Reference-wave solution for the two-frequency propagator in a statistically homogeneous random medium

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Spatial and temporal structures of ultrawide-band high-frequency fields can be appreciably affected by random changes of the medium parameters characteristic of almost all geophysical environments. The dispersive properties of random media cause distortions in the propagating signal, particularly in pulse broadening and time delay. Theoretical analysis of pulsed signal propagation is usually based on spectral decomposition of the time-dependent signal and the analysis of the two-frequency mutual coherence function. In this work we present a new reference-wave method and apply it to solving the equation of the two-frequency mutual coherence function propagator. This method is based on embedding the problem into a higher-dimensional space and is accompanied by the introduction of additional coordinates. Choosing a proper transform of the extended coordinate system allows us to emphasize "fast" and "slow" varying coordinates which are consequently normalized to the scales specific to a given type of problem. Such scaling usually reveals the important expansion parameter defined as a ratio of the characteristic scales and allows us to present the equation being solved as a hierarchy of terms having a decreasing order of expansion with respect to this parameter. We present an analytical result for the two-frequency mutual coherence function propagating in a random medium with arbitrary refractive index fluctuations and show that when approximating the transverse structure function of the medium by a quadratic form, the solution reduces to the exact result derived previously. Extension of the reference-wave method to the analysis of the pulse distortion effects is considered.

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I. INTRODUCTION

Attempts to improve the resolving abilities of modern radar, sonar, and other wave-based remote sensing systems have stimulated the ongoing trend of the exploitation of ultrawide-band signals [1]. This trend also finds extensive support in communication engineering because of the expanding demands for high-data-rate communication channels. The spatial and temporal structures of ultrawide-band high-frequency fields can be appreciably affected by changes of the medium parameters characteristic of almost all geophysical environments. For example, the propagation of electromagnetic waves is influenced by scattering and absorption induced by the prevailing meteorological conditions, especially precipitation, by the effects of fluctuations in the refractivity of the atmosphere, and by the electronic concentration in the ionosphere. The occurrence of most such changes is unpredictable, requiring the application of stochastic analysis.

The dispersive properties, characteristic of random media, cause distortions in the propagating signal, particularly in pulse broadening and time delay [2]. Theoretical analysis of pulsed signal propagation, especially in a dispersive medium, must be based on spectral decomposition of the time-dependent signal in order to solve a reduced equation for the time-harmonic field. Consequently, the space-time correlation properties are expressed as spectral integrals of the statistical moments of the wave field for different frequencies and different positions. For example, the second-order space-time statistical moment and the average intensity of pulsed signal can be expressed as a spectral integral of the spatial two-frequency mutual coherence function (TFMCF).

In order to describe the time evaluation of pulses of simple form, the mean arrival time and average pulse width are sufficient. The temporal moments can be estimated without needing to solve the equation for the TFMCF by using the technique commonly used in quantum mechanics [3], which has also been adopted for random propagation problems [4–6]. This technique is based on knowledge of the derivatives of the TFMCF for zero-frequency separation. It has, however, a limited applicability and is suitable only for the description of a simple form of pulse. In general, a description based on the mutual two-frequency coherence function is required.

The exact solutions for the TFMCF have been obtained for only two limiting cases. The first is valid for the regime of weak intensity fluctuations [2], and the second is based on the approximation of the transverse structure function by a quadratic form [7–10]. A quantitative extension of the strong fluctuation case can be made by using the extended Huygens-Fresnel principle [11]. A two-scale asymptotic expansion procedure has been efficiently applied to solving the two-point coherence equations in complicated environments [12–15]. This method has also been extended to the case of the bichromatic coherence equation [16]. It was shown that although the approach does not lead to a universal solution, it provides a good approximation for limited propagation distances and over a narrow frequency separation range. In some cases the TFMCF equation can be transformed into a separable form and solved by using the modal expansion for various source excitations [17–19]. The similarity between the equation for the parabolic-wave amplitude and Schrödinger's equation describing the movement of a quantum particle lead to the possibility of a description of wave propagation in random media by using the Feynman path-integral solutions [20–23]. The path-integral solutions have been applied for construction of the expression of the TFMCF and evaluated using the cumulant technique [24]. The path-integral expression of the TFMCF has been also evaluated by using the variational principle [25]. In the Ref. [26] the TFMCF has been computed by using the iterative expansion of an integral equation.

This work is based on a reference-wave method previously developed for solving parabolic-type wave equations [27]. Here we apply it to solving the equation of the two-frequency mutual coherence function. This method is based on embedding the problem into a higher-dimensional space and is accompanied by the introduction of additional coordinates. Choosing a proper transform of the extended coordinate system allows us to emphasize ''fast'' and ''slow'' varying coordinates which are subsequently normalized to the scales specific to a given type of problem. This scaling usually reveals the important expansion parameter defined as a ratio of the characteristic scales and allows us to present the equation being solved as a hierarchy of terms having a decreasing order of expansion with respect to this parameter. A similar approach has been taken in the two-scale expansion procedure [12–15]. An equation for the paired field measure was derived by using the parabolic-wave equation in a random medium for the field component and its complex conjugate, so that both components appear in the resulting equation symmetrically. This symmetry is preserved further when new transverse sum and difference coordinates are defined. We emphasize that in this work a nonsymmetric paired field function is defined. The first component is a solution of the equation governing propagation in a perturbed medium, while the second is a solution of a nonperturbed deterministic equation. A solution of the deterministic equation, in principle, can be found. Application of the embedding methods here, because of a lack of symmetry, requires that the transforms of the coordinate system will also be asymmetric.

