Noise-induced enhancement of fluctuation and spurious synchronization in uncoupled type-I intermittent chaotic systems

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We study the dynamics of a pair of two *uncoupled* identical type-I intermittent chaotic systems driven by *common* random forcing. We first observe that the degree of the fluctuation of the local expansion rate of orbits to perturbations of a single system as a function of the noise intensity shows a convex curve and takes its maximum value at a certain noise intensity, whereas the Liapunov exponent itself monotonically increases in this range. Furthermore, it is numerically demonstrated that this nontrivial enhancement of fluctuation causes that two orbits with different initial conditions may synchronize each other after a finite interval of time. As pointed out by Pikovsky [Phys. Lett. A **165**, 33 (1992)], since the Liapunov exponent of the present system is positive, the synchronization that we observed is a numerical artifact due to the finite precision of calculations. The spurious noise-induced synchronization in an ensemble of uncoupled type-I intermittent chaotic systems are numerically characterized and the relations between these features and the fluctuation properties of the local expansion rate are also discussed.

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I. INTRODUCTION

It is well known that the sensitivity to the perturbations of the initial condition is one of the most generic features of chaotic dynamical systems. In chaotic dynamical systems, two orbits with slightly different initial conditions in the state space separate exponentially with time and become uncorrelated with each other. So, uncoupled identical chaotic systems never synchronize each other unless each system has an identical initial condition. Of course, in the presence of couplings among elements, the synchronization of elements can arise under some suitable conditions and the issues of synchronization in coupled chaotic systems have been attracting considerable attention of many researchers. However, several researchers recently reported counterintuitive examples that an ensemble of uncoupled identical chaotic systems driven by common external noise can also synchronize each other, that is, the distance between orbits of systems driven by the same noise collapse to a single noisy orbit with time evolution [1–6]. This noise-induced synchronization phenomenon is an illustrative example that the interplay between internal nonlinear deterministic evolution law and external random fluctuation can introduce more ''order'' in the dynamics.

The Liapunov exponents which quantitatively characterize the sensitive dependence on the initial condition in the deterministic case may also be suitably defined, and its sign gives a criterion whether the synchronization in an ensemble of uncoupled identical systems driven by common noise appears or not [2]. If the largest Liapunov exponent of the single system is negative, then two orbits with slightly different initial conditions will eventually synchronize each other [2]. The dependence of the largest Liapunov exponent on external noise in chaotic dynamical systems was first reported by Matsumoto and Tsuda on a discrete one-dimensional dynamical system which is associated with the Belousov-Zhabotinsky chemical reaction (the BZ map) [7]. They studied the effects of noise on the BZ map and observed that with a small amount of noise a chaotic orbit changes into a periodic one smeared with noise, which is indicated by the negativity of the largest Liapunov exponent. This ordering effect of noise in chaotic dynamical systems is called ''noise-induced order'' (NIO) [7]. Since the BZ map, which is a one-dimensional map, consists of very steep and flat regions and has a strong nonuniformity and the weakness to external perturbations in the dynamics, Matsumoto and Tsuda claimed that NIO is attributed to this strong nonuniformity of the BZ map, and thus NIO is not observed in the logistic map which has weak nonuniformity [8]. However, if the parameter of the logistic map is located near a periodic window which exists densely everywhere in the parameter space, then there is a possibility that some kind of ordering effects caused by noise appear even in the case of the logistic map. In our previous paper [9], we investigated the effect of noise on the logistic map and the Rössler oscillator near a periodic window exhibiting type-I intermittency and numerically observed that the degree of temporal regularity of the time series increases with the increase of the noise intensity and attains its maximum at a certain noise intensity. Such resonant phenomenon is called ''coherence resonance'' (CR) [10]. It was also shown in Ref. [9] that the Liapunov exponent as a function of noise intensity shows a concave curve and takes a minimum value at the same noise intensity which generates the maximal temporal regularity of the time series. In this sense, CR in type-I intermittency reported by us is also a kind of NIO.

The aim of the present paper is to characterize the effects

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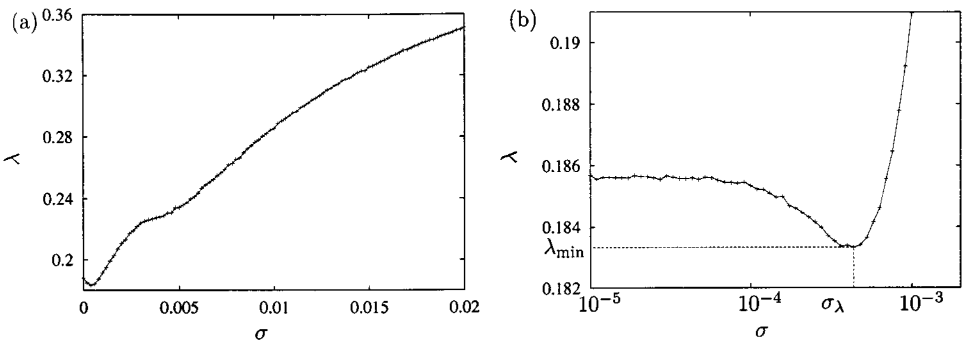

FIG. 1. Liapunov exponent λ for the noisy logistic map (1) as a function of σ; (b) shows an enlargement of (a) for small σ.

of external noise on chaotic dynamical systems which exhibit type-I intermittent chaos from the view point of synchronization. In particular, we focus on the dependence of both the average value and the degree of fluctuation of the local expansion rate, and consider how the correlation between orbits of a pair of uncoupled identical type-I intermittent chaotic systems driven by common random forcing depends on the noise. We numerically demonstrate that the degree of the fluctuation of the local expansion rate of orbits as a function of the noise intensity shows a convex curve and takes its maximum at a noise intensity which is different from that where the average value of the local expansion rate, i.e., the Liapunov exponent, achieves the minimum. Moreover, we show that the noise-induced synchronization also occurs in the range of the noise intensity that coincides with that where an anomalous enhancement of the fluctuation of the local expansion rate is observed. According to Pikovsky's criterion [2], since the Liapunov exponent is positive, this noise-induced synchronization is a *spurious* one due to a numerical round off in the calculations of orbits.

The present paper is organized as follows. In Sec. II, we introduce the logistic map driven by random noise and characteristics of the stability of the system to perturbation that play important roles in noise-induced synchronization. In Sec. III, we observe the time evolution of the distance between two orbits started with different initial conditions and driven by common random forcing, and observe that the synchronization of orbits appears for a certain range of the noise intensity. It is also numerically demonstrated that the average relaxation time needed for the achievement of synchronization grows exponentially with the numerical precision level, which implies the observed synchronization is spurious. Moreover, we investigate how the relaxation process depends on the noise intensity and discuss its relation with the fluctuation of the local expansion rate. A summary and concluding remarks are given in Sec. IV.

