that in recent years two general, equivalent, quantization techniques have been developed for causal linear absorptive dielectrics [8–10].

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II. MAXWELL'S EQUATIONS

In the present case the set of macroscopic, classical, Maxwell's equations for an isotropic, linear, absorptive dielectric is given by (we set ε_0, μ_0, and c equal to 1 and $\boldsymbol{H}=\boldsymbol{B}$)

$$\partial_t \boldsymbol{D}(\mathbf{x},t)=\partial_{\mathbf{x}}\times\boldsymbol{B}(\mathbf{x},t),\quad \partial_t\boldsymbol{B}(x,t)=-\partial_{\mathbf{x}}\times\boldsymbol{E}(\mathbf{x},t),$$

$$\partial_{\mathbf{x}}\cdot\boldsymbol{D}(x,t)=0,\quad \partial_{\mathbf{x}}\cdot\boldsymbol{B}(x,t)=0,$$

$$\boldsymbol{D}(\mathbf{x},t)=\boldsymbol{E}(\mathbf{x},t)+\boldsymbol{P}(\mathbf{x},t),$$

$$\boldsymbol{P}(\mathbf{x},t)=\int_{t_0}^{t} ds\,\chi(\mathbf{x},t-s)\boldsymbol{E}(\mathbf{x},s), \tag{2.1}$$

with t_0 some initial time which can be taken as $t_0=-\infty$. We assume that $\chi(\mathbf{x},t)=0$ for $t<0$ (causality) and also $\chi(\mathbf{x},0)=0$ (no instantaneous current surges at the initial time t_0). In addition we require that $\chi(\mathbf{x},t)$ and $\partial_t\chi(\mathbf{x},t)=\chi'(\mathbf{x},t)$ are finite. Denoting Laplace transforms with a hat,

$$\hat{f}(z)=\int_0^{\infty} dt\,\exp[izt]f(t),\quad \mathrm{Im}\,z\geqslant 0, \tag{2.2}$$

so

$$\hat{\chi}(\mathbf{x},z)=\int_0^{\infty} dt\,\exp[izt]\chi(\mathbf{x},t), \tag{2.3}$$

we have for the complex electric permeability

$$\varepsilon(\mathbf{x},\omega)=1+\hat{\chi}(\mathbf{x},\omega+i0) \tag{2.4}$$

and we note that, since $\chi(\mathbf{x},0)=0$,

$$\hat{\chi}'(\mathbf{x},z)=iz\hat{\chi}(\mathbf{x},z). \tag{2.5}$$

We rewrite the first of Eqs. (2.1) as

$$\partial_t\boldsymbol{E}(\mathbf{x},t)=\partial_{\mathbf{x}}\times\boldsymbol{B}(\mathbf{x},t)-\boldsymbol{J}(\mathbf{x},t),$$

$$\boldsymbol{J}(\mathbf{x},t)=\int_{t_0}^{t} ds\,\chi'(\mathbf{x},t-s)\boldsymbol{E}(\mathbf{x},s). \tag{2.6}$$

The electromagnetic energy

$$\mathcal{E}_{\mathrm{em}}(t)=\frac{1}{2}\int d\mathbf{x}\{\boldsymbol{E}(\mathbf{x},t)^2+\boldsymbol{B}(\mathbf{x},t)^2\} \tag{2.7}$$

satisfies

$$\partial_t\mathcal{E}_{\mathrm{em}}(t)=-\int d\mathbf{x}\;\;\boldsymbol{J}(\mathbf{x},t)\cdot\boldsymbol{E}(\mathbf{x},t)$$
$$=-\int d\mathbf{x}\int_{t_0}^{t} ds\,\chi'(\mathbf{x},t-s)\boldsymbol{E}(\mathbf{x},s)\cdot\boldsymbol{E}(\mathbf{x},t). \tag{2.8}$$