The outline of this work is as follows. The problem is formulated in Sec. II. In Sec. III, we present the reference-wave method and apply it to the equation of the two-frequency mutual coherence function. We show that in the case of a quadratic structure function, the expressions derived by the reference-wave method are identical to the results obtained from an exact analytic solution of the equation of the mutual two-frequency coherence function. In Sec. IV, we propose extending the reference-wave method to the analysis of the intensity fluctuations of time-dependent signals.

II. FORMULATION OF THE PROBLEM

We consider a constant-background medium on which is superimposed a weak random part $\tilde{n}(\mathbf{r},\sigma)$. Such a random medium is represented by the refractive index

$$N(\mathbf{r},\sigma)=1+\tilde{n}(\mathbf{r},\sigma). \tag{1}$$

The statistical properties of the medium are assumed to be described by the δ-correlated correlation function of the refractive index fluctuations:

$$B_n(\mathbf{r}_1-\mathbf{r}_2,\sigma_1-\sigma_2)=A_n(\mathbf{r}_1-\mathbf{r}_2)\,\delta(\sigma_1-\sigma_2). \tag{2}$$

We start with a pulsed-wave source that radiates in some preferred direction a pulse

$$f(t)=f_0(t)\exp(i\omega_0 t), \tag{3a}$$

with the spectrum envelope

$$F(\omega)=\int_{-\infty}^{\infty}f(t)\exp(-i\omega t)dt \tag{3b}$$

centered around the frequency ω_0, which determines the important frequency band. We assume that the high-frequency propagation conditions are satisfied—i.e., $\Delta\omega\ll\omega_0$, with $\Delta\omega$ being the radiation bandwidth of the time-dependent signal (3), and $k\ell_n\gg 1$, where $k=\omega/c$ is the radiation wave number.

When the dispersive contributions of the background medium are weak, propagation of high-frequency time-harmonic signals in spatially inhomogeneous media is intuitively related to the geometrical ray trajectories representing the paths of energy flux transfer. We base our solution on the parabolic approximation along a straight background ray in a ray-centered coordinate system $\mathbf{R}=\{\mathbf{r},\sigma\}$, where σ measures the range along the preferred direction and $\mathbf{r}=\{x,y\}$ is a two-dimensional radius vector in the transverse plane perpendicular to that direction. Extracting from the high-frequency field the main phase variation along some reference ray,

$$U(\mathbf{r},\sigma)=u(\mathbf{r},\sigma)\exp(-ik\sigma). \tag{4}$$

The spatial distribution of the source is assumed to be specified at some initial plane σ_0 perpendicular to the propagation direction and characterized by the field function $u_0(\mathbf{r}_0,\sigma_0)$.

In order to perform the analysis of the propagation of the time-dependent signals by taking into account the dispersive properties of the random medium, it is suitable to decompose the initial excitation into the spectral form and to consider the propagation of each time harmonic component $u(\mathbf{r},\sigma|\omega)$ separately.

The total field at an arbitrary time t and location $\{\mathbf{r},\sigma\}$ can be represented as a superposition

$$\Psi(\mathbf{r},\sigma,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)u(\mathbf{r},\sigma|\omega)\exp[i\omega t-k(\omega)\sigma]d\omega. \tag{5}$$

The space-time correlation properties of the propagating signal are determined from the correlation function

$$\langle\Psi(\mathbf{r}_1,\sigma,t)\Psi^*(\mathbf{r}_2,\sigma,t)\rangle = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}d\omega_1\int_{-\infty}^{\infty}d\omega_2\Phi(\omega_1,\omega_2,\sigma),$$

$$\Gamma_{1,2}(\mathbf{r}_1,\mathbf{r}_2,\sigma)\exp[i(\omega_1-\omega_2)t], \tag{6}$$

where

$$\Phi(\omega_1,\omega_2,\sigma)=F(\omega_1)F^*(\omega_2)\exp\{-i[k(\omega_1)-k(\omega_2)]\sigma\} \tag{6a}$$

is the bilinear spectrum of the transient plane wave propagating in a homogeneous medium and measured at a distance σ from the source. The function

$$\Gamma_{1,2}(\mathbf{r}_1,\mathbf{r}_2,\sigma)=\langle U(\mathbf{r}_1,\sigma|\omega_1)U^*(\mathbf{r}_2,\sigma|\omega_2)\rangle \tag{7}$$

is the mutual two-frequency coherence function [2]. The mean intensity variation is obtained from Eq. (5) by taking the same location for both signal components:

$$\langle I(\mathbf{r},\sigma,t)\rangle=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}d\omega_1\int_{-\infty}^{\infty}d\omega_2\Phi(\omega_1,\omega_2,\sigma) \times\Gamma_{1,2}(\mathbf{r},\mathbf{r},\sigma)\exp[i(\omega_1-\omega_2)t]. \tag{8}$$

According to Eqs. (6) and (8), the mean shape of the propagating signal is determined by two factors. The first, the mutual spectrum $\Phi(\omega_1,\omega_2,\sigma)$, accounts for the distortion caused by the dispersive character of the unperturbed medium. The second, represented by the TFMCF, describes the loss of coherence between different spectral components because of the scattering of the random refractive index fluctuations. In this work we concentrate on the second factor. We neglect completely the influence of the dispersive properties of the medium on the strength of random scattering because of the above narrow-band assumption. It has been shown that $\Gamma_{12}(\mathbf{r}_1,\mathbf{r}_2,\sigma)$ is a solution of the following equation:

$$\frac{\partial\Gamma_{12}(\mathbf{r}_1,\mathbf{r}_2,\sigma)}{\partial\sigma}=\left(\frac{i}{2k_1}\nabla^2_{\mathbf{r}_1}-\frac{i}{2k_2}\nabla^2_{\mathbf{r}_2}\right)\Gamma_{12}(\mathbf{r}_1,\mathbf{r}_2,\sigma) -F(\mathbf{r}_1-\mathbf{r}_2,k_1,k_2)\Gamma_{12}(\mathbf{r}_1,\mathbf{r}_2,\sigma), \tag{9}$$