II. ENHANCEMENT OF FLUCTUATIONS OF THE LOCAL EXPANSION RATE FOR THE SINGLE NOISY LOGISTIC MAP

As an appropriate illustration, let us consider the following logistic map subjected to additive noise

$$x_{t+1}=1-\mu x_t^2+\sigma\xi_t, \tag{1}$$

where μ is the bifurcation parameter, ξ_t is an independent random variable uniformly distributed over an interval $[-0.5,0.5]$, and σ is the noise intensity. Throughout this paper, all numerical calculations were carried out in double precision. In the case of the noise-free logistic map ($\sigma=0$), the largest periodic window of period three appears at $\mu_c=1.75$ by a saddle-node bifurcation. At a value of μ slightly below μ_c, the time series of the logistic map consists of almost period three cycles intermittently interrupted by short term irregular bursts, i.e., type-I intermittency [11] is observed. In the following, we take $\epsilon\equiv\mu_c-\mu=10^{-4}$.

First, we introduce the Liapunov exponent which measures the *average* local expansion rate of orbits for general one-dimensional noisy maps. Although there are various definitions of the Liapunov exponents for noisy dynamical systems [2,12,13], here we use the following one

$$\lambda=\lim_{T\to\infty}(1/T)\sum_{t=0}^{T-1}\ln|F'(x_t)|, \tag{2}$$

where F' denotes the slope of the deterministic part of the one-dimensional noisy map. This definition is formally same as the case of the noise-free deterministic one-dimensional maps, but a sequence $\{x_t\}$ is now an orbit driven by external noise. The value calculated from Eq. (2) does not depend on the initial condition x_0 and a specific realization $\{\xi_t\}$ of noise

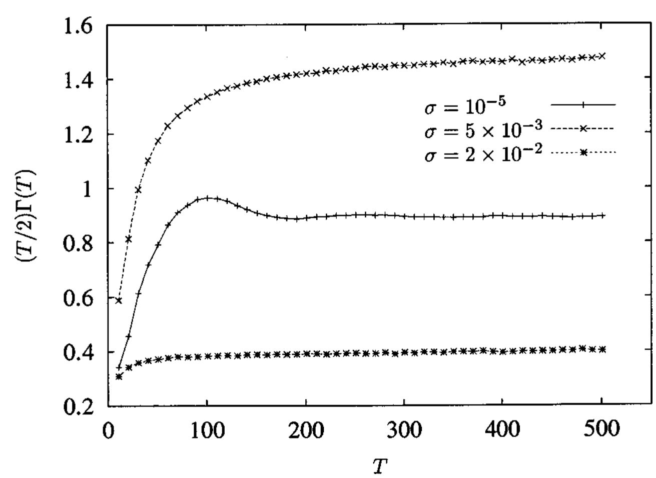

FIG. 2. $(T/2)\Gamma(T)$ vs the coarse graining time scale T for three different values of the noise intensity σ; $(T/2)\Gamma(T)$ for large T gives the diffusion constant D in Eq. (7).

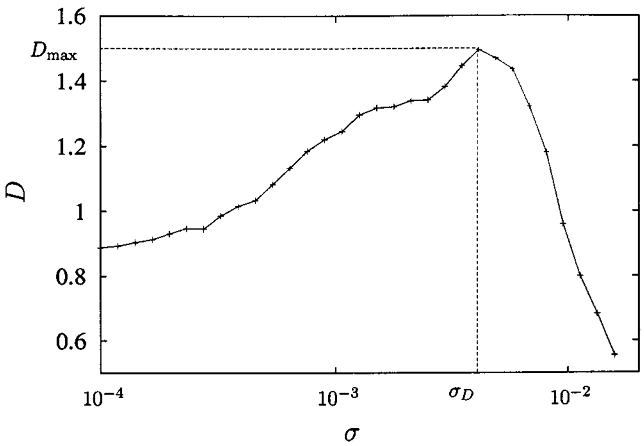

FIG. 3. The diffusion constatnt D as a funciton of the noise intensity σ.

process if the system is ergodic. In the case of noisy dynamical systems, the meaning of the sign of λ in Eq. (2) is not always clear as far as considering only the dynamics of each individual orbit. However, when we consider behavior of an *ensemble* of identical systems driven by common noise, λ in Eq. (2) plays a prominent role in synchronization problem as pointed out by Pikovsky [2]. Let us consider the following pair of one-dimensional maps subjected to the same noise:

$$x_{t+1}=F(x_t)+\xi_t,$$

$$y_{t+1}=F(y_t)+\xi_t. \tag{3}$$

It is easily found that $x_t=y_t\equiv u_t$ is an invariant subspace of Eqs. (3). In order to characterize the stability of the synchronized state $x=y$, by linearizing the dynamics with respect to the distance $v_t=|x_t-y_t|$ between two orbits at time t around $v=0$, we have

$$v_{t+1}=|F'(u_t)|v_t. \tag{4}$$

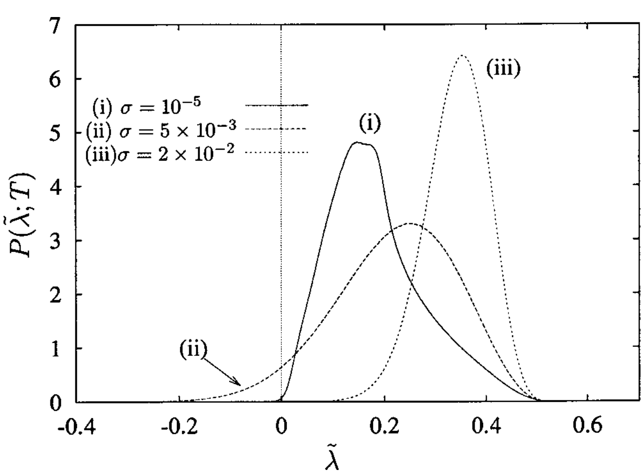

FIG. 4. The probability density function $P(\tilde{\lambda};T)$ of the finite-time Liapunov exponent $\tilde{\lambda}$ with $T=200$ for three different values of the noise intensity σ. A negative tail of $P(\tilde{\lambda};T)$ is observed at the intermediate noise intensity (ii).

