Symplectic evolution of Wigner functions in Markovian open systems

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(Received 11 April 2003; revised manuscript received 22 August 2003; published 23 January 2004)

The Wigner function is known to evolve classically under the exclusive action of a quadratic Hamiltonian. If the system also interacts with the environment through Lindblad operators that are complex linear functions of position and momentum, then the general evolution is the convolution of a non-Hamiltonian classical propagation of the Wigner function with a phase space Gaussian that broadens in time. We analyze the consequences of this in the three generic cases of elliptic, hyperbolic, and parabolic Hamiltonians. The Wigner function always becomes positive in a definite time, which does not depend on the initial pure state. We observe the influence of classical dynamics and dissipation upon this threshold. We also derive an exact formula for the evolving linear entropy as the average of a narrowing Gaussian taken over a probability distribution that depends only on the initial state. This leads to a long time asymptotic formula for the growth of linear entropy. We finally discuss the possibility of recovering the initial state.

DOI: 10.1103/PhysRevE.69.016204 PACS number(s): 05.45.−a, 03.65.Yz

I. INTRODUCTION

The correspondence between classical and quantum mechanics of closed dynamical systems is most perfect for quadratic Hamiltonians. In this case, the classical evolution is linear, like its quantum counterpart, and generates an orbit within the group of symplectic (linear, canonical) transformations in phase space [1]. These are directly linked to the corresponding quantum metaplectic group [2]. Indeed, the evolution operator in any of the usual representations is merely the complex exponential of the classical generating function [3]. Of course, quadratic Hamiltonians are a very special case, but they include the ubiquitous harmonic oscillator, the parabolic potential barrier, and the free particle, which form adequate starting points for the analysis of more complex motion.

The Weyl representation of an arbitrary quantum operator $\hat{A}$ is

$$A(\mathbf{x})\equiv\int dq'\left\langle q+\frac{q'}{2}|\hat{A}|q-\frac{q'}{2}\right\rangle \exp\left(-i\frac{pq'}{\hbar}\right), \qquad (1)$$

that is, $\hat{A}$ is represented in phase space, $\mathbf{x}=(p,q)$, by the Weyl symbol $A(\mathbf{x})$. The Wigner function $W(\mathbf{x})$ is then the Weyl symbol for $\hat{\rho}/2\pi\hbar$, where $\hat{\rho}$ is the density operator. Just as all Weyl symbols, the Wigner function propagates classically under the action of a quadratic Hamiltonian [2,3]:

$$\frac{\partial}{\partial t}W_t(\mathbf{x})=\{H(\mathbf{x}),W_t(\mathbf{x})\}, \qquad (2)$$

introducing the classical Poisson bracket on the right-hand side [1] and $H(\mathbf{x})=\mathbf{x}\cdot\mathbf{H}\mathbf{x}$, the Weyl symbol of the quadratic Hamiltonian. The symbol $\cdot$ stands for the inner scalar product. Hence one has $W_t(\mathbf{x})=W_0(\mathbf{R}_{-t}\mathbf{x})$, where

$$\mathbf{R}_t=\exp(2\mathbf{J}\mathbf{H}t) \qquad (3)$$

is the 2×2 matrix giving the classical Hamiltonian time evolution of a phase space point $\mathbf{x}$. Actually, this propagation is also shared by the Fourier transform of $W_t(\mathbf{x})$,

$$\tilde{A}(\boldsymbol{\xi})=\frac{1}{2\pi\hbar}\int d\mathbf{x}\,\exp\left(-\frac{i}{\hbar}\boldsymbol{\xi}\wedge\mathbf{x}\right)A(\mathbf{x}), \qquad (4)$$

i.e., it is also true that

$$\frac{\partial}{\partial t}\tilde{W}_t(\boldsymbol{\xi})=\{H(\boldsymbol{\xi}),\tilde{W}_t(\boldsymbol{\xi})\}. \qquad (5)$$

Note that $H(\boldsymbol{\xi})$ must be read literally as the classical Hamiltonian $H(\mathbf{x})$ taken at the point $\boldsymbol{\xi}=(\xi_p,\xi_q)$, which is in general different from the chord transform $\tilde{H}(\boldsymbol{\xi})$ of $H(\mathbf{x})$. In other words, one has also $\tilde{W}_t(\boldsymbol{\xi})=\tilde{W}_0(\mathbf{R}_{-t}\boldsymbol{\xi})$. Above we have made use of the wedge product,

$$\boldsymbol{\xi}\wedge\mathbf{x}\equiv\xi_p q-\xi_q p\equiv(\mathbf{J}\boldsymbol{\xi})^T\mathbf{x}\equiv\mathbf{J}\boldsymbol{\xi}\cdot\mathbf{x}, \qquad (6)$$

also defining the transpose of a vector $(.)^T$ and the matrix $\mathbf{J}$. The semiclassical background for the symplectic invariance of both the Wigner function and its Fourier transform is that the Weyl phase space coordinate $\mathbf{x}$ may be associated to pairs of points in phase space, $\mathbf{x}_\pm$, by $\mathbf{x}=(\mathbf{x}_+ +\mathbf{x}_-)/2$. The conjugate variable to this center is the chord $\boldsymbol{\xi}=\mathbf{x}_+-\mathbf{x}_-$. The linear motion of both the chord $\boldsymbol{\xi}$ and the center $\mathbf{x}$ is the same as for each individual phase space point $\mathbf{x}_+$ or $\mathbf{x}_-$. We will refer $\tilde{W}(\boldsymbol{\xi})$ as the chord function as in Ref. [4], though it is also known as the characteristic function in quantum optics.

The question that we address is to what extent can the simplicity and generality of symplectic motion of closed quantum systems be incorporated within the description of systems whose coupling to the environment cannot be ignored. In this case the evolution is no longer unitary, unless the full Hamiltonian of the system combined with the environment is taken into account. All the same, a certain mea-

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sure of generality is restored by the assumption that the density operator is governed by a Markovian master equation [5],

$$\frac{\partial\hat{\rho}}{\partial t}=-\frac{i}{\hbar}[\hat{H},\hat{\rho}]-\frac{1}{2\hbar}\sum_j\ 2\hat{L}_j\hat{\rho}\hat{L}_j^\dagger-\hat{L}_j^\dagger\hat{L}_j\hat{\rho}-\hat{\rho}\hat{L}_j^\dagger\hat{L}_j. \tag{7}$$

If a further assumption is made that each Lindblad operator $\hat{L}_j$ is a linear function of $\hat{p}$ and $\hat{q}$, we will show that the evolution of $\widetilde{W}_t(\boldsymbol{\xi})$ is the product of the classically evolved $\widetilde{W}_0(\boldsymbol{\xi})$ with $g_t(\boldsymbol{\xi})$, a Gaussian centered on $\boldsymbol{\xi}=0$, which has diminishing width. One can then generalize in a straightforward way the exact solution given by Agarwal [6] for the Wigner function. This is a convolution of the Fourier transform of $g_t(\boldsymbol{\xi})$ with the classically evolved $W_0(\mathbf{x})$. In other words the Wigner function is coarse grained by a widening Gaussian window.

A simple example of symplectic evolution of an open quantum system is that of a dust particle interacting with air molecules, or radiation, so that in the absence of gravity $\hat{H}=\hat{p}^2/2m$ and the interaction with the environment depends basically on the particle's position: $\hat{L}=\eta\hat{q}$, where η is the coupling parameter. This example is discussed by Giulini *et al.* in Ref. [7]. Another important example is that of an optical field, say an arbitrary superposition of coherent states interacting with thermal photons. In terms of real variables, the internal Hamiltonian is just $\hat{H}=\omega(\hat{p}^2+\hat{q}^2)/2$, i.e., the harmonic oscillator. The Lindblad operators in this case are known to be $\gamma(\overline{n}+1)\hat{a}/2$ and $\gamma\overline{n}\hat{a}^\dagger/2$, where $\hat{a}^\dagger$ and $\hat{a}$ are the usual creation and annihilation operators, $\overline{n}$ is the average number of thermal photons at the frequency ω of the cavity mode at temperature T and γ is the decay rate [8].