Noting that $\chi'(\mathbf{x},t)=0$ for $t<0$, we obtain

$$\mathcal{E}_{\mathrm{em}}(t)-\mathcal{E}_{\mathrm{em}}(t_0)$$
$$=-\int d\mathbf{x}\int_{t_0}^{t} ds\int_{t_0}^{s} du\,\chi'(\mathbf{x},s-u)\boldsymbol{E}(\mathbf{x},u)\cdot\boldsymbol{E}(\mathbf{x},s)$$
$$=-\int d\mathbf{x}\int_{t_0}^{t} ds\int_{t_0}^{t} du\,\chi'(\mathbf{x},s-u)\boldsymbol{E}(\mathbf{x},u)\cdot\boldsymbol{E}(\mathbf{x},s). \tag{2.9}$$

Making the Fourier decomposition

$$\chi'(\mathbf{x},t)=\int d\omega\,\exp[-i\omega t]\rho(\mathbf{x},\omega)$$
$$=\int d\omega\,\cos(\omega t)\rho(\mathbf{x},\omega),\quad t\geqslant 0, \tag{2.10}$$

where the integral is over $\mathbb{R}$ and $\rho(\mathbf{x},\omega)$ can be assumed to be even in ω, we then have

$$\mathcal{E}_{\mathrm{em}}(t)-\mathcal{E}_{\mathrm{em}}(t_0)=-\int d\mathbf{x}\int_{t_0}^{t} ds\int_{t_0}^{t} du\int d\omega\rho(\mathbf{x},\omega)$$
$$\times\exp[i\omega(s-u)]\boldsymbol{E}(\mathbf{x},s)\cdot\boldsymbol{E}(\mathbf{x},u)$$
$$=-\int d\mathbf{x}\int d\omega\rho(\mathbf{x},\omega)$$
$$\times\left|\int_{t_0}^{t} ds\,\exp[i\omega s]\boldsymbol{E}(\mathbf{x},s)\right|^2, \tag{2.11}$$

indicating a monotonous decay for $\rho(\mathbf{x},\omega)\geqslant 0$, which is the standard dissipativity requirement. We can express (see Ref. [9]) $\hat{\chi}(\mathbf{x},z)$ in terms of $\rho(\mathbf{x},\omega)$ through

$$\hat{\chi}(\mathbf{x},z)=\frac{1}{z}\int d\omega\,\frac{\rho(\mathbf{x},\omega)}{\omega-z}=\int d\omega\,\frac{\rho(\mathbf{x},\omega)}{\omega^2-z^2}. \tag{2.12}$$

Then

$$\mathrm{Im}\,\varepsilon(\mathbf{x},\omega)=\frac{1}{2i}\{\hat{\chi}(\mathbf{x},\omega+i0)-\hat{\chi}(\mathbf{x},\omega-i0)\}=\frac{\pi\rho(\mathbf{x},\omega)}{\omega} \tag{2.13}$$

and

$$\chi'(\mathbf{x},0)=\int d\omega\rho(\mathbf{x},\omega), \tag{2.14}$$

which we assumed to be finite. Thus we arrive at the usual situation that for absorptive systems $\mathrm{Im}\,\varepsilon(\mathbf{x},\omega)>0$ for $\omega>0$ but note that $\mathrm{Im}\,\varepsilon(\mathbf{x},\omega)<0$ for $\omega<0$. Let now $t_0=-\infty$ and $t=+\infty$. Then, with $\tilde{\boldsymbol{E}}$ the Fourier transform of $\boldsymbol{E}$,

$$\Delta\mathcal{E}_{\mathrm{em}}=\mathcal{E}_{\mathrm{em}}(\infty)-\mathcal{E}_{\mathrm{em}}(-\infty)$$
$$=-\int d\mathbf{x}\int d\omega\rho(\mathbf{x},\omega)|\tilde{\boldsymbol{E}}(\mathbf{x},\omega)|^2. \tag{2.15}$$

We see that this quantity can vanish if $\rho(\mathbf{x},\omega)=0$ on some interval Δ and $\boldsymbol{E}(\mathbf{x},t)$ is such that it does not have Fourier components in some ω-interval Δ.