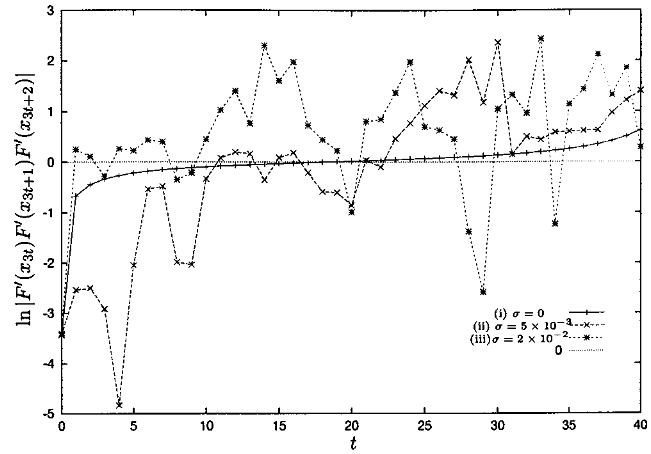

FIG. 5. Typical realizations of time series of the local expansion rate of the noisy logistic map with (i) $\sigma=0$, (ii) $\sigma=5\times10^{-3}$, and (iii) $\sigma=2\times10^{-2}$.

Thus, the deviation v_t from the synchronized state $v=0$ is exponentially expanded or contracted as $v_t=\exp(\lambda t)v_0$. Thus, the sign of λ in Eq. (2) determines whether synchronization between two *nearby* orbits driven by common noise occurs or not. Here it should be noted that since the sign of λ in Eq. (2) only gives the local stability of the synchronized state $v=0$, the negativity of the Liapunov exponent is a *necessary* condition for the appearance of synchronization in uncoupled dynamical systems driven by common noise. We also note that the Liapunov exponent for noisy dynamical systems defined in Eq. (2) measures the sensitivity to perturbations of the initial condition but not that to the realization of noise process. Paladin *et al.* proposed a measure which quantifies the rate of divergence of two nearby orbits evolving under two *different* noise realizations as the "complexity" of noisy dynamical systems and claimed its importance in physics literature [13].

Now let us investigate the dependence of the Liapunov exponent λ of the noisy logistic map (1) on the noise intensity σ. Figures 1 show the Liapunov exponent λ as a function of the noise intensity σ. At each noise intensity σ, λ is numerically calculated by averaging over a time series with length 10^9. One finds from Fig. 1 (a) that λ increases monotonically for relatively large σ [12]. However, for small σ, $\lambda(\sigma)$ shows a concave curve which has its minimum value $\lambda_{\rm min}$ at σ_λ as shown in Fig. 1(b). This decrease of λ with the increase of σ implies that the portion of time spent in the contracting region $\{x\colon\ |F'(x)|<1\}$ of an orbit increases as σ is increased. In our previous study [9], we observed that a coherence measure β which characterizes the temporal regularity of orbits of the noisy logistic map (1) shows a resonant phenomenon against the noise intensity σ, which is called the coherence resonance, in the same range of the noise intensity as that where $\lambda(\sigma)$ shows a concave structure in Fig. 1(b).

It would not be sufficient to characterize the whole nature of the stability of the system only by the average value λ. In particular, refined measures characterizing the *fluctuation* of the local expansion rate of orbits are needed for systems which exhibit strong non-Gaussian temporal evolution such

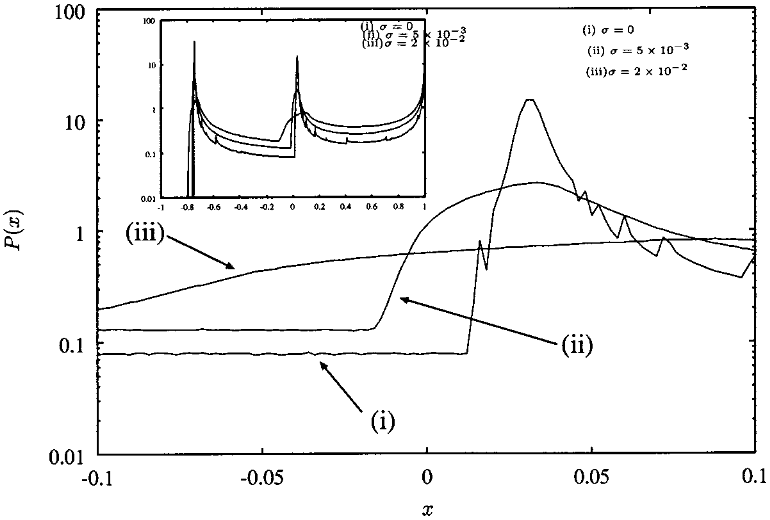

FIG. 6. The probability density functions of the noisy logistic map around $x=0$ with (i) $\sigma=0$, (ii) $\sigma=5\times10^{-3}$, and (iii) $\sigma=2\times10^{-2}$. Inset figure is those of the full interval $[-1,1]$

as intermittent fluctuation. The *thermodynamic formalism* [14] that is based on the large deviation theory [15] in mathematics literature is very useful to capture the characteristics of scaling properties exhibited in the local singular structures of strange attractors or fluctuations of the local expansion rate of chaotic orbits and has been developed in the context of the theory of deterministic chaotic dynamical systems [16]. It would be natural to introduce formally the notions of the thermodynamic formalism also for noisy dynamical systems. As mentioned above, although the value of λ in Eq. (2) does not depend on the choice of an initial condition and a specific realization of noise process, the *finite-time* Liapunov exponent

$$\tilde{\lambda}_T = (1/T)\sum_{t=0}^{T-1} \ln|F'(x_t)|, \tag{5}$$

which is a measure of exponential expansion or contraction rate averaged over finite T steps, may take various values depending on them. Let us divide a time series with length MT into M segments of equal length T and denote $\tilde{\lambda}_m(T) = (1/T)\Sigma_{t=0}^{T-1}\ln|F'(x_{mT+t})|$. Although it is possible to observe any higher moment of the finite-time Liapunov exponent, it would be natural to observe the variance

$$\Gamma(T) = \lim_{M\to\infty} (1/M)\sum_{m=0}^{M-1} [\tilde{\lambda}_m(T) - \lambda]^2, \tag{6}$$

as a simple characteristic of the fluctuation of the local expansion rate of orbits over T steps. By the law of large numbers, $\Gamma(T)$ converges to zero in the limit $T\to\infty$ and it is naturally expected that the "diffusion constant"

$$D \equiv \lim_{T\to\infty} (T/2)\Gamma(T) \tag{7}$$

exists, which gives an effective asymptotic feature of the fluctuation of the local expansion rate of two nearby orbits as the diffusion process [17].