Recently Diósi and Kiefer (DK) [9] showed in the case of the first example that the Wigner function of any pure state becomes positive within a definite time. Thus the Markovian interaction with the environment has the effect of erasing the interference fringes characteristic of quantum coherence and from then on the effect of coarse graining on the Wigner function is not distinguishable from that of a classical Liouville distribution. How general is the DK scenario? In Sec. II we present the exact solution of the Lindblad equation for general quadratic $\hat{H}$ and arbitrary complex linear Lindblad operators $\hat{L}_j=\lambda_j\hat{q}+\mu_j\hat{p}$. In Sec. III we explain the underlying classical structure of the solution. Then, in Sec. IV, we use the properties of the convolution to generalize the positivity time of DK in the case of arbitrary quadratic Hamiltonian and non-Hermitian Lindblad operators. Furthermore we make the much stronger statement that the Wigner function cannot be positive before this threshold, unless the initial distribution is a Gaussian. We discuss the consequences of this statement through the example of a bath of photons. In Sec. V we specify the behavior of the positivity threshold for each type of quadratic Hamiltonian. It turns out that a nonzero dissipation coefficient implies that positivity is reached exponentially fast. The positivity threshold is in general inversely proportional to the dissipative coefficient. However it becomes inversely proportional to the Lyapunov exponent if the latter is greater than this coefficient, that is in the case of a hyperbolic system (i.e., the inverted oscillator) in the weak coupling limit. Then positivity can be reached much faster than in the corresponding elliptic case (i.e., the harmonic oscillator with the same coupling constants). Though all the formulas presented here are appropriate for a single degree of freedom, the generalization to higher dimensions is discussed in this section. In Sec. VI we derive a general formula for the growth of the linear entropy $(1-\mathrm{Tr}\hat{\rho}_t^2)$, with respect to the initial density operators, for each choice of the quadratic master equation. We also show that for long times the growth of linear entropy attains a universal form. Finally in Sec. VII we point out that this general solution is obviously reversible, giving a very synthetic inversion formula which generalizes previous work about quantum state reconstruction [10].

The generalization of the convolution as exact solution of the master equation (7) when $\hat{H}$ is not quadratic, or for nonlinear $\hat{L}_j$, is not obvious. However, the approximate semiclassical theory developed by one of the present authors [11] has no such constraint. Its compatibility with the present theory is the subject of a companion paper [12]. A simpler version of the present work, for the restricted case of hermitian $\hat{L}_j$, can be accessed in Ref. [13].

II. EXACT SOLUTION IN THE QUADRATIC CASE WITH DISSIPATION

We derive here the exact solution of the Lindblad equation in the case where the Hamiltonian is quadratic and the Lindblad operators are complex linear forms in $\hat{q}$ and $\hat{p}$.

Taking the Weyl-Wigner transform of Eq. (7), i.e., associating the Weyl symbol $A(\mathbf{x})$ to each operator $\hat{A}$, as defined in Eq. (1), and using the product rules [14] for operators, we obtain,

$$\begin{aligned}\frac{\partial W_t}{\partial t}(\mathbf{x})=&\{H(\mathbf{x}),W_t(\mathbf{x})\}+\sum_j\ (\mathbf{J}\mathbf{l}_j''\cdot\mathbf{l}_j')\\&\times\left[\mathbf{x}\cdot\frac{\partial W_t}{\partial\mathbf{x}}(\mathbf{x})+2W_t(\mathbf{x})\right]\\&+\frac{\hbar}{2}\sum_j\ \left\{\mathbf{J}\mathbf{l}_j'\cdot\left(\frac{\partial^2W_t}{\partial\mathbf{x}^2}(\mathbf{x})\right)\mathbf{J}\mathbf{l}_j'\right.\\&\left.+\mathbf{J}\mathbf{l}_j''\cdot\left(\frac{\partial^2W_t}{\partial\mathbf{x}^2}(\mathbf{x})\right)\mathbf{J}\mathbf{l}_j''\right\}.\end{aligned} \tag{8}$$

Here $L_j(\mathbf{x})=\mathbf{l}_j'\cdot\mathbf{x}+i\mathbf{l}_j''\cdot\mathbf{x}$ are the Weyl symbols of the non-Hermitian linear Lindblad operators. We use the notation

$$\mathbf{x}=\begin{pmatrix}p\\q\end{pmatrix}\quad\text{and}\quad\mathbf{l}'=\begin{pmatrix}\lambda'\\\mu'\end{pmatrix},\quad\mathbf{l}''=\begin{pmatrix}\lambda''\\\mu''\end{pmatrix}. \tag{9}$$

It is well known that in this case $H(\mathbf{x})$ and $L_j(\mathbf{x})$ can be identified as the classical variables corresponding to $\hat{H}$ and

$\widehat{L}_j$. Note that if $\mathbf{l}''=0$ then the second term in Eq. (8) cancels. It will become clear that this term is responsible for dissipation in the evolution of $W_t(\mathbf{x})$ and we define the dissipation coefficient $\alpha=\Sigma_j(\mathbf{J}\mathbf{l}_j''\cdot\mathbf{l}_j')$, which is zero in Ref. [13].

It is actually easier to solve the evolution equation for the chord function $\widetilde{W}_t(\boldsymbol{\xi})$,

$$\frac{\partial\widetilde{W}_t}{\partial t}(\boldsymbol{\xi})=\{H(\boldsymbol{\xi}),\widetilde{W}_t(\boldsymbol{\xi})\}-\alpha\boldsymbol{\xi}\cdot\frac{\partial\widetilde{W}_t}{\partial\boldsymbol{\xi}}(\boldsymbol{\xi})-\frac{1}{2\hbar}\sum_j[(\mathbf{l}_j'\cdot\boldsymbol{\xi})^2+(\mathbf{l}_j''\cdot\boldsymbol{\xi})^2]\widetilde{W}_t(\boldsymbol{\xi}), \tag{10}$$

as derived in Appendix A. We guess a solution of the form

$$\widetilde{W}_t(\boldsymbol{\xi})=\widetilde{W}_0(\boldsymbol{\xi}_{-t})\exp\left(-\frac{1}{2\hbar}\sum_j\int_0^t[(\mathbf{l}_j'\cdot\boldsymbol{\xi}_{t'-t})^2+(\mathbf{l}_j''\cdot\boldsymbol{\xi}_{t'-t})^2]dt'\right), \tag{11}$$

where $\boldsymbol{\xi}_t$ is a linear evolution of $\boldsymbol{\xi}$, which will be explicited *a posteriori*, such that

$$\boldsymbol{\xi}_0=\boldsymbol{\xi}. \tag{12}$$

Then, inserting the form (11) of $\widetilde{W}_t$ in Eq. (10) and dividing both sides by the exponential of Eq. (11) leads us to the following left part:

$$\begin{aligned}&-\frac{\partial\widetilde{W}_0}{\partial\boldsymbol{\xi}}(\boldsymbol{\xi}_{-t})\cdot\dot{\boldsymbol{\xi}}_{-t}-\frac{1}{2\hbar}\sum_j[(\mathbf{l}_j'\cdot\boldsymbol{\xi})^2+(\mathbf{l}_j''\cdot\boldsymbol{\xi})^2]\widetilde{W}_0(\boldsymbol{\xi}_{-t})\\&-\frac{1}{2\hbar}\widetilde{W}_0(\boldsymbol{\xi}_{-t})\sum_j\int_0^t[2(\mathbf{l}_j'\cdot\boldsymbol{\xi}_{t'-t})\mathbf{l}_j'\cdot(-\dot{\boldsymbol{\xi}}_{t'-t})\\&+2(\mathbf{l}_j''\cdot\boldsymbol{\xi}_{t'-t})\mathbf{l}_j''\cdot(-\dot{\boldsymbol{\xi}}_{t'-t})]dt',\end{aligned} \tag{13}$$