III. MATHEMATICAL PROPERTIES

We note that $\hat{\chi}(\mathbf{x},z)$ as given by Eq. (2.3) is analytic in the upper half plane, which, of course, is a direct consequence of the causality condition

$$\chi(\mathbf{x},t)=0, \quad t<0. \tag{3.1}$$

Introducing the measure $m(\mathbf{x},d\omega)$ according to

$$m(\mathbf{x},d\omega)=\rho(\mathbf{x},\omega)d\omega, \tag{3.2}$$

we have

$$f(\mathbf{x},z)\equiv z\hat{\chi}(\mathbf{x},z)=-i\hat{\chi}'(\mathbf{x},z)=\int m(\mathbf{x},d\omega)\frac{1}{\omega-z}, \tag{3.3}$$

which also has this analyticity property and moreover

$$\mathrm{Im}\, z\hat{\chi}(\mathbf{x},z)\geqslant 0, \quad \mathrm{Im}\, z>0. \tag{3.4}$$

Also $m(\mathbf{x},d\omega)=m(\mathbf{x},-d\omega)\geqslant 0$ and

$$\int_{\mathbb{R}} m(\mathbf{x},d\omega)=\chi'(\mathbf{x},0), \ \text{finite}. \tag{3.5}$$

Moreover, with $y>0$ and using $\chi(\mathbf{x},0)=0$, we have

$$\begin{aligned}
\sup_{y>0}|f(\mathbf{x},iy)| &= \sup_{y>0} y|iy\hat{\chi}(\mathbf{x},iy)| \\
&= \sup_{y>0} y\left| y\int_0^\infty dt\, \exp[-yt]\chi(\mathbf{x},t)\right| \\
&= \sup_{y>0} y\left| \int_0^\infty dt\{\partial_t \exp[-yt]\}\chi(\mathbf{x},t)\right| \\
&= \sup_{y>0} y\left| \int_0^\infty dt\, \exp[-yt]\chi'(\mathbf{x},t)\right| \\
&= \sup_{y>0}\left| \int_0^\infty du\, \exp[-u]\chi'\left(\mathbf{x},\frac{u}{y}\right)\right| <\infty,
\end{aligned} \tag{3.6}$$

where sup indicates the supremum or least upper bound of a function. We can ask whether or not Eq. (3.3) is the most general form of such a function $f(\mathbf{x},z)$. In fact it is not. The general form is a so-called Herglotz function (a useful summary of the properties of Herglotz functions can be found in Ref. [11]). The actual theorem we have in mind states the following [11].

Theorem: The function $f(z)$ with z in the open upper half plane $\hat{\mathbb{C}}$ has the representation

$$f(z)=\int_{\mathbb{R}}\sigma(d\lambda)\frac{1}{\lambda-z}$$

for σ a non-negative finite measure if and only if f is analytic on $\hat{\mathbb{C}}$, $\mathrm{Im}\, f\geqslant 0$ and $\sup_{y>0} y|f(iy)|<\infty$.

We see that our system falls into this class (in mathematical terms m is an absolutely continuous measure) but more general measures are allowed. Thus m can also have point measure and singular continuous contributions (for these notions see Ref. [12], p. 22). We disregard the last. Point measures $m_p(\mathbf{x},d\lambda)$ can be represented in terms of δ functions

$$\begin{aligned}
m_p(\mathbf{x},d\lambda) &= \sum_{j=0}^{\infty} m_j(\mathbf{x})\,\delta(\lambda-\lambda_j)d\lambda \\
&= m_0(\mathbf{x})\,\delta(\lambda)d\lambda+\sum_{j=1}^{\infty} m_j(\mathbf{x})\,\delta(\lambda-\lambda_j)d\lambda,
\end{aligned} \tag{3.7}$$