$$\Gamma_{12}(\mathbf{r}_1,\mathbf{r}_2,\sigma_0)=\Gamma^0_{12}(\mathbf{r}_1,\mathbf{r}_2), \tag{9a}$$

with

$$F(\mathbf{s},k_1,k_2)=\frac{1}{2}[(k_1^2+k_2^2)A_n(0)-2k_1k_2A_n(s)] \tag{10a}$$

$$=\frac{1}{2}[(k_1-k_2)^2A_n(0)+k_1k_2D_n(s)], \tag{10b}$$

where

$$D_n(s)=2[A_n(0)-A_n(s)] \tag{11}$$

is the structure function of the medium's fluctuations which are assumed to be δ correlated along the main propagation direction.

III. REFERENCE-WAVE SOLUTION

In order to solve Eq. (10) governing the propagation of the two-frequency coherence function it is convenient to transfer to the center of mass and difference coordinates

$$\mathbf{p}=\frac{\mathbf{r}_1+\mathbf{r}}{2},\quad \mathbf{s}=\mathbf{r}_1-\mathbf{r}_2. \tag{12}$$

In the new coordinates Eq. (9) becomes

$$\frac{\partial\Gamma(\mathbf{p},\mathbf{s},\sigma)}{\partial\sigma}=\frac{i}{2}\frac{k_1+k_2}{k_1k_2}\nabla_{\mathbf{p}}\cdot\nabla_{\mathbf{s}}\Gamma(\mathbf{p},\mathbf{s},\sigma) -\frac{i}{8}\frac{k_1-k_2}{k_1k_2}\nabla_p^2\Gamma(\mathbf{p},\mathbf{s},\sigma) -\frac{i}{2}\frac{k_1-k_2}{k_1k_2}\nabla_s^2\Gamma(\mathbf{p},\mathbf{s},\sigma) -F(\mathbf{s},k_1,k_2)\Gamma(\mathbf{p},\mathbf{s},\sigma), \tag{13}$$

$$\Gamma(\mathbf{p},\mathbf{s},\sigma_0)=\Gamma_0(\mathbf{p},\mathbf{s}). \tag{13a}$$

Next, we introduce the average wave number k, the difference wave number Δk,

$$k=\frac{k_1+k_2}{2},\quad \Delta k=k_1-k_2, \tag{14}$$

and the relative frequency mistuning parameter

$$\Omega=\frac{k_1-k_2}{k_1+k_2}=\frac{\Delta k}{2k}. \tag{15}$$

Using these parameters and a point source boundary condition, Eq. (14) reduces to the following form:

$$\frac{\partial\Gamma(\mathbf{p},\mathbf{s},\sigma)}{\partial\sigma}=\frac{i}{k}\frac{1}{1-\Omega^2}\nabla_{\mathbf{p}}\cdot\nabla_{\mathbf{s}}\Gamma(\mathbf{p},\mathbf{s},\sigma) -\frac{i}{k}\frac{\Omega}{1-\Omega^2}\nabla_s^2\Gamma(\mathbf{p},\mathbf{s},\sigma) -\frac{i}{4k}\frac{\Omega}{1-\Omega^2}\nabla_p^2\Gamma(\mathbf{p},\mathbf{s},\sigma) -k^2F_s(s,\Omega)\Gamma(\mathbf{p},\mathbf{s},\sigma), \tag{16}$$

$$\Gamma(\mathbf{p},\mathbf{s},\sigma_0)=\delta(\mathbf{p}-\mathbf{p}_0)\,\delta(\mathbf{s}-\mathbf{s}_0), \tag{16a}$$

with the scattering function

$$F_s(s,\Omega)=\left[2\Omega^2A_n(0)+\frac{1}{2}(1-\Omega^2)D_n(s)\right]. \tag{17}$$

We consider in parallel the following equation for the function:

$$\frac{\partial\Psi(\mathbf{p}_1,\mathbf{s}_1,\sigma)}{\partial\sigma}=-\frac{i}{k}\frac{1}{1-\Omega^2}\nabla_{\mathbf{p}_1}\cdot\nabla_{\mathbf{s}_1}\Psi(\mathbf{p}_1,\mathbf{s}_1,\sigma)+\frac{i}{4k}\frac{\Omega}{1-\Omega^2}\nabla^2_{p_1}\Psi(\mathbf{p}_1,\mathbf{s}_1,\sigma)+\frac{i}{k}\frac{\Omega}{1-\Omega^2}\nabla^2_{s_1}\Psi(\mathbf{p}_1,\mathbf{s}_1,\sigma), \tag{18a}$$

$$\Psi(\mathbf{p}_1,\mathbf{s}_1,\sigma_0)=\delta(\mathbf{p}_1-\mathbf{p}_{10})\,\delta(\mathbf{s}_1-\mathbf{s}_{10}). \tag{18b}$$

Next, we define a product

$$\Pi(\mathbf{p},\mathbf{s},\mathbf{p}_1,\mathbf{s}_1,\sigma)=\Gamma(\mathbf{p},\mathbf{s},\sigma)\Psi(\mathbf{p}_1,\mathbf{s}_1,\sigma). \tag{19}$$