which must be equal to the following right part:

$$\begin{aligned}&-2\mathbf{JH}\boldsymbol{\xi}_{-t}\cdot\frac{\partial\widetilde{W}_0}{\partial\boldsymbol{\xi}}(\boldsymbol{\xi}_{-t})-\alpha\boldsymbol{\xi}_{-t}\cdot\frac{\partial\widetilde{W}_0}{\partial\boldsymbol{\xi}}(\boldsymbol{\xi}_{-t})\\&-\frac{1}{2\hbar}\sum_j[(\mathbf{l}_j'\cdot\boldsymbol{\xi})^2+(\mathbf{l}_j''\cdot\boldsymbol{\xi})^2]\widetilde{W}_0(\boldsymbol{\xi}_{-t})-\frac{1}{2\hbar}\widetilde{W}_0(\boldsymbol{\xi}_{-t})\\&\times\sum_j\int_0^t2[(\mathbf{l}_j'\cdot\boldsymbol{\xi}_{t'-t})\mathbf{l}_j'\cdot(-2\mathbf{JH}\boldsymbol{\xi}_{t'-t}-\alpha\boldsymbol{\xi}_{t'-t})\\&+(\mathbf{l}_j''\cdot\boldsymbol{\xi}_{t'-t})\mathbf{l}_j''\cdot(-2\mathbf{JH}\boldsymbol{\xi}_{t'-t}-\alpha\boldsymbol{\xi}_{t'-t})]dt'.\end{aligned} \tag{14}$$

We have used

$$\boldsymbol{\xi}\cdot\frac{\partial}{\partial\boldsymbol{\xi}}[\widetilde{W}_0(\boldsymbol{\xi}_t)]=\boldsymbol{\xi}_t\cdot\frac{\partial\widetilde{W}_0}{\partial\boldsymbol{\xi}}(\boldsymbol{\xi}_t), \tag{15}$$

and

$$\{H(\boldsymbol{\xi}),\widetilde{W}_t(\boldsymbol{\xi})\}=-2\mathbf{JH}\boldsymbol{\xi}\cdot\frac{\partial\widetilde{W}_t}{\partial\boldsymbol{\xi}}(\boldsymbol{\xi}). \tag{16}$$

Hence the ansatz (11) is a solution of Eq. (10) if $\boldsymbol{\xi}_t$ fulfills

$$\dot{\boldsymbol{\xi}}_t=2\mathbf{JH}\boldsymbol{\xi}_t+\alpha\boldsymbol{\xi}_t. \tag{17}$$

Thus, we can write explicitly

$$\boldsymbol{\xi}_t=e^{\alpha t}\mathbf{R}_t\boldsymbol{\xi}, \tag{18}$$

where $\mathbf{R}_t$, defined in Eq. (3), gives the purely Hamiltonian evolution and the dissipation term α leads to a classical non-Hamiltonian expansion ($\alpha>0$) or contraction ($\alpha<0$) of the chord variable $\boldsymbol{\xi}$. One should be aware that although the Hamiltonian part of the evolution of $\boldsymbol{\xi}$ is shared with that of the phase space point $\mathbf{x}$, the effect of dissipation is inverted, as it will be explained soon.

The argument of the exponential in Eq. (11) is a quadratic form in $\boldsymbol{\xi}$, so the solution can be written as

$$\widetilde{W}_t(\boldsymbol{\xi})=\widetilde{W}_0(\boldsymbol{\xi}_{-t})\exp\left(-\frac{1}{2\hbar}\boldsymbol{\xi}\cdot\mathbf{M}(t)\boldsymbol{\xi}\right) \tag{19}$$

with $\mathbf{M}(t)$ a real, time dependent 2×2 matrix, which can naturally be decomposed into

$$\mathbf{M}(t)=\sum_j M_j(t)=\sum_j\int_0^t dt'\,e^{2\alpha(t'-t)}\mathbf{R}_{t'-t}^T\mathbf{l}_j\mathbf{l}_j^T\mathbf{R}_{t'-t}, \tag{20}$$

so that each Lindblad operator contributes a Gaussian to Eq. (19).

Now, back into the Weyl-Wigner representation by using Eq. (4), one obtains the solution of Eq. (8),

$$W_t(\mathbf{x})=\frac{1}{2\pi\hbar}e^{2\alpha t}\int W_0[e^{\alpha t}\mathbf{R}_{-t}(\mathbf{x}-\mathbf{y})]\frac{1}{\sqrt{\det\mathbf{M}_J(t)}}\times\exp\left(-\frac{1}{2\hbar}\mathbf{y}\cdot\mathbf{M}_J(t)^{-1}\mathbf{y}\right)d\mathbf{y}, \tag{21}$$

where we have defined

$$\begin{aligned}\mathbf{M}_J(t)&=-\mathbf{JM}(t)\mathbf{J},\\\mathbf{M}_J(t)^{-1}&=-\mathbf{JM}(t)^{-1}\mathbf{J}\end{aligned} \tag{22}$$

with the symplectic matrix $\mathbf{J}$ defined in Eq. (6). $W_0(\mathbf{x})$ is the initial Wigner function and the convolution Gaussian turns into a Dirac δ function as t goes to 0. We have equivalently

$$W_t(\mathbf{x})=w_t(\mathbf{x}_{-t}) \tag{23}$$

$$=\frac{1}{2\pi\hbar\sqrt{\det\mathbf{M}_J(t)}}\int W_0(\mathbf{y})\times\exp\left(-\frac{1}{2\hbar}(\mathbf{y}-\mathbf{x}_{-t})\cdot\widetilde{\mathbf{M}}(t)^{-1}(\mathbf{y}-\mathbf{x}_{-t})\right)d\mathbf{y}, \tag{24}$$

with

$$\mathbf{x}_t = e^{-\alpha t}\mathbf{R}_t\mathbf{x} \tag{25}$$

and

$$\tilde{\mathbf{M}}(t) = -e^{2\alpha t}\mathbf{R}_{-t}^T\mathbf{J}M(t)\mathbf{J}\mathbf{R}_{-t} = -\mathbf{M}_J(-t). \tag{26}$$

Hence the solution is a convolution with a Gaussian which broadens in time, composed with a backwards non-Hamiltonian evolution of the phase space variable **x**. As mentioned earlier, the Hamiltonian part of this classical evolution is the same as in the chord space, whereas the dissipative part has the opposite effect: dissipation ($\alpha > 0$) will shrink the phase space variable, and thus expand the distribution W_t.

This solution, which has been derived in the case of an homogeneous quadratic Hamiltonian, can be generalized easily to a quadratic Hamiltonian with a linear part. One then has to be aware that the matrix $\mathbf{R}_t$, which appears in the exponential Lindbladian damping, strictly corresponds to the classical motion of the *chord*, which is determined by the *homogeneous* part of the Hamiltonian H. This remark is important, for instance, in the parabolic case, say a particle with a linear potential, where the motion of the chord disregards the potential.

Since $|\det\tilde{\mathbf{M}}(t)|$ grows with time, one can conclude, following DK [9], that the solution (21) becomes positive after a certain time. Indeed it is, after rescaling the variable, the convolution of the initial Wigner function with a Gaussian of broadening size, which smoothes out oscillations around zero. It is the property of symplectic invariance of the Wigner function that DK employ to prove strict positivity in a specific case that is now extended to its broader context. Moreover we shall give in the much stronger result Sec, IV that the Wigner function cannot be positive before the DK time, which does not depend on the initial pure state, unless it is a coherent state. We then give the general behavior of this positivity threshold for different dynamics in Sec. V, namely when **H** is elliptic, as for the harmonic oscillator, hyperbolic, as for the scattered particle, or in the parabolic intermediate case which includes the system studied in Ref. [9]. The following section explains the formal correspondence between this problem and a classical Brownian motion described by a Langevin equation, as Agarwal [6] did for his solution.