where we have set $\lambda_0=0$ [so $m_0(\mathbf{x})=0$ if this value does not occur]. The corresponding contribution to $\chi'(\mathbf{x},t)$ is then

$$\begin{aligned}
\chi'_p(\mathbf{x},t) &= \int_{\mathbb{R}} m_p(\mathbf{x},d\lambda)\exp[-i\lambda t] \\
&= m_0(\mathbf{x})+\sum_{j=1}^{\infty} m_j(\mathbf{x})\exp[-i\lambda_j t].
\end{aligned} \tag{3.8}$$

Since $\chi'_p(\mathbf{x},t)$ must be real, we have

$$\chi'_p(\mathbf{x},t)=m_0(\mathbf{x})+\sum_{j=1}^{\infty} m_j(\mathbf{x})\cos\lambda_j t, \tag{3.9}$$

and again we can assume $m_p(\mathbf{x},-d\lambda)=m_p(\mathbf{x},d\lambda)$, so we can write

$$\begin{aligned}
m_p(\mathbf{x},d\lambda) &= m_0(\mathbf{x})\,\delta(\lambda)d\lambda+\frac{1}{2}\sum_{j=1}^{\infty} m_j(\mathbf{x}) \\
&\quad\times\{\delta(\lambda-\lambda_j)+\delta(\lambda+\lambda_j)\}d\lambda \\
&= m_0(\mathbf{x})\,\delta(\lambda)d\lambda+\sum_{j=1}^{\infty} m_j(\mathbf{x})|\lambda_j|\,\delta(\lambda^2-\lambda_j^2)d\lambda.
\end{aligned} \tag{3.10}$$

From this we see that we can assume $\lambda_j>0$ for $j>0$ as we shall do in the following. We note further that if m_0 vanishes

$$\chi_p(\mathbf{x},t)=\sum_{j=1}^{\infty} m_j(\mathbf{x})\frac{\sin\lambda_j t}{\lambda_j}, \quad m_0=0. \tag{3.11}$$

Let us assume that we only have point measures and that $m_0(\mathbf{x})=0$. Then Eq. (2.15) becomes

$$\begin{aligned}
\Delta\mathcal{E}_{\mathrm{em}} &= \mathcal{E}_{\mathrm{em}}(\infty)-\mathcal{E}_{\mathrm{em}}(-\infty) \\
&= -\int d\mathbf{x}\int m(\mathbf{x},d\lambda)|\tilde{\boldsymbol{E}}(\mathbf{x},\lambda)|^2 \\
&= -2\int d\mathbf{x}\sum_{j=1}^{\infty} m_j|\tilde{\boldsymbol{E}}(\mathbf{x},\lambda_j)|^2, \quad m_0=0,
\end{aligned} \tag{3.12}$$

and we see that $\Delta\mathcal{E}_{\rm em}$ vanishes if we can produce a field $\boldsymbol{E}(\mathbf{x},t)$ such that its Fourier components vanish in the points λ_j for x inside the material. Instead of Eq. (2.13) we have

$$\mathrm{Im}\,\varepsilon(\mathbf{x},\lambda)=\frac{\pi\lambda}{|\lambda|}\sum_{j=1}^{\infty} m_j(\mathbf{x})\,\delta(\lambda^2-\lambda_j^2),\quad m_0=0, \tag{3.13}$$

which vanishes outside the points λ_j. We come back to the case $m_0\neq 0$ in the discussion section.

In practical cases the $m_j(\mathbf{x})$'s are usually constant over specific space regions. For instance, for a photonic crystal with a vacuum background, $\varepsilon(\mathbf{x},\lambda)=1$ and $m_j(\mathbf{x})=0$ for $\mathbf{x}$ in the background, whereas $m_j(\mathbf{x})=m$ constant over the material outside the background.