The equation for $\Pi(\mathbf{p},\mathbf{s},\mathbf{p}_1,\mathbf{s}_1,\sigma)$ is

$$\frac{\partial\Pi}{\partial\sigma}=\frac{i}{k}\frac{1}{1-\Omega^2}(\nabla_{\mathbf{p}}\cdot\nabla_{\mathbf{s}}-\nabla_{\mathbf{p}_1}\cdot\nabla_{\mathbf{s}_1})\Pi-\frac{i}{4k}\frac{\Omega}{1-\Omega^2}\times(\nabla^2_p-\nabla^2_{p_1})\Pi-\frac{i}{k}\frac{\Omega}{1-\Omega^2}(\nabla^2_s-\nabla^2_{s_1})\Pi-k^2F_s(\mathbf{s},\Omega)\Pi, \tag{20}$$

with

$$\Pi(\mathbf{p},\mathbf{s},\mathbf{p}_1,\mathbf{s}_1,\sigma_0)=\delta(\mathbf{p}-\mathbf{p}_0)\,\delta(\mathbf{s}-\mathbf{s}_0)\,\delta(\mathbf{p}_1-\mathbf{p}_{10})\,\delta(\mathbf{s}_1-\mathbf{s}_{10}). \tag{20a}$$

Now we introduce the following new coordinates:

$$\mathbf{u}=\mathbf{s}_1,\quad \mathbf{q}=\mathbf{s}-\mathbf{s}_1,\quad \mathbf{v}=\mathbf{p}_1,\quad \mathbf{w}=\mathbf{p}-\mathbf{p}_1. \tag{21}$$

In the new coordinates Eq. (20) for the function

$$\bar{\Pi}(\mathbf{u},\mathbf{q},\mathbf{v},\mathbf{w},\sigma)=\Pi(\mathbf{w}+\mathbf{v},\mathbf{q}+\mathbf{u},\mathbf{v},\mathbf{u},\sigma) \tag{22}$$

becomes

$$\frac{\partial\bar{\Pi}}{\partial\sigma}=\frac{i}{k}\frac{1}{1-\Omega^2}(\nabla_{\mathbf{w}}\cdot\nabla_{\mathbf{u}}+\nabla_{\mathbf{v}}\cdot\nabla_{\mathbf{q}}-\nabla_{\mathbf{v}}\cdot\nabla_{\mathbf{u}})\bar{\Pi}-\frac{i}{2k}\frac{\Omega}{1-\Omega^2}(\nabla_{\mathbf{w}}\cdot\nabla_{\mathbf{v}}+4\nabla_{\mathbf{q}}\cdot\nabla_{\mathbf{u}})\bar{\Pi}+\frac{i}{4k}\frac{\Omega}{1-\Omega^2}\times(\nabla^2_v+4\nabla^2_u)\bar{\Pi}-k^2F_s(\mathbf{q}+\mathbf{u},\Omega)\bar{\Pi}, \tag{23}$$

with

$$\bar{\Pi}(\mathbf{u},\mathbf{q},\mathbf{v},\mathbf{w},\sigma_0)=\delta(\mathbf{w}+\mathbf{v}-\mathbf{p}_0)\,\delta(\mathbf{q}+\mathbf{u}-\mathbf{s}_0)\,\delta(\mathbf{v}-\mathbf{p}_{10})\times\delta(\mathbf{u}-\mathbf{s}_{10}). \tag{23a}$$

The next step is a transfer to the spectral domain with respect to the **u** and **v** coordinates. Such a transfer is realized by the following transformations:

$$\Lambda(\boldsymbol{\rho},\mathbf{q},\boldsymbol{\eta},\mathbf{w},\sigma)=\int\int d^2u\,d^2v\,\bar{\Pi}_1(\mathbf{u},\mathbf{q},\mathbf{v},\mathbf{w},\sigma)\times\exp\{-ik(1-\Omega^2)[\boldsymbol{\rho}\cdot\mathbf{u}+\boldsymbol{\eta}\cdot\mathbf{v}]\}, \tag{24a}$$

$$\bar{\Pi}_1(\mathbf{u},\mathbf{q},\mathbf{v},\mathbf{w},\sigma)=\left\{\frac{k(1-\Omega^2)}{2\pi}\right\}^4\int\int d^2\rho\,d^2\eta\,\Lambda\times(\boldsymbol{\rho},\mathbf{q},\boldsymbol{\eta},\mathbf{w},\sigma)\exp\{ik(1-\Omega^2)\times[\boldsymbol{\rho},\mathbf{u}+\boldsymbol{\eta}\cdot\mathbf{v}]\}. \tag{24b}$$

The equation for $\Lambda(\boldsymbol{\rho},\mathbf{q},\boldsymbol{\eta},\mathbf{w},\sigma)$ is

$$\frac{\partial\Lambda}{\partial\sigma}+\left\{\left[\boldsymbol{\rho}-\frac{\Omega}{2}\boldsymbol{\eta}\right]\cdot\nabla_{\mathbf{w}}+(\boldsymbol{\eta}-2\Omega\boldsymbol{\rho})\cdot\nabla_{\mathbf{q}}\right\}\Lambda=ik(1-\Omega^2)\left[\boldsymbol{\rho}\cdot\boldsymbol{\eta}-\frac{\Omega}{4}(\eta^2+4\rho^2)\right]\Lambda-k^2F_s\left(\mathbf{q}+\frac{i}{k(1-\Omega^2)}\nabla_{\boldsymbol{\rho}},\Omega\right)\Lambda, \tag{25}$$

$$\Lambda(\boldsymbol{\rho},\mathbf{q},\boldsymbol{\eta},\mathbf{w},\sigma_0)=\delta(\mathbf{w}+\mathbf{p}_{10}-\mathbf{p}_0)\,\delta(\mathbf{q}+\mathbf{s}_{10}-\mathbf{s}_0)\times\exp\{-ik(1-\Omega^2)[\boldsymbol{\rho}\cdot\mathbf{s}_{10}+\boldsymbol{\eta}\cdot\mathbf{p}_{10}]\}. \tag{25a}$$