III. CLASSICAL CORRESPONDENCE

Since Eq. (8) is a Fokker-Planck equation, it can be interpreted as the evolution equation for the probability distribution of a classical Brownian motion defined by a Langevin equation. This correspondence gives a simple classical interpretation of the problem. Hence the decoherence may be seen as a diffusion induced by some random force, and dissipation can be interpreted as a classical viscosity, although it is always accompanied by another diffusive term. The only feature which cannot be assigned a classical meaning is the Wigner function itself, which, as a pseudoprobability distribution, can have negative values.

One can check easily, see, for instance, Ref. [15], that the following Langevin equation:

$$\dot{p} = -\frac{\partial H}{\partial q}(\mathbf{x}) - \alpha p + \sqrt{\hbar}\sum_m [\lambda'_m f_m(t) + \lambda''_m g_m(t)],$$

$$\dot{q} = \frac{\partial H}{\partial p}(\mathbf{x}) - \alpha q + \sqrt{\hbar}\sum_m [\mu'_m f_m(t) + \mu''_m g_m(t)], \tag{27}$$

induces Eq. (8) as a Fokker-Planck counterpart. The "Brownian forces" $f_m(t)$ and $g_m(t)$ verify

$$\langle f_m(t')f_n(t'')\rangle = \delta_{m,n}\delta(t'-t''),$$

$$\langle g_m(t')g_n(t'')\rangle = \delta_{m,n}\delta(t'-t''),$$

$$\langle f_m(t')g_n(t'')\rangle = 0. \tag{28}$$

$f_m(t)$ correspond to the diffusion induced by the nondissipative real part of the Lindblad operators, whereas $g_m(t)$ correspond to the diffusion induced by dissipation. It can easily be verified that the Fokker-Planck equation is symplectically invariant, so one is allowed to perform the following change of coordinates:

$$\overline{p} = p - \frac{\alpha}{2\mathbf{H}_{11}}q,$$

$$\overline{q} = q, \tag{29}$$

to the above Langevin equation, which then turns into the following more intuitive form, where the dissipation depends only on the momentum:

$$\dot{p} = -\frac{\partial \overline{H}}{\partial q}(\mathbf{x}) - \overline{\alpha}p + \sqrt{\hbar}\sum_m [\overline{\lambda'_m} f_m(t) + \overline{\lambda''_m} g_m(t)],$$

$$\dot{q} = \frac{\partial \overline{H}}{\partial p}(\mathbf{x}) + \sqrt{\hbar}\sum_m [\mu'_m f_m(t) + \mu''_m g_m(t)]. \tag{30}$$

Here, the matrix for the transformed Hamiltonian $\overline{H}$ is

$$\begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{12} & \mathbf{H}_{22} + \dfrac{\alpha^2 + 4\alpha\mathbf{H}_{12}}{4\mathbf{H}_{11}} \end{pmatrix}, \tag{31}$$

$\overline{\lambda'_m} = \lambda'_m - (\alpha/2\mathbf{H}_{11})\mu'_m$ and, respectively, for $\overline{\lambda''_m}$, and $\overline{\alpha} = 2\alpha$.

From this classical picture we can interpret the general behavior of the solution of the Lindblad equation. In the case of a closed system, remember that the Wigner function undergoes a Liouville, unitary, propagation. Now the system is coupled to an environment, i.e., there are nonzero Lindblad operators. If dissipation of energy is neglected, these operators are Hermitian, so there is no imaginary part of the vectors **l**. Then the effect of the environment over the system can be interpreted as a diffusion process corresponding to random forces in the Langevin equation. Formally, it corresponds to the initial Wigner function being convoluted with a Gaussian which broadens with time. If one now takes into account the dissipation induced by the environment, allowing the Lindblad operator to be non-Hermitian, a viscous term appears in Eqs. (27) and (30), meaning that the classical trajectories on which the distribution travels are drifted to lesser or higher energy, according to the sign, $+$ or $-$, of the "viscosity" α. Indeed, the dissipative linear motion governed by the nonrandom terms of Eqs. (27) is just that of (25). One should be aware that it is only a formal classical scheme, and that the viscous term might have a purely quantum origin. For instance, in the case of photons in a cavity with dissipation, whose Lindblad operators are explained in Sec. V A, the viscosity is related to spontaneous emission, which breaks the symmetry between emission and absorption. Then the classical trajectories of the above equation spiral in towards the origin, although a semiclassical theory would lead to no dissipation. The opposite case would be an amplified cavity, where the trajectories would spiral out. Note that this viscous term always goes along with a supplement of diffusion, which can be interpreted as a consequence of the fluctuation-dissipation "theorem" [16].

IV. UNIQUE POSITIVITY TIME FOR ANY INITIAL STATE

We have seen in Sec. II that, because of the Lindbladian part of the master equation, the pure state Wigner function is convoluted with a Gaussian whose width grows with time. It has been pointed out by DK [9] that at the time t_p at which the width of the Gaussian reaches $\hbar$, that is, when it becomes the Weyl representation of some coherent or squeezed state, then its convolution with the initial Wigner function $W_0(\mathbf{x})$ is a Husimi function [17,2,4] of the initial state, or a Q function in the language of quantum optics. It is a well known property that the Husimi function is positive, so we have $W_{t_p}\geqslant 0$, and obviously, since the Gaussian is strictly broadening, $W_t>0$ for $t>t_p$. It has already been emphasized by Leonhardt *et al.* [18] that the form of the Wigner function after interaction with a dissipative environment can be read as an intermediate phase space distribution $W(\mathbf{x},t=0,s)$, with an s which depends on the dissipation rate. Thus $s=0$ corresponds to the (initial) Wigner function and $s=-1$ to the Husimi function of DK (in Ref. [18] the role of the environment is played by the imperfections of a beam separator).

We shall now prove that the Wigner function can never be positive before the positivity threshold t_p, unless it is positive from the beginning, that is, unless the initial state is a Gaussian state. Indeed, if an initial pure state $|\psi_0\rangle$ is not a Gaussian, then it was shown by Tatarskiĭ [19] that the initial Wigner function $W_0(\mathbf{x})$ has negative parts. But it is also true that non-Gaussian Husimi functions necessarily have zeroes, as shown in Appendix B. Since W_{t_p} is, up to a symplectic transform, a Husimi function, there exists $\mathbf{x}_0$ such that

$$W_{t_p}(\mathbf{x}_0)=0. \tag{32}$$

Let us now investigate the case $t<t_p$. Then $W_t(\mathbf{x})$ given by Eq. (24) is a convolution of $W_0(\mathbf{x}_{-t})$ with a Gaussian of width smaller than $\hbar$. The point is that one can then convolute again with $\exp[-(\kappa/\hbar)\mathbf{x}_{-t}\cdot\tilde{\mathbf{M}}(t)^{-1}\mathbf{x}_{-t}]$, with the real parameter κ chosen so that the output is also a Husimi function $Q(\mathbf{x}_{-t})$. To show this we refer to the simple general relation:

$$\int d\mathbf{y}\,\exp[-(\mathbf{x}-\mathbf{y})\cdot\mathbf{M}(\mathbf{x}-\mathbf{y})]\exp(-\mathbf{y}\cdot\kappa\mathbf{M}\mathbf{y}) = \frac{\pi}{(1+\kappa)\sqrt{\det\mathbf{M}}}\exp\left(-\mathbf{x}\cdot\frac{\kappa}{1+\kappa}\mathbf{M}\mathbf{x}\right). \tag{33}$$

This Husimi function can be identified with W_{t_p} through a symplectic transform $\mathbf{x}_{-t}\rightarrow\mathbf{x}'$, so we have

$$Q(\mathbf{x}_0')=\int d\mathbf{y}\;w_t(\mathbf{x}_0'-\mathbf{y})\exp\left(-\frac{\kappa}{2\hbar}\mathbf{y}\cdot\tilde{\mathbf{M}}(t)^{-1}\mathbf{y}\right)=0. \tag{34}$$

Now it is obvious from Eq. (34) that w_t, hence W_t, must have a negative part for all $t<t_p$.