Figure 2 shows the dependence of $(T/2)\Gamma(T)$ on T of the noisy logistic map (1) for three different values of σ. Here, we have taken $M=10^6$ for the numerical calculation of Eq. (6) for each T. It is clearly found from Fig. 2 that $(T/2)\Gamma(T)$ converges to a constant with the increase of T, which confirms the assumption of the existence of the asymptotic value D in Eq. (7). Furthermore, it is also found that the limiting value of $(T/2)\Gamma(T)$ is maximized at an intermediate noise intensity $\sigma=5\times10^{-3}$. The dependence of the diffusion constant D on the noise intensity σ is shown in Fig. 3, where D is estimated as $D=(1000/2)\Gamma(1000)$ with $M=10^6$ for each noise intensity σ. It is clearly found from Fig. 3 that external noise with a suitable intensity enhances the fluctuation of local expansion rate of orbits more than that of the noise-free case and the diffusion constant D is a convex function of the noise intensity σ taking its maximum value $D_{\max}$ at σ_D. A large value of D suggesting the "nonuniformity" [7,8,17] of the dynamics implies the possibility of emergence of a noise-induced order. It should be noted that the value of σ_D is almost ten times larger than that of σ_λ in Fig. 1(b), which implies that the possible ordering effect caused by the enhancement of fluctuation of the local expansion rate of orbits

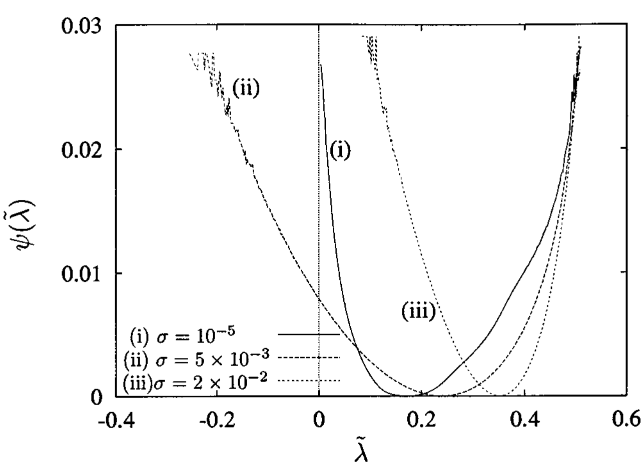

FIG. 7. The fluctuation spectra $\psi(\tilde{\lambda})$ of the local expansion rate of the noisy logistic map for three different values of the noise intensity σ. The fluctuation spectrum $\psi(\tilde{\lambda})$ for $\tilde{\lambda}\leq 0$ plays an essential role in the noise-induced synchronization.

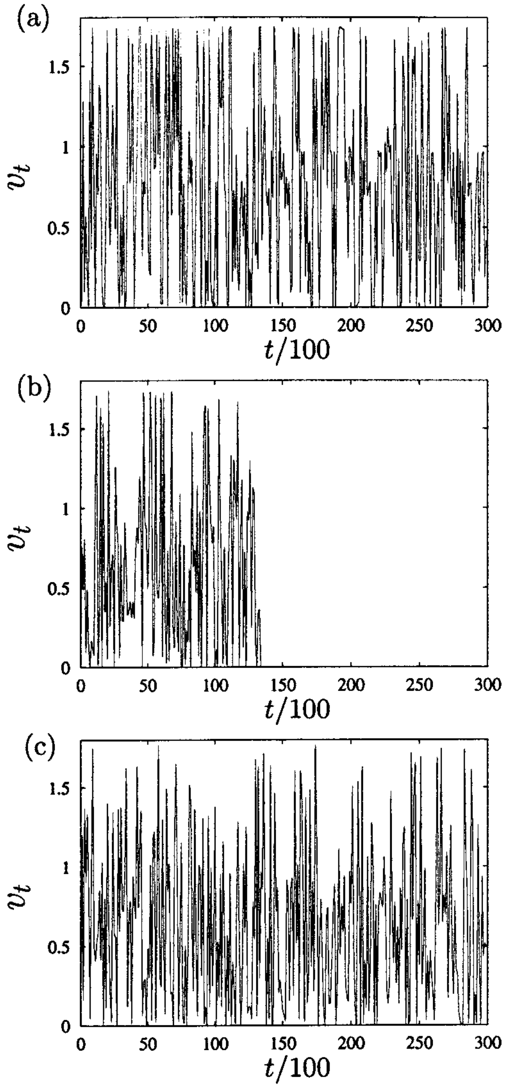

FIG. 8. Typical time series of the distance $v_t = |x_t(1) - x_t(2)|$ between two orbits of two uncoupled noisy logistic maps (11) with (a) $\sigma = 10^{-5}$, (b) $\sigma = 5\times10^{-3}$, and (c) $\sigma = 2\times10^{-2}$. The noise-induced synchronization is observed in (b).

is different from that in the case of CR in type-I intermittency [9].

Because the diffusion constant D only gives an information of the "small" fluctuation around the average value λ, it would be needed to introduce a measure in order to characterize the "large" fluctuation that is a deviation far from λ. The large deviation principle which is an asymptotic theory in probability theory is concerned with this problem. The large deviation property is characterized by

$$P(\tilde{\lambda};T) \sim \exp[-T\psi(\tilde{\lambda})], \tag{8}$$

where $P(\tilde{\lambda};T)$ denotes the probability density function of the finite-time Liapunov exponent defined as

$$P(\tilde{\lambda};T) \equiv \lim_{M\to\infty} (1/M) \sum_{m=0}^{M-1} \delta(\tilde{\lambda} - \tilde{\lambda}_m(T)), \tag{9}$$

and the scaling function $\psi(\tilde{\lambda})$ is called the *fluctuation spectrum* [16]. Note that $\psi(\tilde{\lambda})$ is a concave function taking its minimum value zero at $\tilde{\lambda} = \lambda$ and can be approximated by a quadratic function around λ as

$$\psi(\tilde{\lambda}) = (1/2D)(\tilde{\lambda} - \lambda)^2, \tag{10}$$

if the central limit theorem holds.