Equations (24) and (25) can be solved by applying asymptotic analysis. Such analysis assumes coordinate scaling with respect to the radiation wavelength and the length scales characteristic of the refractive index spatial variation. The characteristic scales help us to expose the "fast" and "slow" variables with consequent ordering of terms and presenting the equation as an expansion into the power series of the expansion parameter $\varepsilon=1/(k\ell_s)$ that represents the order of the single-scattering angle. Formally, this also can be done to the nonscaled equation by expanding it into the inverse power series of the wave number k. Expanding the scattering function operator

$$F_s\left(\mathbf{q}+\frac{i}{k(1-\Omega^2)}\nabla_{\boldsymbol{\rho}},\Omega\right)$$

into the power series of $[i/k(1-\Omega^2)]\nabla_{\boldsymbol{\rho}}$ leads to the following first-order partial differential equation:

$$\frac{\partial\Lambda}{\partial\sigma}+\left\{\left[\boldsymbol{\rho}-\frac{\Omega}{2}\boldsymbol{\eta}\right]\cdot\nabla_{\mathbf{w}}+(\boldsymbol{\eta}-2\Omega p)\cdot\nabla_{\mathbf{q}}\right\}\Lambda-ik\nabla_{\mathbf{q}}A_n(\mathbf{q})\cdot\nabla_{\boldsymbol{\rho}}\Lambda=ik(1-\Omega^2)\left[\boldsymbol{\rho}\cdot\boldsymbol{\eta}-\frac{\Omega}{4}(\eta^2+4\rho^2)\right]\Lambda-k^2F_s(\mathbf{q},\Omega)\Lambda, \tag{26}$$

$$\Lambda(\boldsymbol{\rho},\mathbf{q},\boldsymbol{\eta},\mathbf{w},\sigma_0)=\delta(\mathbf{w}+\mathbf{p}_{10}-\mathbf{p}_0)\,\delta(\mathbf{q}+\mathbf{s}_{10}-\mathbf{s}_0)\times\exp\{-ik(1-\Omega^2)[\boldsymbol{\rho}\cdot\mathbf{s}_{10}+\boldsymbol{\eta}\cdot\mathbf{p}_{10}]\},\tag{26a}$$

where we have used the explicit form of the scattering function in Eq. (17c). The third term in the left-hand side of Eq. (26) is of a smaller order of magnitude than the other terms. We retain it because it represents the phase information along the rays and can be very important in the exponential phase terms of the solution. However, it is less important in the amplitude terms, and there it can be neglected.

Equation (26) can be solved by the method of characteristics. Its characteristic equation are

$$\frac{d\mathbf{q}}{d\zeta}=\boldsymbol{\eta}-2\Omega\boldsymbol{\rho},\quad \mathbf{q}(\zeta=\sigma)=\mathbf{q}_\sigma,\tag{27a}$$

$$\frac{d\boldsymbol{\rho}}{d\zeta}=-ik\boldsymbol{\nabla}_{\mathbf{q}}A_n(\mathbf{q}),\quad \boldsymbol{\rho}(\zeta=\sigma)=\boldsymbol{\rho}_\sigma,\tag{27b}$$

$$\frac{d\mathbf{w}}{d\zeta}=\boldsymbol{\rho}-\frac{\Omega}{2}\boldsymbol{\eta},\quad \mathbf{w}(\zeta=\sigma)=\mathbf{w}_\sigma,\tag{27c}$$

$$\frac{d\boldsymbol{\eta}}{d\zeta}=0,\quad \boldsymbol{\eta}(\zeta=\sigma)=\boldsymbol{\eta}_\sigma,\tag{27d}$$

$$\frac{d\Lambda}{d\zeta}=\left\{ik(1-\Omega^2)\left[\boldsymbol{\rho}\cdot\boldsymbol{\eta}-\frac{\Omega}{4}(\eta^2+4\rho^2)\right]-k^2F_s(\mathbf{q},\Omega)\right\}\Lambda.\tag{27e}$$

The boundary condition for Eq. (27e) is the same as in Eq. (26a). When the solutions of the characteristic equations (27a)–(27d) are known, Eq. (27e) can be expressed as

$$\Lambda=\Lambda(\boldsymbol{\rho}(\sigma_0),\mathbf{q}(\sigma_0),\boldsymbol{\eta}(\sigma_0),\mathbf{w}(\sigma_0),\sigma_0)\exp\left\{ik(1-\Omega^2)\int_{\sigma_0}^{\sigma}d\zeta\left[\boldsymbol{\rho}(\zeta)\cdot\boldsymbol{\eta}(\zeta)-\Omega\left(\rho^2(\zeta)+\frac{\eta^2(\zeta)}{4}\right)\right]\right\}\times\exp\left\{-k^2\int_{\sigma_0}^{\sigma}d\zeta F_s(\mathbf{q}(\zeta),\Omega)\right\}.\tag{28}$$

The expression presented by Eq. (28) can be simplified by using the explicit solutions for the characteristics:

$$\begin{aligned}\Lambda(\boldsymbol{\rho}_\sigma,\mathbf{q}_\sigma,\boldsymbol{\eta}_\sigma,\mathbf{w}_\sigma,\sigma,\sigma_0)&=\delta(\mathbf{w}(\sigma_0)-\mathbf{p}_0)\,\delta(\mathbf{q}(\sigma_0)-\mathbf{s}_0)\exp\{-ik(1-\Omega^2)[\boldsymbol{\rho}(\sigma_0)\cdot\mathbf{s}_{10}+\boldsymbol{\eta}(\sigma_0)\cdot\mathbf{p}_{10}]\}\\&\times\exp\left\{ik\frac{(1-\Omega^2)}{4Q}[(1-\Omega^2)\eta_\sigma^2-(\boldsymbol{\eta}_\sigma-2\Omega\boldsymbol{\rho}_\sigma)^2]\Delta\sigma\right\}\\&\times\exp\left\{-k^2[(1+\Omega^2)A_n(0)+(1-\Omega^2)A_n(\mathbf{q}_\sigma)]\Delta\sigma+2k^2(1-\Omega^2)\int_{\sigma_0}^{\sigma}d\zeta A_n(\mathbf{q}(\zeta))\right\},\end{aligned}\tag{29}$$