Let us emphasize the striking consequence of this result: the positivity time does not depend on the initial distribution, as long as it is not a Gaussian one. The following example shall illustrate this remarkable property in a more transparent way.

Example

We start from the familiar superposition of two coherent states, i.e., ground states of the harmonic oscillator, displaced to the phase space points $\mathbf{x}_\zeta=(0,\pm\zeta)$: $|\psi_0\rangle=(|\zeta\rangle+|-\zeta\rangle)/\sqrt{2}$, in the context of photons in a cavity with dissipation. The Wigner function, which is a particular case of Eq. (21), is the sum of three terms; two of these correspond to the coherent states taken independently, and the third term comes from their interference:

$$W_t(\mathbf{x})=W_\zeta(\mathbf{x})+W_{-\zeta}(\mathbf{x})+W_i(\mathbf{x}), \tag{35}$$

with

$$W_\zeta(\mathbf{x})=\frac{2N}{\pi\beta_t}\exp\left(-\frac{2}{\beta_t}p^2\right)\exp\left(-\frac{2}{\beta_t}(q-e^{-\gamma t/2}\zeta)^2\right) \tag{36}$$

and

$$W_i(\mathbf{x})=\frac{4N}{\pi\beta_t}\exp\left(-\frac{2}{\beta_t}(p^2+q^2)\right)\times\exp\left[-2\left(1-\frac{e^{-\gamma t}}{\beta_t}\right)\zeta^2\right]\cos\left(\frac{4e^{\gamma t/2}}{\beta_t}\zeta p\right), \tag{37}$$

where γ, defined in the introduction, corresponds to 2α, and $\beta_t=2\bar{n}[1-\exp(-\gamma t)]+1$. The function is normalized by $N=[1+\exp(-\zeta^2/\hbar)]^{-1}$.Obviously the minimum values are concentrated on the line $q=0$, where the expression simplifies into

$$W_t(p,q=0)=\frac{4N}{\pi\beta_t}\exp\left(-\frac{2}{\beta_t}p^2\right)\left[\exp\left[-2\left(1-\frac{e^{-\gamma t}}{\beta_t}\right)\zeta^2\right]\right.$$
$$\left.\times\cos\left(\frac{4e^{\gamma t/2}}{\beta_t}\zeta p\right)+\exp\left(-2\frac{e^{-\gamma t}}{\beta_t}\zeta^2\right)\right]. \tag{38}$$

The Wigner function becomes positive when

$$1-\frac{e^{-\gamma t}}{\beta_t}=\frac{e^{-\gamma t}}{\beta_t}, \tag{39}$$

that is, at $t_p=1/\gamma\ln[1+1/(2\bar{n}+1)]$, which indeed does not depend on the initial spacing ζ of the two coherent states. On the other hand, the position $(p_m,0)$ of the closest zero, given by the first minimum of the cosine at that time,

$$p_m=\frac{\beta_t}{4\sqrt{2}\zeta}, \tag{40}$$

gets further away as ζ becomes smaller. This shows that the negative regions that remain until t_p may be so shallow as to be practically irrelevant.

The chord representation reveals how the positivity threshold is related to a more reasonable estimate of the decoherence time. The expression of the initial chord function in our example is

$$\widetilde{W}_0(\boldsymbol{\xi})=\frac{1}{2\pi\hbar}\exp\left(-\frac{\boldsymbol{\xi}^2}{4\hbar}\right)\exp\left(-\frac{i\boldsymbol{\xi}\wedge\mathbf{x}_\zeta}{\hbar}\right)+\text{c.c.}$$
$$+\frac{1}{2\pi\hbar}\exp\left(-\frac{(\boldsymbol{\xi}-\mathbf{x}_\zeta)^2}{4\hbar}\right)$$
$$+\frac{1}{2\pi\hbar}\exp\left(-\frac{(\boldsymbol{\xi}+\mathbf{x}_\zeta)^2}{4\hbar}\right). \tag{41}$$

The first two terms, distributed around the origin, correspond to the two coherent states taken separately, whereas the last two terms, distributed around $\mathbf{x}_\zeta$ and $-\mathbf{x}_\zeta$, describe the quantum interference between them. The positivity time corresponds to the moment when the Gaussian in Eq. (19) damps everything outside a region of size $\hbar$ in the chord space. However, both interference terms in this example will be damped much sooner if ζ is large enough, indicating an overall loss of coherence.

It should be remarked that one just has to study a specific example, here the twin coherent states, to get the positivity time for any non-Gaussian initial pure state.

V. BEHAVIOR OF THE POSITIVITY THRESHOLD

Positivity is attained when the determinant of the matrix $\widetilde{\mathbf{M}}(t)$, that is the determinant of $\mathbf{M}(-t)$, is equal to 1/4. Then the expression of the solution (24) is indeed a Husimi function. The matrix $\mathbf{M}$ is defined by Eq. (20), so, to calculate it, we notice that $\mathbf{JH}$, in the expression (3) of $\mathbf{R}_t$, can be diagonalized in most cases, that is when its two eigenvalues are finite and different. We then define the matrix $\mathbf{P}$ such that

$$2\mathbf{JH}=\mathbf{P}^{-1}\mathbf{DP} \tag{42}$$

with

$$\mathbf{D}=\begin{pmatrix}\sigma & 0\\ 0 & -\sigma\end{pmatrix}, \tag{43}$$

and $\sigma=2\sqrt{-\det\mathbf{H}}$. Note that since $\mathbf{H}$ is symmetric, then $\mathbf{JH}$ has a null trace, and so has $\mathbf{D}$. The dissipation parameter α and σ are basic elements for the description of the evolution of Markovian quadratic open systems.

Then, using Eq. (20) one can easily derive the expression of $\mathbf{M}$,

$$\mathbf{M}(t)=\mathbf{P}^T\begin{pmatrix}\frac{1-e^{-2(\sigma+\alpha)t}}{2(\alpha+\sigma)}\mathbf{A}_{11} & \frac{1-e^{-2\alpha t}}{2\alpha}\mathbf{A}_{12}\\ \frac{1-e^{-2\alpha t}}{2\alpha}\mathbf{A}_{21} & \frac{1-e^{2(\sigma-\alpha)t}}{2(\alpha-\sigma)}\mathbf{A}_{22}\end{pmatrix}\mathbf{P}, \tag{44}$$

where $\mathbf{A}$ is defined by

$$\mathbf{A}=(\mathbf{P}^{-1})^T\sum_j[\mathbf{l}_j'(\mathbf{l}_j')^T+\mathbf{l}_j''(\mathbf{l}_j'')^T]\mathbf{P}^{-1}. \tag{45}$$

The corresponding quadratic form actually depends on the classical motion of the chord, determined by the homogeneous part $\mathbf{x}\cdot\mathbf{Hx}$ of the Hamiltonian. One then has to separate different cases: the elliptic case, $\det\mathbf{H}>0$, the parabolic case, $\det\mathbf{H}=0$, and the hyperbolic case, $\det\mathbf{H}<0$.