The distribution function $P(\tilde{\lambda};T)$ of Eq. (1) for $T = 200$ and several different values of σ are shown in Fig. 4. For each σ, $M = 10^8$ values of $\tilde{\lambda}_T$ with $T = 200$ are taken for the histogram approximation of $P(\tilde{\lambda};T)$ with 10^3 bins of $\tilde{\lambda} \in [-0.4,0.6]$. $T = 200$ is large enough such that the average value λ is located at the peak of $P(\tilde{\lambda};T)$ and we can see from Fig. 4 that the location of the peak shifts right monotonically as σ is increased as shown in Fig. 1(a). A pronounced tail of the probability distribution taking negative values of $\tilde{\lambda}$ appears at an intermediate noise intensity $\sigma = 5\times10^{-3}$. This negative tail means that the difference between two nearby orbits decreases by a factor of $e^{200\tilde{\lambda}} < 1$ in 200 successive iterations. Let us mention briefly why the fluctuations of local expansion rate of orbits in type-I intermittency can be enhanced by a certain amount of external noise. Figure 5 shows typical realizations of time series of the local expansion rate $\ln|F'(x_{3t})F'(x_{3t+1})F'(x_{3t+2})|$ of an orbit which starts with $x_0 = 0.001$ that is located near the position of the maximum of the logistic map $x = 0$, where $\ln|F'(0)| = -\infty$, for three different noise intensities. Note that in order to see the dynamics around the channel clearly, $\ln|F'(x_{3t})F'(x_{3t+1})F'(x_{3t+2})|$ is considered instead of $\ln|F'(x_t)|$ for the present channel of period three. When the dynamics of the logistic map exhibits type-I intermittency, one of the channels of the map corresponding to the laminar motion starts around the maximum of the map, so if noise is absent or small enough, the orbit starting with an initial condition near $x = 0$ spends long time at the channel and therefore the value of the finite-time Liapunov exponent $\tilde{\lambda}$ that is averaged over this time interval becomes negative as shown by the curves (i) and (ii) in Fig. 5. Figure 6 shows the stationary probability density $P(x)$ for three different noise intensities. When noise is absent, a typical orbit is seldom to visit a neighborhood of $x = 0$ as shown by the curve (i) in Fig. 6, so the probability that $\tilde{\lambda}$ takes a negative value is very small. If noise is introduced, the dynamics of orbits which pass near the singular point $x = 0$ changes depending on its intensities. In the case where the noise intensity is sufficiently small, the influence of noise can be neglected and the result is similar to the noiseless case. On the other hand, in the case of sufficiently large noise intensity, although the probability density $P(x)$ spreads and the probability takes values around $x = 0$ becomes large in comparison with the noiseless case, the large noise destroys the coherence of laminar motion as shown by the curve (iii) in Fig. 5, which implies that the orbit does not have negative finite-time Liapunov exponent. However, for the intermediate intensity of noise, the probability density $P(x)$ is modified to have a large value around $x = 0$ by the introduction of noise, and the laminar motion is still undestroyed. Thus the orbits starting around $x=0$ that have negative finite-time Liapunov exponents appear with a certain amount of probability. In summary, the probability taking negative values of $\tilde{\lambda}$ increases for an intermediate noise intensity and therefore the nonmonotonic change of the fluctuation of the local expansion rate against the noise intensity can be observed as shown in Fig. 4.

The fluctuation spectrum $\psi(\tilde{\lambda})$ represents more detailed information on the asymptotic fluctuation of the local expansion rate of orbits than D. Figure 7 shows the numerically obtained fluctuation spectra $\psi(\tilde{\lambda})$ corresponding to $P(\tilde{\lambda};T)$ in Fig. 4 for three different values of σ. Here, each $\psi(\tilde{\lambda})$ in Fig. 7 is obtained from $P(\tilde{\lambda};T)$ with $T=500$ and $M=10^8$ as $\psi(\tilde{\lambda})=-(1/T)\ln[P(\tilde{\lambda};T)/P_{\max}]$, where $P_{\max}=\max_{\tilde{\lambda}} P(\tilde{\lambda};T)$. In the case of a small noise intensity $\sigma=10^{-5}$ [Fig. 7(i)], one finds that although the curve $\psi(\tilde{\lambda})$ can be approximated by a quadratic function (10) around λ, there exists a linearlike slope on the right-hand side of $\psi(\tilde{\lambda})$, which quantitatively characterize the transition between the laminar motions and the turbulent bursts of type-I intermittency [18]. On the other hand, in the cases of $\sigma=5\times10^{-3}$ and 2×10^{-2} [Fig. 7(ii) and 7(iii)], noise intensities are large enough so that the singularity as shown in Fig. 7(i) is not observed in $\psi(\tilde{\lambda})$. However, at an intermediate noise intensity $\sigma=5\times10^{-3}$, $\psi(\tilde{\lambda})$ spreads towards the negative values of $\tilde{\lambda}$, which yields nontrivial noise-induced ordering effects in an ensemble of uncoupled identical type-I intermittent elements as it will be discussed in the following section.

III. NOISE-INDUCED SPURIOUS SYNCHRONIZATION IN UNCOUPLED LOGISTIC MAPS AND ITS STATISTICAL PROPERTIES

In this section, we consider the following pair of two uncoupled identical logistic maps (1) driven by common random forcing

$$x_{t+1}(j)=1-\mu x_t^2(j)+\sigma\xi_t, \quad j=1,2. \tag{11}$$

Here, we take $\mu=\mu_c-10^{-4}$ so that each orbit of Eqs. (11) exhibits type-I intermittency. In the noise-free case, almost all pairs of two orbits of Eqs. (11) never synchronize each other because of the positive Liapunov exponent. However, there is a possibility that the sensitivity on initial conditions of chaotic systems is suppressed by an introduction of noise and this suppression may yield nontrivial correlation between two orbits with different initial conditions. Thus, we begin by observing temporal evolution of the distance $v_t=|x_t(1)-x_t(2)|$ between two orbits of Eqs. (11) and its dependence on the noise intensity σ. Figures 8 show typical realizations of v_t of Eqs. (11) for three different values of σ. Here, v_t is plotted for every 100th step of t. One can find from Figs. 8 that the variable v_t continues to fluctuate with time for both small and large values of σ [Figs. 8(a) and 8(c)], while, for an intermediate noise intensity [Fig. 8(b)], the variable v_t suddenly collapses to zero after a certain step of time indicating the noise-induced synchronization between two orbits of Eqs. (11). The detailed plot of v_t for $\sigma=5\times10^{-3}$ around the onset time of the synchronization [$t/100\sim130$ in Fig. 8(b)] is shown in Figs. 9. One can see from Fig. 9(a) that the variable v_t takes small values intermittently and finally approaches to zero.

Since the Liapunov exponent of the noisy logistic map is always positive as shown in Fig. 1, the synchronization which we observed in Fig. 8(b) and Figs. 9 is an outcome due to the finite precision in numerical calculations as discussed in Refs. [4–6]. Since all numerical calculations in our present study are carried out with double precision, if the difference v becomes less than the accuracy 10^{-16} at a certain time $t=t_*$, then we have $v_t\equiv0$ for all $t>t_*$ and this may yield a misleading conclusion that complete synchronization is achieved. The dynamics of the variable $\ln v_t$ shows an anomalous diffusive motion with an absorbing wall at $\ln10^{-16}\sim-36.84$ as shown in Fig. 9(b).