where $\Delta\sigma=\sigma-\sigma_0$. In order to find the desired solution we perform the inverse transform (25b) and set $\mathbf{u}=\mathbf{0}$ and $\mathbf{v}=\mathbf{0}$:

$$\begin{aligned}\overline{\Pi}_1(\mathbf{u}=\mathbf{0},\mathbf{q}_\sigma,\mathbf{v}=\mathbf{0},\mathbf{w}_\sigma,\sigma)&=\left[\frac{k(1-\Omega^2)}{2\pi}\right]^4\int\int d^2\rho_\sigma d^2\eta_\sigma\,\delta(\mathbf{w}(\sigma_0)-\mathbf{p}_0)\,\delta(\mathbf{q}(\sigma_0)-\mathbf{s}_0)\\&\times\exp\left\{ik\frac{(1-\Omega^2)}{4\Omega}[(1-\Omega^2)\eta_\sigma^2-(\boldsymbol{\eta}_\sigma-2\Omega\boldsymbol{\rho}_\sigma)^2]\Delta\sigma\right\}\exp\left\{-2k^2\left[\Omega^2A_n(0)-\frac{1}{4}(1-\Omega^2)D_n(\mathbf{q}_\sigma)\right]\Delta\sigma\right\}\\&\times\exp\left\{-k^2(1-\Omega^2)\int_{\sigma_0}^{\sigma}d\zeta D_n[\mathbf{q}(\zeta)]\right\}.\end{aligned}\tag{30}$$

The solution for $\mathbf{w}(\zeta)$ and $\boldsymbol{\eta}(\zeta)$ can be obtained directly from the characteristic equations (27). This leads to

$$\mathbf{w}(\zeta)=\frac{1}{2\Omega}[\mathbf{q}_\sigma-\mathbf{q}(\zeta)]+\mathbf{w}_\sigma+\frac{1-\Omega^2}{2\Omega}\boldsymbol{\eta}_\sigma(\zeta-\sigma),\tag{31a}$$

$$\boldsymbol{\eta}=\boldsymbol{\eta}_\sigma.\tag{31b}$$

Performing the $\boldsymbol{\eta}_\sigma$ integration in Eq. (30) and changing the integration variable to $\boldsymbol{\rho}=\boldsymbol{\eta}_\sigma-2\Omega\boldsymbol{\rho}_\sigma$, we obtain

$$\overline{\Pi}_1(\mathbf{u}=0,\mathbf{q}_\sigma,\mathbf{v}=0,\mathbf{w}_\sigma,\sigma)=\left[\frac{k^2(1-\Omega^2)}{4\pi^2\Delta\sigma}\right]^2\int\int d^2\rho\,\delta(\mathbf{q}(\sigma_0)-\mathbf{s}_0)\exp\left\{ik\frac{(1-\Omega^2)}{4\Omega}[(1-\Omega^2)\eta_\sigma^2-\rho^2]\Delta\sigma\right\}$$
$$\times\exp\left\{-2k^2\left[\Omega^2A_n(0)-\frac{1}{4}(1-\Omega^2)D_n(\mathbf{q}_\sigma)\right]\Delta\sigma\right\}$$
$$\times\exp\left\{-k^2(1-\Omega^2)\int_{\sigma_0}^{\sigma}d\zeta D_n(\mathbf{q}(\zeta))\right\}, \tag{32}$$

with

$$\boldsymbol{\eta}_\sigma=\frac{1}{(1-\Omega^2)\Delta\sigma}\{[\mathbf{q}_\sigma-\mathbf{s}_0]+2\Omega(\mathbf{w}_\sigma-\mathbf{p}_0)\}. \tag{32a}$$

The desired solution for the mutual two-frequency coherence function can, in principle, be obtained directly from Eq. (32). This requires the extraction of the solution for the reference wave propagating along characteristics (27) from Eq. (32). The modulus of the amplitude of this reference wave can be obtained as a square root of the expression in Eq. (32) without the scattering term:

$$\Psi(\mathbf{p}=0,\mathbf{s}=0,\mathbf{u}=0,\mathbf{v}=0,\sigma)=\left[\frac{k^2(1-\Omega^2)}{4\pi^2\Delta\sigma}\right]\left|\int\int d^2\ \rho\,\delta(\mathbf{q}(\sigma_0))\exp\left[-\frac{ik\Delta\sigma(1-\Omega^2)\rho^2}{4\Omega}\right]\right|^{1/2}. \tag{33}$$

For $\Omega=0$, the result of Eq. (33) with Eq. (32) reduces to the exact solution of the second-order coherence function equation [2].