Since the Wigner function is symplectically invariant, one just has to treat the simplest expression in each case, respectively, the harmonic oscillator, $H(\mathbf{x})=p^2/2+q^2/2$, the particle in a linear potential, $H(\mathbf{x})=p^2/2+q$, and the scattered particle, $H(\mathbf{x})=pq$, which is symplectically equivalent to $p^2/2-q^2/2$.[1]

A. Harmonic oscillator

In this case, the Hamiltonian reads

$$H(\mathbf{x})=\omega\left(\frac{p^2}{2}+\frac{q^2}{2}\right) \tag{46}$$

[1] The reduction of a linear Lindblad operator under a symplectic transformation $\mathbf{x}'=\mathbf{Cx}$, is especially simple in the Wigner or in the chord representation. If, instead of Eq. (9), we set $L_j(\mathbf{x})=\boldsymbol{\ell}_j\wedge\mathbf{x}$, then the invariance is obtained by taking $l_j'=\mathbf{C}l_j$. Of course, one must also use the invariance of the classical Hamiltonian $H(\mathbf{x}')=H(\mathbf{x})$.

and the matrix $\mathbf{P}$ is

$$\mathbf{P}=\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix}. \tag{47}$$

Then the determinant of $\mathbf{M}(-t)$ reads

$$\det\mathbf{M}(-t)=\frac{e^{4\alpha t}-2e^{2\alpha t}\cos 2\omega t+1}{4(\alpha^2+\omega^2)}\mathbf{A}_{11}\mathbf{A}_{22} - \frac{e^{4\alpha t}-2e^{2\alpha t}+1}{4\alpha^2}\mathbf{A}_{12}\mathbf{A}_{21}, \tag{48}$$

where in this case the matrix $\mathbf{A}$ is complex:

$$\mathbf{A}=\frac{1}{2}\sum_j \begin{pmatrix} (\lambda_j')^2-(\mu_j')^2+2i\lambda_j\mu_j & -i(\lambda_j')^2-i(\mu_j')^2 \\ -i(\lambda_j')^2-i(\mu_j')^2 & -(\lambda_j')^2+(\mu_j')^2+2i\lambda_j\mu_j \end{pmatrix}+(\text{idem with }\ \lambda_j''\ \text{ and }\ \mu_j''), \tag{49}$$

by using Eq. (9).

In the dissipative case, that, is for $\alpha>0$, this determinant diverges exponentially fast, and positivity is attained in a time of the order of $1/|\alpha|$. Let us take for instance the bath of photons, then the coefficients of the Lindblad operators are

$$\mathbf{l}_1'=\begin{pmatrix} 0 \\ \sqrt{\dfrac{\gamma(\bar{n}+1)}{2}} \end{pmatrix},\quad \mathbf{l}_1''=\begin{pmatrix} \sqrt{\dfrac{\gamma(\bar{n}+1)}{2}} \\ 0 \end{pmatrix},$$

$$\mathbf{l}_2'=\begin{pmatrix} 0 \\ \sqrt{\dfrac{\gamma\bar{n}}{2}} \end{pmatrix},\quad \mathbf{l}_2''=\begin{pmatrix} -\sqrt{\dfrac{\gamma\bar{n}}{2}} \\ 0 \end{pmatrix}, \tag{50}$$

and the friction $\alpha=\gamma/2$. Then

$$\det\mathbf{M}(-t)=\frac{(e^{\gamma t}-1)^2}{4}(2\bar{n}+1)^2, \tag{51}$$

which equals 1/4 at $t=1/\gamma\ln(1+1/(2\bar{n}+1))$, as was previously mentioned. Note that, although the coarse graining grows for ever, the size of the Wigner distribution reaches a finite limit, for the rescaled function (24) is the expression of $W(\mathbf{x})$ and not $W(\mathbf{x}_{-t})$.

If, on the other hand, $\alpha<0$ then the determinant reaches its limit in a time also of the order of $1/|\alpha|$. We conjecture that this limit has a lower bound greater than 1/4. Here rescaling $\mathbf{x}_{-t}\to\mathbf{x}$ now implies that $W_t(\mathbf{x})$ spreads with no bound.

B. Scattered particle

The simplest form of the Hamiltonian for a particle scattered by a parabolic barrier is

$$H(\mathbf{x})=\omega pq. \tag{52}$$

Then the matrix $\mathbf{P}$ is just identity and the matrix $\mathbf{A}$ is real, so the determinant reads

$$\det\mathbf{M}(-t)=\frac{e^{4\alpha t}-2e^{2\alpha t}ch2\omega t+1}{4(\alpha^2-\omega^2)}\mathbf{A}_{11}\mathbf{A}_{22} - \frac{e^{4\alpha t}-2e^{2\alpha t}+1}{4\alpha^2}\mathbf{A}_{12}\mathbf{A}_{21}, \tag{53}$$

As long as $\alpha>-\omega$ it grows exponentially, so positivity is always reached. When $\alpha<-\omega$ the determinant has again a finite asymptotic value. Then the positivity threshold is of the order of $1/(|\alpha|-\omega)$, which is greater than the corresponding elliptic case, with identical Lindblad operators.

The main difference with the elliptic case appears in the weak coupling limit $|\alpha|\ll\omega$. Whereas positivity of the elliptic system will then be attained in a time which is still inversely proportional to the coupling with the environment, the positivity threshold of the hyperbolic system will saturate at $1/\omega$. We conjecture that this will also be the case in a more general chaotic system.

C. Particle in a linear potential

We now study the intermediate case, which can be represented by the Hamiltonian

$$H(\mathbf{x})=\frac{p^2}{2}+q. \tag{54}$$

This degenerate case does not follow our general form for the matrix $\mathbf{M}$, so one has to treat it separately, taking care of the linear term (cf. remark of Sec. II). One should remember here that the motion of the chord is given by the free motion of the particle, which corresponds to

$$\mathbf{R}_t=\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}. \tag{55}$$

Then the damping matrix is

$$\mathbf{M}(t)=t\sum_j \begin{pmatrix} \frac{e^{-2\alpha t}}{2\alpha}Q_{11}^{(2)}-\frac{1}{2\alpha}Q_{11}^{(0)} & \frac{e^{-2\alpha t}}{2\alpha}Q_{12}^{(1)}-\frac{1}{2\alpha}Q_{12}^{(0)} \\ \frac{e^{-2\alpha t}}{2\alpha}Q_{12}^{(1)}-\frac{1}{2\alpha}Q_{12}^{(0)} & \left(\frac{e^{2-\alpha t}}{2\alpha}-\frac{1}{2\alpha}\right)Q_{22}^{(0)} \end{pmatrix}, \tag{56}$$

where $Q_{ij}^{(d)}$ are the polynomials of degree d in t and of degree 2 in the coupling constants, say $(\lambda_j',\lambda_j'',\mu_j',\mu_j'')$. Let us take for instance one Lindblad operator with $(\mathbf{l}')^T=(0,\sqrt{D'})$ and $(\mathbf{l}'')^T=(\epsilon\sqrt{D''},0)$, with $\epsilon=\pm 1$. Then one has

$$\det\mathbf{M}(-t)=\frac{1}{4}\left[e^{4\epsilon\bar{D}t}\left(1+\frac{1}{4(D'')^2}\right)-e^{2\epsilon\bar{D}t}\left(\frac{D'}{D''}t^2+\frac{1}{2(D'')^2}+2\right)+1+\frac{1}{4(D'')^2}\right], \tag{57}$$

with $\bar{D}=\sqrt{D'D''}$. The limit is always greater than 1/4, with the usual exponential contrast between the dissipative case ($\epsilon=1$) and the excited case ($\epsilon=-1$). In Ref. [9], DK study this example with no dissipation, and they find a positivity time of the order of $1/\sqrt{D'}$. On the following table, one can read different values of the positivity threshold for $D'=2$, and check that the limit $D''\to 0$ is attained gradually:

D''	0	0.1	1	10	100
$\epsilon=-1$	0.930	0.640	0.244	0.077	0.022
$\epsilon=1$	0.930	1.040	1.025	0.752	0.400.

(58)

However for large values of D'', the positivity threshold will behave like $1/\sqrt{D'D''}$.

So far the discussion has been restricted to the case of a single degree of freedom. Besides trivial changes of factors of $2\pi\hbar$, the basic form of the solutions of the Lindblad equation in the chord representation (19) and for the Wigner function (21) remain unchanged in the case of n degrees of freedom. The evolution matrix $\mathbf{R}_t$ now has the dimension $(2n)\times(2n)$, but it can again be simplified by symplectic transformations. It will often decompose into blocks corresponding to the above examples. If every Lindblad vector $\mathbf{l}_j$ is defined for a single block, then its contribution to $\mathbf{M}(t)$ will be of the same form as that for a single degree of freedom, but otherwise each case must be examined separately. Also, four-dimensional blocks may arise, corresponding to hyperbolic spiral motion, as well as singular cases analyzed by Arnold [1].