In the literature of noise-induced synchronization, the paper by Maritan and Banavar (MB) [3] has aroused a lot of controversy. They studied the logistic map under the influence of noise: $x_{t+1}=4x_t(1-x_t)+\xi_t$, where ξ_t denotes a random number uniformly distributed in an interval $[-W,W]$. They claimed that if the value of W is large enough, then two orbits which started with different initial conditions driven by a common random sequence $\{\xi_t\}$ eventually converge to the same orbit [3]. However, this claim was heavily criticized by Pikovsky [4], and he pointed out that the Liapinov exponent of the model of MB is always positive and thus led a conclusion that the noise-induced synchronization that MB discovered was a numerical artifact

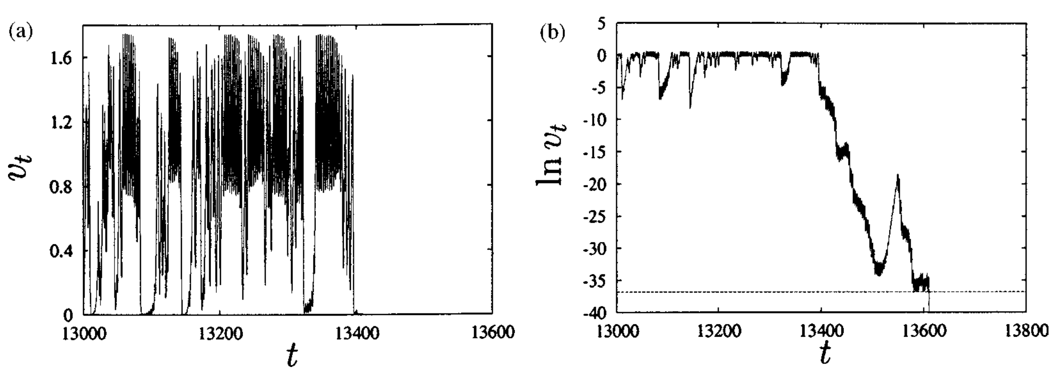

FIG. 9. A detailed plot of Fig. 8 (b) around the onset of the noise-induced synchronization. (b) shows $\ln v_t$ instead of v_t in (a).

due to lack of precision of calculations. Following Pikovsky's suggestion, Longa *et al.* [5] performed a systematic analysis and obtained quantitative features of relaxation time until which two orbits become numerically identical, confirming the suggestion of Pikovsky. In Ref. [3], a random number ξ_t at time t is adjusted such that the variable x_t is confined within a unit interval [0,1], and the resulting sequence $\{\xi_t\}$ becomes state dependent and effectively biased noise which does not have zero mean. Herzel and Freund [6] showed that the introduction of biased noise with nonzero mean gives a possibility that the finite-time Liapunov exponent which is averaged over a time series with finite length can become negative, which leads to spurious synchronization. Although we also investigate the noisy logistic map in the present paper, there are several points of difference from the study of MB: First, we take the parameter value of the logistic map near a periodic window such that the dynamics exhibits type-I intermittency, whereas MB take the parameter value at which the dynamics exhibits the fully developed chaos in the noise-free case. Second, in the study of MB random numbers used as external noise are state dependent whereas in the case of our present study state independent random numbers are used, and noise-induced synchronization in the model of MB is achieved with the considerably large noise intensity compared with our model. Third, the noise-induced enhancement of fluctuation of the local expansion rate plays an essential role in the occurrence of synchronization in our model.

In order to detect how the degree of synchronization in Eqs. (11) depends on the noise intensity, we evaluate the synchronization ratio $\rho = N/M$ for a long time interval $T_{\max}$ of observation, where N is the number of pairs of initial conditions of Eqs. (11) with which the numerical synchronized state is reached within $T_{\max}$ iterations and M is the total number of trials of numerical simulations. We take $M = 10^3$ and $T_{\max} = 10^6$ and, in order to control the precision level of numerical synchronization explicitly, we provide a threshold $\Delta = 10^{-L}$ and the onset time t_* of the synchronization is determined such that the distance v_t between two orbits gets smaller than Δ for the first time at t_*. That is, the time t_* is regarded as the *first passage time* needed for the diffusive motion of the variable $\ln v_t$ to pass through $\ln \Delta = -L \ln 10$. Figure 10 shows the ratio ρ as a function of the noise intensity σ for three different threshold values. The spurious synchronization with positive λ is observed in the region $\sigma = 10^{-3} \sim 10^{-2}$, which is nearly same as that where the fluctuation of the local expansion rate is amplified (Fig. 3).

Furthermore, we evaluate the average relaxation time τ needed for the spurious synchronization to occur and investigate how this relaxation time depends on the noise intensity σ and the numerical precision level L. Figure 11 shows the average relaxation time τ of Eqs. (11) as a function of the numerical precision level L for three different values of σ. For each level L, the value of τ is numerically determined by averaging over the first passage time t_* of 10^4 randomly chosen pairs of initial conditions of Eqs. (11). It is found from Fig. 11 that the average relaxation time τ increases exponentially with the increase of the level L as

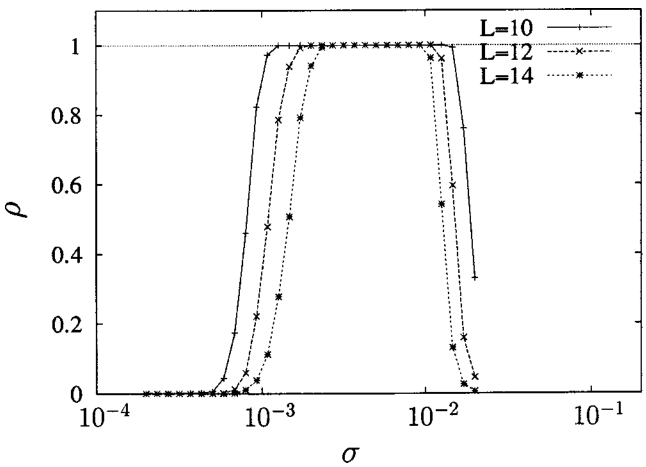

FIG. 10. Synchronization ratio ρ as a function of the noise intensity σ for three different values of the numerical precision level L.

$$\tau \sim \exp(\alpha L), \tag{12}$$

where the exponent α can be considered to be a prominent characteristic of the noise-induced spurious synchronization of Eqs. (11). One can also see from Fig. 11 that the value of α depends on σ. For an intermediate noise intensity $\sigma = 5 \times 10^{-3}$ [Fig. 11(ii)], the exponent is smaller than that for both large and small noise intensities (Figs. 11(i) and 11(iii)), which implies that there is a certain suitable range of σ where the noise-induced synchronization is apt to occur. The exponent α estimated by a linear least square fitting in Fig. 11 for each value of σ is shown in Fig. 12. The function α of σ, as shown in Fig. 12, clearly plots a concave curve taking its minimum at $\sigma \sim 5 \times 10^{-3}$.