IV. THE DRUDE-LORENTZ MODEL

A well-known model for the electric susceptibility is the Drude-Lorentz model. Disregarding the $\mathbf{x}$ dependence for the moment, we now have

$$\hat{\chi}(\omega)=-\sum_j \frac{\Omega_j^2}{\omega^2-\omega_j^2+i\gamma_j\omega},\quad \gamma_j,\Omega_j>0, \tag{4.1}$$

leading to

$$\rho(\omega)=\frac{\omega^2}{\pi}\sum_j \frac{\gamma_j\Omega_j^2}{(\omega^2-\omega_j^2)^2+\gamma_j^2\omega^2}. \tag{4.2}$$

Here the γ_j's determine the degree of absorption. The Drude model is the special case where there is only a single ω_j, which equals zero. Thus [see also Eq. (1.2)]

$$\hat{\chi}_{\rm Drude}(\omega)=-\frac{\Omega^2}{\omega(\omega+i\gamma)},$$

$$\rho_{\rm Drude}(\omega)=\frac{1}{\pi}\frac{\gamma\Omega^2}{\omega^2+\gamma^2}. \tag{4.3}$$

Next we suppose that in Eq. (4.1),

$$\omega_j>\frac{\gamma_j}{2}>0. \tag{4.4}$$

Then we can write

$$\omega^2-\omega_j^2+i\omega\gamma_j=\left(\omega-\hat{\omega}_j+i\frac{\gamma_j}{2}\right)\times\left(\omega+\hat{\omega}_j+i\frac{\gamma_j}{2}\right),\hat{\omega}_j\sqrt{\omega_j^2-\gamma_j^2} \tag{4.5}$$

and

$$\hat{\chi}(\omega)=-\sum_j \frac{\Omega_j^2}{2\omega+\gamma_j}\left\{\frac{1}{\omega-\hat{\omega}_j+\frac{i\gamma_j}{2}}+\frac{1}{\omega+\hat{\omega}_j+\frac{i\gamma_j}{2}}\right\}. \tag{4.6}$$

We now take the limit $\gamma\downarrow 0$, i.e., the ''nonabsorptive'' limit. Then $\hat{\omega}_j\to\omega_j$ and (P denotes a Cauchy principal value)

$$\hat{\chi}(\omega)=-\sum_j \Omega_j^2\left\{\mathrm{P}\ \frac{1}{\omega^2-\omega_j^2}+i\pi\frac{\omega}{|\omega|}\sum_j \Omega_j^2\,\delta(\omega^2-\omega_j^2)\right\}. \tag{4.7}$$

Hence

$$\rho(\omega)=\sum_j \Omega_j^2\omega_j\,\delta(\omega^2-\omega_j^2). \tag{4.8}$$

Comparing this with Eq. (3.10) for $\lambda_0=0$, we can identify λ_j with ω_j and m_j with Ω_j^2,

$$\lambda_j=\omega_j,\quad m_j=\Omega_j^2. \tag{4.9}$$

Thus we conclude that in the limit $\gamma\downarrow 0$ we do not obtain a purely dispersive system. It also follows from our results in Sec. III that this limiting system has the proper mathematical behavior.

V. DISCUSSION

A. Results

We found that the Drude-Lorentz model has a limit in which $\varepsilon(\omega)$ is real, except for a number of discrete frequencies ω_j. Absorption can only take place at these frequencies and the Kramers-Kronig relations are still valid. However, for $\omega=\omega_j$ the electric field vanishes inside the material, so absorption is absent. On the other hand, the fact that $\varepsilon(\mathbf{x},\omega)$ has an imaginary part is crucial for the existence of a Lagrange-Hamilton formalism and its quantization.