VI. THE GROWTH OF LINEAR ENTROPY

Besides considering the positivity of the Wigner function, we can use the exact solution (19) to investigate the growth of linear entropy

$$S_t=1-\mathrm{Tr}\hat{\rho}_t^2. \tag{59}$$

This is only zero for a pure state, just as for the von Neumann entropy

$$\mathbb{S}_t=-\mathrm{Tr}\hat{\rho}_t\ln\hat{\rho}_t. \tag{60}$$

Since the solution is simpler in the chord representation, we make use of the following relations[2] for operators $\hat{A},\hat{B},\ldots$ represented by chord functions $\tilde{A}(\boldsymbol{\xi}),\tilde{B}(\boldsymbol{\xi}),\ldots$

$$\hat{A}\to\tilde{A}(\boldsymbol{\xi}), \tag{61}$$

$$\hat{A}^\dagger\to\tilde{A}(-\boldsymbol{\xi})^*, \tag{62}$$

$$\mathrm{Tr}\hat{A}=\tilde{A}(\boldsymbol{\xi}=0), \tag{63}$$

$$\mathrm{Tr}\hat{A}\hat{B}=\frac{1}{2\pi\hbar}\int d\boldsymbol{\xi}\tilde{A}(\boldsymbol{\xi})\tilde{B}(-\boldsymbol{\xi}). \tag{64}$$

Therefore, using $\hat{\rho}^\dagger=\hat{\rho}$ and $\tilde{\rho}_t(\boldsymbol{\xi})=2\pi\hbar\ \tilde{W}_t(\boldsymbol{\xi})$, we obtain

$$\mathrm{Tr}\hat{\rho}_t=2\pi\hbar\ \tilde{W}_t(\boldsymbol{\xi}=0)=1, \tag{65}$$

and

$$\mathrm{Tr}\hat{\rho}_t^2=2\pi\hbar\int d\boldsymbol{\xi}\ |\tilde{W}_t(\boldsymbol{\xi})|^2. \tag{66}$$

In the case of the solution (19) of the Lindblad equation, we thus have

$$\tilde{W}_0(\boldsymbol{\xi}=0)=\frac{1}{2\pi\hbar} \tag{67}$$

and

$$2\pi\hbar\ \int d\boldsymbol{\xi}\ |\tilde{W}_0(\boldsymbol{\xi})|^2=1 \tag{68}$$

if the initial state is pure.

After changing the variables of integration in Eq. (66), replacing $\boldsymbol{\xi}_{-t}$ in Eq. (19) by the initial chord $\boldsymbol{\xi}'$, we obtain

$$\mathrm{Tr}\hat{\rho}_t^2=2\pi\hbar e^{2\alpha t}\int d\boldsymbol{\xi}'|\tilde{W}_0(\boldsymbol{\xi}')|^2\exp\left(\frac{1}{\hbar}\boldsymbol{\xi}'\cdot\mathbf{M}(-t)\boldsymbol{\xi}'\right), \tag{69}$$

[2]Note that there is a misprint in formula (6.24) of Ref. [4].

where $\mathbf{M}(-t)$, defined in Eq. (20), is negative definite.

Thus $\mathrm{Tr}\hat{\rho}_t^2$ is just the average of a rescaled Gaussian whose width narrows in time in a manner that depends exclusively on the particular form of $H(\mathbf{x})$ and the linear Lindblad operators $L_j(\mathbf{x})$. The initial state merely determines the probability density employed in the calculation of the average. A general asymptotic behavior can be predicted for this formula, because the width of the Gaussian generally shrinks as t grows. If the contraction is sufficient, the expression (69) will tend to

$$\mathrm{Tr}\hat{\rho}_t^2 \simeq 2\pi\hbar\, e^{2\alpha t}\, |\tilde{W}_0(\mathbf{0})|^2 \int d\boldsymbol{\xi}\, \exp\left[\frac{1}{\hbar}\boldsymbol{\xi}\cdot\mathbf{M}(-t)\boldsymbol{\xi}\right] \simeq \frac{\pi\hbar e^{2\alpha t}}{\sqrt{\det\mathbf{M}(-t)}}, \tag{70}$$

by using Eq. (67). It can be explained, using Eqs. (43) and (44), in terms of the dissipation α and the basic Hamiltonian parameter σ:

$$\mathrm{Tr}\hat{\rho}_t^2 \simeq \frac{\pi\hbar}{\sqrt{\dfrac{e^{-4\alpha t}-2e^{-2\alpha t}\left(\dfrac{e^{2\sigma t}+e^{-2\sigma t}}{2}\right)+1}{4(\alpha^2+\sigma^2)}\mathbf{A}_{11}\mathbf{A}_{22}-\dfrac{e^{-4\alpha t}-2e^{-2\alpha t}+1}{4\alpha^2}\mathbf{A}_{12}\mathbf{A}_{21}}}. \tag{71}$$

If the underlying classical system is elliptic, $\mathrm{Re}(\sigma)=0$, one can distinguish two situations. In the excited case, $\alpha<0$, $\mathrm{Tr}\hat{\rho}_t^2$ converges to zero. In the dissipative case, $\alpha>0$, it converges to a finite value, which is not surprising since the system then reaches a thermal equilibrium [6]. For instance in the case of a bath of photons, one has

$$\mathrm{Tr}\hat{\rho}_t^2 \rightarrow \frac{4\pi\hbar}{2\bar{n}+1}. \tag{72}$$

If the system is hyperbolic, the "Lyapunov exponent" $\omega=\mathrm{Re}(\sigma)\neq 0$. The consequence is to shift the definition of the above dichotomy. Indeed, $\mathrm{Tr}\hat{\rho}_t^2$ has a nonzero limit in the more restricted range $\alpha>\omega$.

The decoherence time, for the decay of $\mathrm{Tr}\hat{\rho}_t^2$, is in general inversely proportional to the coupling strength. However in the weak coupling limit, $|\alpha|\ll\omega$, it is $1/\omega$ in the hyperbolic case, whereas it is $1/|\alpha|$ in the elliptic one. Hence the decoherence time defined by the linear entropy is here consistent with the positivity threshold. This is a strong support to the thesis that positive Lyapunov exponents accelerate decoherence [20].

The asymptotic formula (70) holds only for those cases where all the eigenvalues of $\mathbf{M}(-t)$ tend to infinity. Counterexamples are the elliptic case with $\alpha<0$, since the determinant then has a finite limit, and the hyperbolic case, for $\alpha<-\omega$. Moreover, though $-\omega<\alpha<\omega$ leads to a determinant which tends to infinity, one of the eigenvalues will indeed diverge whereas the other one, corresponding to the unstable direction, will have a finite limit.

VII. REVERSIBILITY OF THE SOLUTION

Although decoherence is usually associated with a loss of information, which goes along with the growth of entropy, several techniques have been developed in quantum optics to recover the initial information after interaction of the system with the environment. In general these consist of reconstructing the quantum state of a lossy cavity by using mathematical inversion formulas [10] or directly by experimental processes [21]. We show here that the reversibility of the Wigner function results from the deconvolution of its evolution (24), or even simpler, as a division (19) in the chord space. Indeed, one has

$$\tilde{W}_t(\boldsymbol{\xi})=\tilde{W}_0(e^{-\alpha t}\mathbf{R}_{-t}\boldsymbol{\xi})\tilde{G}_t(\boldsymbol{\xi}), \tag{73}$$

which can easily be inverted as

$$\tilde{W}_0(\boldsymbol{\xi})=\frac{\tilde{W}_t(e^{\alpha t}\mathbf{R}_t\boldsymbol{\xi})}{\tilde{G}_t(e^{\alpha t}\mathbf{R}_t\boldsymbol{\xi})}. \tag{74}$$

This is a generalization and a simplification of the inversion formula of Ref. [10] since the loss induced by beam splitter is a particular case of our general formalism.