Generally speaking, the reference-wave amplitude is a complex function and can be retrieved only in some particular cases. Then the expression for the two-frequency propagator can be extracted directly: i.e.,

$$g_{1,2}(\mathbf{p},\mathbf{s},\sigma|\mathbf{p}_0,\mathbf{s}_0,\sigma_0)$$
$$=\overline{\Pi}_1(\mathbf{u}=0,\mathbf{q}_\sigma=\mathbf{s},\mathbf{v}=\mathbf{0},\mathbf{w}_\sigma=\mathbf{p},\sigma)/\Psi(\mathbf{u}=\mathbf{0},\mathbf{v}=\mathbf{0},\sigma). \tag{34}$$

For a source having an arbitrary spatial distribution $\Gamma_{1,2}(\mathbf{p}_0,\mathbf{s}_0,\sigma_0)$, the two-frequency response at the observation plane σ can be obtained directly from the propagation relation

$$\Gamma_{1,2}(\mathbf{p},\mathbf{s},\sigma)=\int\int d^2p_0d^2s_0g_{1,2}(\mathbf{p},\mathbf{s},\sigma|\mathbf{p}_0,\mathbf{s}_0,\sigma_0)$$
$$\times\Gamma_{1,2}(\mathbf{p}_0,\mathbf{s}_0,\sigma_0). \tag{35}$$

In order to obtain an approximate result for the solution $\Psi(\mathbf{u},\mathbf{v},\sigma)$, we note that it can be obtained by solving Eqs. (25) and/or (26) without the scattering term along the characteristics (27), which leads to the solution (28) in which the scattering term is omitted. In principle, Eqs. (27a) and (27b) can be solved by direct differentiation of Eq. (27a) with respect to the coordinate ζ. Then the use of Eq. (27b) leads to the equation

$$\frac{d^2\mathbf{q}}{d\zeta^2}-2ik\Omega\boldsymbol{\nabla}_{\mathbf{q}}A_n(\mathbf{q})=\mathbf{0}. \tag{36}$$

It must be noted that $\Psi(\mathbf{u},\mathbf{v},\sigma)$ represents an amplitude term specified by the transport of the initial condition along the characteristics. In most of the practical cases, the scattering term in the characteristic equation (27b) is very small and can be neglected. In such a case, the solutions of Eqs. (27) and (35) are straight rays, and Eq. (23) for Π is factorized as a product of two reference waves. Then the reference-wave amplitude can be determined directly from Eq. (18) and is equal to

$$\Psi(\mathbf{u}=\mathbf{0},\mathbf{v}=\mathbf{0},\sigma)=\frac{k^2(1-\Omega^2)}{4\pi^2\Delta\sigma^2}. \tag{37}$$

The above approximation becomes even more relevant for large values of $\mathbf{q}$ exceeding the correlation length ℓ_n, for which the scattering term in Eqs. (27b) and (36) vanishes because of the rapid decay of the correlation function $A_n(\mathbf{q})$.

The correction for $q\ll\ell_n$ can be estimated by expanding the scattering function $A_n(s)$ into the power series of s. Then the structure function $D_n(s)$ in Eq. (12) can be approximated by a quadratic term

$$D_n(r)=2A_0\left(\frac{r}{\ell_n}\right)^2. \tag{38}$$

This structure function allows us to obtain analytical solutions for the characteristic equations (27). This solution for the coordinate $\mathbf{q}(\zeta)$ is given by the following formula:

$$\mathbf{q}(\zeta)=\mathbf{q}_\sigma\cos[\alpha(\zeta-\sigma)]+(\boldsymbol{\rho}_\sigma/\alpha)\sin[\alpha(\zeta-\sigma)], \tag{39}$$

where

$$\alpha = 2\sqrt{ik\Omega A_0}/\ell_n. \tag{40}$$

Now, substituting the solution $\mathbf{q}(\sigma_0)$ from Eq. (39) into Eq. (33), we can extract the exact expression for the reference wave:

$$\Psi(\mathbf{u}=0,\mathbf{v}=\mathbf{0},\sigma) = \frac{k^2(1-\Omega^2)}{4\pi^2\Delta\sigma}\frac{\alpha}{\sin(\alpha\Delta\sigma)}. \tag{41}$$

For the limit $\alpha \to 0$, the expression in Eq. (39) approaches that of a homogeneous medium.

The expression for the two-frequency propagator can be deduced directly from Eq. (32):

$$\begin{aligned} g_{1,2}(\mathbf{p},\mathbf{s},\sigma|\mathbf{p}_0,\mathbf{s}_0,\sigma_0) &= (1-\Omega^2)\left(\frac{k^2}{2\pi\Delta\sigma}\right)^2\frac{\alpha\Delta\sigma}{\sin(\alpha\Delta\sigma)} \\ &\quad\times\exp(-2k^2\Omega A_0\Delta\sigma)\exp\left\{\frac{ik}{4\Omega gD\sigma}[(\mathbf{s}-\mathbf{s}_0) \right.\\ &\quad\left.+2\Omega(\mathbf{p}-\mathbf{p}_0)]^2\right\}\exp\left\{\frac{ik\alpha(1-\Omega^2)}{4\Omega\sin(\alpha\Delta\sigma)}\right.\\ &\quad\left.\times[(s^2+s_0^2)\cos(\alpha\Delta\sigma)-2(\mathbf{s}\cdot\mathbf{s}_0)]\right\}. \end{aligned} \tag{42}$$

Integrating Eq. (34) with Eq. (42) and the plane-wave initial condition, we obtain the following result:

$$\begin{aligned} \Phi(\mathbf{s},\sigma) &= \frac{1}{\cos(\alpha\Delta\sigma)}\exp\left\{-2k^2\Omega^2A_0\Delta\sigma\right.\\ &\quad\left.+\frac{ik(1-\Omega^2)}{4\Omega}s^2\alpha\tan(\alpha\Delta\sigma)\right\}. \end{aligned} \tag{43}$$

This result coincides with the exact solution for the mutual coherence function obtained for the two-frequency plane wave propagating in a quadratic medium [7,8].

We note that the solutions (42) and (43) are exact solutions of Eq. (17), which gives us confidence that there is no additional phase term in the expressions (39) and (41) for the reference wave.