In particular, band gaps can exist [2,3]. Note that in Ref. [4] we found that the latter do not occur in absorptive regions, i.e., regions where $\mathrm{Im}\,\varepsilon(\omega)\neq 0$. More generally we can have other situations where $m(\mathbf{x},\Delta)=0$ and hence $\mathrm{Im}\,\varepsilon(\omega)=0$ in some interval Δ. Then, according to the discussion in Sec. III, the Kramers-Kronig relations can still be valid.

From a practical point of, view our results mean that we do not have to be concerned about causality violations if, in a finite interval, $\varepsilon(\mathbf{x},\omega)$ is real, dispersive.

In Ref. [9] we already discussed the quantization procedure for the point measure case but we did not make the connection with the limiting case of a Drude-Lorentz model. There we concentrated upon the smooth case of Sec. II. The main difference is that there we obtain a coupling with a continuum of harmonic oscillator modes which now changes into a coupling with a discrete set of harmonic oscillators.

B. The Drude case

The Drude case is somewhat special. Now Eq. (4.4) does not hold but we can take the limit $\gamma \downarrow 0$ in Eq. (4.3), resulting in

$$\rho(\omega) = \Omega^2 \delta(\omega). \tag{5.1}$$

The Drude model is pathological in that $\varepsilon(\omega)$ becomes infinite in the static limit $\omega = 0$. Note that if we had kept the contribution $m_0(\mathbf{x})\delta(\omega)$ to $m(\mathbf{x}, d\omega)$, then the corresponding contribution to $\mathrm{Im}\,\varepsilon(\mathbf{x},\omega)$ is undefined. But

$$\lim_{\gamma \downarrow 0} \omega^2 \hat{\chi}_{\mathrm{Drude}}(\omega) = -\Omega^2, \tag{5.2}$$

which is finite and real. Here we note that the transverse Green's function associated with the Helmholtz equation for a spatially homogeneous medium is

$$G(\mathbf{x},\mathbf{y},z) = \frac{\exp[i\sqrt{z^2\varepsilon(z)}\Delta]}{4\pi\Delta}, \quad \boldsymbol{\Delta} = \mathbf{x} - \mathbf{y}, \tag{5.3}$$

which leads to

$$G(\mathbf{x},\mathbf{y},0) = \frac{\exp[-\Omega\Delta]}{4\pi\Delta} \tag{5.4}$$

in the limit $\gamma \downarrow 0$. Since the local density of states is proportional to $\mathrm{Im}\,G(\mathbf{x},\mathbf{y},\omega)$, taken in $\mathbf{x} = \mathbf{y}$, we see that it is actually zero in $\omega = 0$. The situation is similar in the photonic crystal case [4] although the proof is not trivial. Thus there are no particular problems with the Green's function at frequency zero in the limiting case. The whole issue is rather academic since the Drude model is in practice only used for frequencies well above zero and we can set ω_0 small but finite.

C. Outlook

It seems natural to consider a perturbative approach with the γ_j's as a small parameter. In particular this is interesting for photonic crystals possessing a band gap Δ for $\gamma = 0$. Then the density of states $N(\omega)$ vanishes for ω in the gap but this changes for nonzero γ. We expect that for small γ it is still small and that perturbation theory to leading order in γ will give a fairly accurate result. We are at present investigating this situation. Some preliminary results are given in Ref. [13].

Another interesting issue is the corresponding nonlinear situation. In Ref. [7] the scalar version of Eq. (1.7), extended with a nonlinear term is discussed for finite γ. Thus, in our notation,

$$\partial_t^2 x(t) + \gamma \partial_t x(t) + \omega_0^2 x(t) = \frac{e}{m} E(t) + \kappa x(t)^2, \tag{5.5}$$

where the anharmonic term $\kappa x(t)^2$ is added. This is a rather *ad hoc* approach, in particular the damping term being linear. An alternative is to calculate the quadratic (or higher) response of a material system to a driving electric field and study the limiting behavior of the ensuing nonlinear susceptibility.

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