Now, let us investigate the noise-induced synchronization in Eqs. (11) in connection with the fluctuation property of the local expansion rate of orbits of Eq. (1). The occurrence of the noise-induced synchronization in Eqs. (11) has its origin in the increase of the probability that the finite-time Li-

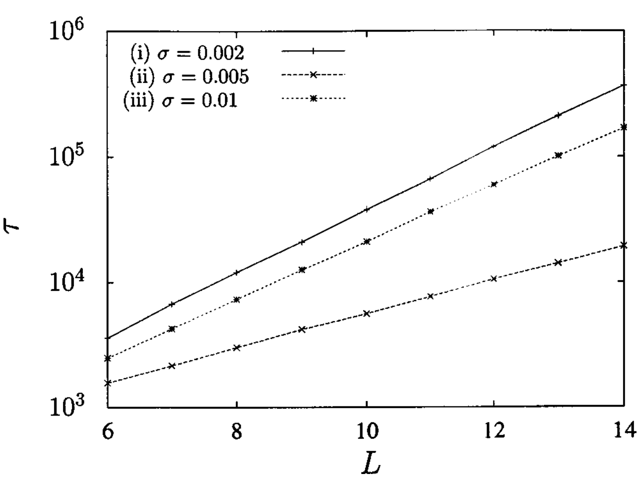

FIG. 11. Average relaxation time τ for the onset of the noise-induced synchronization as a function of the numerical precision level L for three different values of the noise intensity σ.

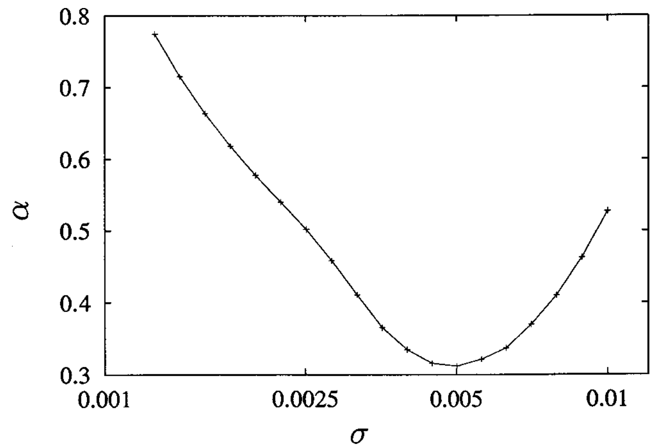

FIG. 12. Characteristic exponent α in Eq. (12) of the noise-induced synchronization as a function of the noise intensity σ.

apunov exponent $\tilde{\lambda}$ takes negative values with a certain range of noise intensity as shown in Fig. 4, and this fact enables us to conclude that the characteristic exponent α in Eq. (12) can be quantitatively expressed by the fluctuation spectrum $\psi(\tilde{\lambda})$ as follows [19]. Let us assume that the initial value $v_0=|x_0(1)-x_0(2)|$ is sufficiently small but larger than the threshold Δ and v_t satisfies

$$v_t<av_0, \quad 1\leq t\leq t_*, \tag{13}$$

with a constant a of order one. Then, v_t is approximated by the linearization

$$v_t=\exp\left(\sum_{n=0}^{t-1}\ln|F'(u_n)|\right)v_0=e^{t\tilde{\lambda}}v_0 \tag{14}$$

for $1\leq t\leq t_*$, where $u_n=[x_n(1)+x_n(2)]/2$. Let P_{t_*} be the probability of appearance of such a path satisfying

$$v_{t_*}<\Delta \text{ and } v_{t_*-1},v_{t_*-2},\dots,v_1>\Delta. \tag{15}$$

If Δ is sufficiently small and t_* is sufficiently large, then the probability that the condition $v_{t_*-1},v_{t_*-1},\dots,v_1<av_0$ is not satisfied under the condition that $v_{t_*}<\Delta$ is considered to not exponentially depend on t_*. Thus, P_{t_*} is approximated as

$$P_{t_*}\sim \mathrm{Prob}\{v_{t_*}<\Delta\}=\mathrm{Prob}\{\tilde{\lambda}<t_*^{-1}\ln(\Delta/v_0)\}$$
$$=\int_{\lambda_{\min}}^{t_*^{-1}\ln(\Delta/v_0)}P(\tilde{\lambda};t_*)d\tilde{\lambda}, \tag{16}$$

where $\lambda_{\min}$ denotes the minimum value of $\tilde{\lambda}$ and $t_*>\lambda_{\min}^{-1}\ln(\Delta/v_0)$ is assumed, otherwise $P_{t_*}=0$. Since $\psi(\tilde{\lambda})$ is a concave function with a minimum at $\lambda>0$, under the condition that Δ/v_0 is sufficiently small and t_* is large enough, the integral in Eq. (16) is evaluated as

$$\int_{\lambda_{\min}}^{t_*^{-1}\ln(\Delta/v_0)}P(\tilde{\lambda};t_*)d\tilde{\lambda}\sim\exp[-\psi(t_*^{-1}\ln(\Delta/v_0))t_*]. \tag{17}$$

That is,

$$P_{t_*}\sim\exp[-\psi(t_*^{-1}\ln(\Delta/v_0))t_*]. \tag{18}$$

As a function of t_*, for small Δ/v_0, P_{t_*} takes a sharp maximum at a value $\tilde{t}$ of t_*, where

$$\psi(t_*^{-1}\ln(\Delta/v_0))t_* \tag{19}$$

achieves its minimum. For an ensemble of orbits with fixed v_0, a $P_{\tilde{t}}\sim\Sigma_{t_*}{}^{\infty}P_{t_*}$ portion of orbits satisfying $v_t<av_0$ experience $v_t<\Delta$ for the first time at $t\simeq\tilde{t}$, while most of the rest portion of orbits experience $v_t>av_0$ at some time $t<\tilde{t}$. For the latter portion of the ensemble of orbits, by the nonlinearity of the dynamics, v_t gets as small as v_0 again after some steps of time, where v_0 is a characteristic scale of the system under which the nonlinearity of the system can be neglected. Thus, the average relaxation time τ is estimated as

$$\tau\sim\sum_{k=1}^{\infty}k\tilde{t}(P_{\tilde{t}})^k\sim\tilde{t}/P_{\tilde{t}}$$
$$\sim\exp[\min_{t_*>\ln(\Delta/v_0)/\lambda_{\min}}\psi(t_*^{-1}\ln(\Delta/v_0))t_*]$$
$$=\exp[-\log(\Delta/v_0)\min_{\lambda_{\min}<\tilde{\lambda}<0}\psi(\tilde{\lambda})/|\tilde{\lambda}|]$$
$$\sim\Delta^{-\nu}, \tag{20}$$

where

$$\nu=\min_{\lambda_{\min}<\tilde{\lambda}<0}\frac{\psi(\tilde{\lambda})}{|\tilde{\lambda}|}. \tag{21}$$