VIII. CONCLUSION

The exact solution of the Markovian master equation for quadratic Hamiltonians and linear complex Lindblad operators has been derived in the form of a convolution for the Wigner function. This involves the classical evolution of the initial Wigner function for the closed system with a phase space Gaussian that is independent of this initial condition, while its width expands in time, depending only on the Hamiltonian and the Lindblad operators. This simple solution allows for the generalization of DK's proof that the Wigner function becomes positive within a definite time t_p. Furthermore we support the much stronger statement that, unless the initial distribution is already a coherent state, the Wigner function must have negative regions before that time

t_p, which then does not depend on the initial state. In Sec. V we have given the behavior of t_p for three basic types of motion of the quadratic Hamiltonian, namely, the elliptic, the hyperbolic, and the parabolic case, verifying that positivity is always reached exponentially fast. The positivity threshold is generally of the order of $1/|\alpha|$, except in the weak coupling regime $|\alpha|\ll\omega$ of a hyperbolic system, where it is of the order of $1/\omega$. One should note that the threshold becomes independent on the Planck constant, if the Lindblad equation is appropriately scaled.

The Fourier transform of the exact solution, the chord function, is even simpler. This is the product of two terms: one is just the non-Hamiltonian classical evolution of the initial chord function with dissipation and the other is again a Gaussian, but with narrowing width. This leads to a simple formula for $\mathrm{Tr}\hat{\rho}^2$ as an average of a shrinking Gaussian, where the probability distribution used to calculate the mean is just the square modulus of the initial chord function. For long times, we use the normalization condition that the chord function is unity at the origin to derive simple rules for the asymptotic growth $\mathrm{Tr}\hat{\rho}^2$ for a general quadratic Hamiltonian. This decays exponentially fast in a time which is of the same order as the positivity threshold. This result is compatible with the arguments presented by Zurek and Paz [20] for exponential growth of linear entropy for chaotic systems. These are classically characterized by local hyperbolicity, where the Lyapunov exponent describes the average effected by a typical orbit that approaches many hyperbolic points. In contrast, the hyperbolic quadratic Hamiltonian defines a linear classical motion, but both will exponentially stretch the Wigner function. Of course, in a chaotic system, the result must be analyzed more deeply since the phase space volume remains finite, which leads to a saturation of entropy even with no dissipation.

We finally point out a very simple inversion formula which allows one to retrieve the initial state of the system, which seems a very transparent way to deal with the topics of quantum state reconstruction.

ACKNOWLEDGMENTS

We thank Luiz Davidovich, Ruinet Matos Filho, and Fabricio Toscano for helpful discussions. We acknowledge financial support from Faperj, CNPq, Pronex, and the Instituto do Milênio for Quantum Information.

APPENDIX A: LINDBLAD EQUATION IN THE CHORD SPACE

From the definition (4) of the chord transform,

$$\tilde{W}(\boldsymbol{\xi})=\frac{1}{2\pi\hbar}\int d\mathbf{x}\exp\left(-\frac{i}{\hbar}\boldsymbol{\xi}\wedge\mathbf{x}\right)W(\mathbf{x}), \tag{A1}$$

applied to the derivatives of W and the product of W with q or p, we set the following transformation rules by using integration by parts:

$$W\rightarrow\tilde{W},$$

$$\frac{\partial W}{\partial\mathbf{x}}\rightarrow\frac{i}{\hbar}\mathbf{J}\boldsymbol{\xi}\tilde{W},$$

$$\mathbf{x}W\rightarrow-\frac{\hbar}{i}\mathbf{J}\frac{\partial\tilde{W}}{\partial\boldsymbol{\xi}},$$

$$\mathbf{x}\cdot\frac{\partial W}{\partial\mathbf{x}}\rightarrow-2\tilde{W}-\boldsymbol{\xi}\cdot\frac{\partial\tilde{W}}{\partial\boldsymbol{\xi}}. \tag{A2}$$

By applying these rules on the following Poisson bracket:

$$\{H(\mathbf{x}),W(\mathbf{x})\}=2(\mathbf{H}_{12}p+\mathbf{H}_{22}q)\frac{\partial W}{\partial p}(\mathbf{x})$$
$$-2(\mathbf{H}_{11}p+\mathbf{H}_{12}q)\frac{\partial W}{\partial q}(\mathbf{x}), \tag{A3}$$

we get

$$\{H(\mathbf{x}),W(\mathbf{x})\}\rightarrow2(\mathbf{H}_{12}\xi_p+\mathbf{H}_{22}\xi_q)\frac{\partial\tilde{W}}{\partial\xi_p}(\mathbf{x})$$
$$-2(\mathbf{H}_{11}\xi_p+\mathbf{H}_{12}\xi_q)\frac{\partial\tilde{W}}{\partial\xi_q}(\mathbf{x}), \tag{A4}$$

that is,

$$\{H(\mathbf{x}),W(\mathbf{x})\}\rightarrow\{H(\boldsymbol{\xi}),\tilde{W}(\boldsymbol{\xi})\}. \tag{A5}$$

In the same way we get

$$\lambda^2\frac{\partial^2W}{\partial q^2}(\mathbf{x})+\mu^2\frac{\partial^2W}{\partial p^2}(\mathbf{x})-2\lambda\mu\frac{\partial^2W}{\partial p\,\partial q}(\mathbf{x})\rightarrow-\frac{1}{\hbar^2}(\mathbf{l}\cdot\boldsymbol{\xi})^2\tilde{W}, \tag{A6}$$

hence the final equation (10) for $\tilde{W}$, with the help of the last line of Eq. (A2).

APPENDIX B: ZEROES OF THE HUSIMI FUNCTION

Defining the complex variable $z(\mathbf{x})=(q+ip)/\sqrt{2\hbar}$, we may express the coherent state $|z\rangle$ [16] as

$$|z\rangle=e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle, \tag{B1}$$

where $|n\rangle$ are the eigenstates of the harmonic oscillator. Hence, the coherent state representation of any pure state $|\psi\rangle$ can be expressed as

$$\langle z|\psi\rangle=e^{-|z|^2/2}F(z^*), \tag{B2}$$

where

$$F(z)=\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\langle n|\psi\rangle \tag{B3}$$

is an entire function, known as the Bargmann function [22]. Since the Husimi function is the square modulus of the coherent state representation, we obtain

$$Q(\mathbf{x})=|\langle z(\mathbf{x})|\psi\rangle|^2=e^{-|z|^2}|F(z)|^2. \tag{B4}$$

Thus the zeroes of the Husimi function coincide with the zeroes of an analytic function. Indeed, we may define the state $|\psi\rangle$ by its Bargmann representation $F(z)$. An important *a priori* restriction is that $F(z)$ is at most of order $r=2$.

To see this, recall that $F(z)$ is of finite order if

$$|F(z)|<e^{|z|^\mu} \tag{B5}$$

for all sufficiently large $|z|$ and the order of this function is $r=\inf\mu$ for which Eq. (B5) holds. But if we use the fact that $|\langle z|\psi\rangle|^2\leqslant 1$ in Eq. (B4), we obtain

$$|F(z)|\leqslant e^{|z|^2/2}<e^{|z|^2}, \tag{B6}$$

so $r\leqslant 2$.

We now make use of the following

Theorem [23]: If $F(z)$ is an entire function of finite order with no zeroes on the plane, then its order is necessarily an integer and $F(z)=e^{P_r(z)}$, where $P_r(z)$ is a polynomial of order r.

In the case of the Bargmann function, then $P_r(z)$ is at most of second order and hence $Q(\mathbf{x})$ given by Eq. (B4) must be a Gaussian in p and q if it represents a normalized function.

Thus, only Gaussian Husimi functions have no zeroes in the phase plane. To a great extent, positions of the isolated zeroes of the Husimi function also restrict the class of admissible pure states through the factorization theorems of Weierstrass and Hadamard [23]. The characterization of the pure states as chaotic or regular by the pattern of zeroes has been extensively studied for the case where the phase space is a torus, because the restriction is then more severe [24].

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