IV. INTENSITY FLUCTUATIONS OF TIME-DEPENDENT SIGNALS

The second-order statistical moments provide us with the average intensity of the propagating signal. In many practical situations it is important to know the distortion of such signals and the higher-order correlation effects. Such information can be obtained by studying the behavior of the normalized intensity variance or the so-called intensity scintillation index [2]:

$$\beta_I^2(\mathbf{r},\sigma,t) = \frac{\langle I^2(\mathbf{r},\sigma,t)\rangle - \langle I(\mathbf{r},\sigma,t)\rangle^2}{\langle I(\mathbf{r},\sigma,t)\rangle^2}. \tag{44}$$

This quantity has been extensively investigated in the frequency domain, but there are no results applicable for time-domain propagation.

The average intensity can be obtained from the two-frequency mutual coherence function investigated in the previous sections. The second-order intensity moment can be computed from the space-time domain fourth-order statistical moment of the field:

$$\begin{aligned} &\langle\hat{U}(\mathbf{r}_1,t)\hat{U}^*(\mathbf{r}_2,t)U(\mathbf{r}_3,t)\hat{U}^*(\mathbf{r}_4,t)\rangle \\ &\quad= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\cdots\int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \Phi(\omega_1,\omega_2) \\ &\qquad\times\Phi(\omega_3,\omega_4)\Gamma_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,\sigma|\omega_1,\omega_2,\omega_3,\omega_4) \\ &\qquad\times\exp[-i(\omega_1-\omega_2+\omega_3-\omega_4)t]. \end{aligned} \tag{45}$$

This leads us to the requirement of solving the equation for the fourth-order multifrequency statistical moment $\Gamma_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,\sigma|\omega_1,\omega_2,\omega_3,\omega_4)$:

$$\begin{aligned} &\frac{\partial\Gamma_4}{\partial\sigma} - \frac{i}{2}\left(\frac{1}{k_1}\nabla_{r_1}^2 - \frac{1}{k_2}\nabla_{r_2}^2 + \frac{1}{k_3}\nabla_{r_3}^2 - \frac{1}{k_4}\nabla_{r_4}^2\right)\Gamma_4 + \frac{1}{8}H_4\Gamma_4 \\ &\quad= 0, \end{aligned} \tag{46}$$

with the scattering function

$$\begin{aligned} H_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) &= (k_1^2+k_2^2+k_3^2+k_4^2)A_n(0) \\ &\quad-2k_1k_2A_n(\mathbf{r}_1-\mathbf{r}_2)+2k_1k_3A_n(\mathbf{r}_1-\mathbf{r}_3) \\ &\quad-2k_1k_4A_n(\mathbf{r}_1-\mathbf{r}_4)-2k_2k_3A_n(\mathbf{r}_2-\mathbf{r}_3) \\ &\quad+2k_2k_4A_n(\mathbf{r}_2-\mathbf{r}_4)-2k_3k_4A_n(\mathbf{r}_3-\mathbf{r}_4). \end{aligned} \tag{46a}$$

Solving Eq. (46) is beyond the scope of the present paper and will be addressed in our future works. It can be noted that the reference-wave method as has been developed for solving the TFMCF equation can be directly applied also to Eq. (46).

V. SUMMARY

In this work, we have presented a reference-wave method and demonstrated its performance by solving the parabolic equation governing the propagation of the two-frequency mutual coherence function. According to the spirit of the method, we defined a nonsymmetric paired field function as a product of two components. The first component of this function is a solution of the equation governing the propagation of the mutual two-frequency coherence function in a randomly inhomogeneous medium, while the second is a solution of a complex-conjugate equation describing propagation in a medium in the absence of fluctuations. The difference between these equations is in the scattering term described by the function F_s in Eq. (17c). Because of the lack of symmetry in the product equation, we applied the asymmetric transform of the coordinate system in order to extract the possible difference in the phase and scattering information tracked along the characteristic trajectories. After the product equation is solved, the desired solution for the mutual two-frequency coherence function can be found presuming that the solution of the nonperturbed equation for the reference wave is known. Here we found, however, two difficulties. The first arises from the fact that the reference component is being tracked along perturbed characteristic trajectories, instead of the straight homogeneous background rays, as a result of the coupling of perturbed and unperturbed equations. The second is the possible loss of phase information while considering a product of two conjugate components.

In order to solve the first discrepancy, we note that the product measure in the absence of scattering represents mainly the amplitude term even when being propagated along the perturbed characteristic trajectories. Moreover, the perturbation term in the characteristic equation (27b) is negligibly small for most practical situations, because of the strength of the scattering of the medium and, in addition, because of the rapid decay of the medium's correlation function with an increase in the separation coordinate q. In this case the solution of the product measure decouples into two two-frequency coherence functions in the absence of scattering, and the expression for the reference wave can be obtained from the unperturbed equation (18). In the case of a small separation argument q in Eq. (27b), the correlation function can be approximated by a quadratic expansion term, which allows one to obtain an exact analytical solution of the coherence equation without any loss of phase information. For weak medium fluctuations, the "quadratic" approximation result also reduces to the expression for the reference wave in a free space. Therefore, for most of the important situations, the expression for the reference wave can be retrieved as a solution of an unperturbed two-frequency coherence equation.

Comparing the result of the reference-wave method with the results obtained in some recent works [24–26], we note that all of them present procedures for computing the statistical characteristics of time-dependent signals propagating in a random medium. It is hard to compare the final formulas, because different tasks have been addressed and different mathematical methods have been used. However, in our opinion application of the reference-wave method has several advantages. First of all, it allows presenting an analytical result, justified on the grounds of asymptotic expansion, for arbitrary spectra of the refractive index fluctuations. Second, although all the procedures mentioned above lead to the exact result in the case of the "quadratic" approximation of the refractive index structure function, only the expression in Eq. (32) solved along the dynamic characteristics (27) allows analysis of the frequency correlation along the propagation path for arbitrary fluctuations spectra.

As noted above, the reference-wave procedure can be directly extended to obtaining solutions for higher-order statistical moments.

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