Therefore, we obtain the following relation between ν and α as

$$\nu=\frac{\alpha}{\ln 10}. \tag{22}$$

The dependence of the exponent ν on the noise intensity σ is shown in Fig. 13. Here, for each σ, the value of ν is estimated by evaluating the fluctuation spectrum $\psi(\tilde{\lambda})$ from the histogram approximation of $P(\tilde{\lambda};T)$ with $T=100$ (plus) and $T=200$ (cross). In Fig. 13, $\alpha/\ln 10$ as a function of σ is also plotted. One can find from Fig. 13 that there is a quite good agreement between $\nu(\sigma)$ estimated with $T=100$ and $\alpha(\sigma)$ in the right-hand side of their minimum values located at $\sigma\sim0.005$, whereas the deviation between two curves of $\nu(\sigma)$ and $\alpha(\sigma)$ is considerably large for $\sigma<0.005$. As shown in Fig. 7, the fluctuation spectrum $\psi(\tilde{\lambda})$ forms a non-Gaussian one and a large coarse graining time scale T is needed for a good approximation of $\psi(\tilde{\lambda})$ by

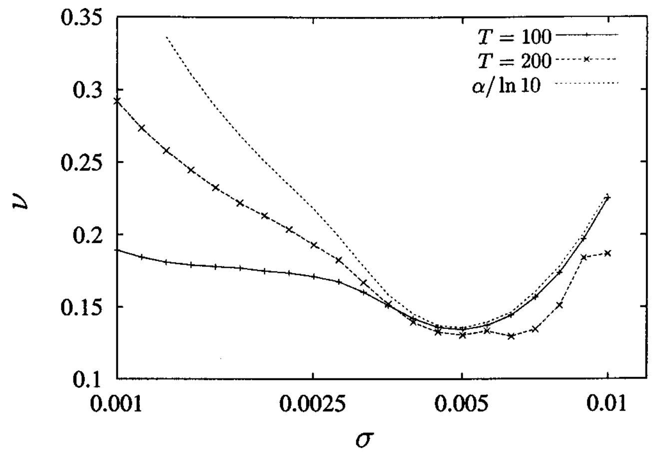

FIG. 13. Characteristic exponent ν in Eq. (21) as a funciton of the noise intensity σ. Each point of ν is estimated by evaluating the fluctuation spectrum $\psi(\tilde{\lambda})$ from the probability density function $P(\tilde{\lambda};T)$ with $T=100(+)$ and $T=200(\times)$. The curve $\alpha/\ln 10$ vs σ is also shown.

$(-1/T)\ln[P(\tilde{\lambda};T)/P_{\max}]$ in Eq. (8). Thus, the coarse graining time scales $T=100$ and 200 are not large enough and the fact that the curve $\nu(\sigma)$ estimated with $T=200$ is a better estimate of $\alpha(\sigma)$ than that with $T=100$ for $\sigma<0.005$ can be explained by the difference of the degree of convergence of $(-1/T)\ln[P(\tilde{\lambda};T)/P_{\max}]$ to $\psi(\tilde{\lambda})$. On the contrary, the shape of $\psi(\tilde{\lambda})$ approaches to a quadratic function with the increase of σ as shown in Fig. 7 and the convergence of $(-1/T)\ln[P(\tilde{\lambda};T)/P_{\max}]$ to $\psi(\tilde{\lambda})$ becomes fast. However, as also shown in Fig. 7, the fluctuation spectrum $\psi(\tilde{\lambda})$ shifts right as σ increases and the location of $\tilde{\lambda}$ which gives the minimum value of $\psi(\tilde{\lambda})/|\tilde{\lambda}|$ for $\tilde{\lambda}<0$ is far deviated from the average value λ. For a large coarse graining time scale T, the realization of $\tilde{\lambda}(T)$ with a value largely deviated from the average value λ becomes a very rare event and in numerical calculations a lot of trials are needed in order to estimate the value of $\psi(\tilde{\lambda}(T))$. So, the results of $\nu(\sigma)$ obtained with $T=200$ is worse than that obtained with $T=100$ for large σ, which explains the difference between $\nu(\sigma)$ obtained with $T=200$ and $\alpha(\sigma)$ for $\sigma>0.005$ in Fig. 7.

IV. SUMMARY AND CONCLUDING REMARKS

In summary, we have numerically demonstrated that in a chaotic system which exhibits type-I intermittency, external noise with an appropriate intensity can affect the fluctuation of the local expansion rate of orbits significantly. Enhancement of the fluctuation has been confirmed as the change of asymptotic quantities such as the diffusion constant D and more generally by the fluctuation spectrum $\psi(\tilde{\lambda})$. A negative tail of the probability distribution of the finite-time Liapunov exponent is generated as a result of this enhancement of the fluctuation, which yields considerable correlation and the spurious synchronization between two orbits of a pair of uncoupled type-I intermittent chaotic systems. Although the issue of the effect of external noise on chaotic systems which exhibit type-I intermittency has been investigated by several researchers in the past decades [20], to our knowledge, our findings in the present paper have not been reported yet.

In Ref. [9], we observed that the phenomenon of CR can naturally appear in type-I intermittency. The region of the noise intensity where the spurious synchronization occurs is separated from that where CR occurs, thus we have concluded that our findings in this paper are practically different kinds of noise-induced ordering effects in type-I intermittency. Moreover, we have introduced an exponent α which is associated with the average relaxation time of the spurious synchronization and investigated its dependence on the noise intensity. We have also discussed the relation between the exponent α and the fluctuation spectrum $\psi(\tilde{\lambda})$ and confirmed its validity by a numerical simulation.

The nontrivial effect of noise as mentioned above is due to the fact that the location of the singular point such as the maximum of the map is close to one of the channels that appear slightly before the saddle-node bifurcation. Such a structure is observed ubiquitously in dynamical systems that exhibit type-I intermittency, so it is naturally expected that our finding in the present study is universally observed in other dynamical systems that exhibit type-I intermittency.

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