Critical dynamics of the kinetic Glauber-Ising model on hierarchical lattices

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The critical dynamics of the kinetic Glauber-Ising model is studied on a family of the diamond-type hierarchical lattices with various branches. By carrying out the time-dependent real-space renormalization-group transformation to the master equation of the systems considered, the dynamic exponent is calculated. We find that the dynamic exponent depends on fractal dimension d_f or the branch number m in a generator, and that it increases with the increase of d_f or m. We notice that for the case of $m=1$ (one-dimensional spin chain, $d_f=1$) our result $z=2$ is the same as the exact result obtained by Glauber, and for the case of $m=2$ (the simplest one in the diamond-type hierarchical lattices, $d_f=2$) the exponent $z=2.626$ is higher than those of the two-dimensional regular lattice and the triangular lattice.

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I. INTRODUCTION

In the field of dynamical critical phenomena of spin systems, our main interest is in the study of the long time behavior of the system which is most strongly affected by critical fluctuations, and an important task is to investigate the scaling behavior of the relaxation time τ of the system. Both theory and experiment show that the relaxation time diverge near the critical point, i.e., there is the phenomenon called critical slowing down. Based on conventional theory [1,2], the relation of τ with ξ can be characterized as $\tau \sim \xi^z$, where ξ is the correlation length ($\xi \to \infty$ when temperature approaches to its critical value) and z is the dynamic exponent.

The time relaxation of spins on both regular (translationally symmetric) lattices and fractal lattices have been extensively investigated in the past three decades. Except the one-dimensional kinetic Glauber-Ising model and the kinetic Gaussian model on regular lattices [3,4], other models (or other systems) have not been exactly solved so far, such as two and three-dimensional kinetic Ising models, the kinetic Potts model, the *XY* model, and the Heisenberg model, etc., or some spin models on other complex structures, e.g., fractals, percolations, and complex networks [5–7], etc. Some approximate methods such as the time-dependent renormalization group (TDRG) [8–11], high-temperature series expansion [12,13], ε expansion [14,15], damage spreading [16,17], Monte Carlo simulations [18–20], and Monte Carlo renormalization-group (RG) calculations [21,22], have been used to study these systems. Among these methods, the time-dependent real-space RG transformation proposed by Achiam and Kosterlitz [8] is a very useful method, especially for fractal systems, because of the self-similar characteristic of fractals. In this aspect, although many works have been done that contain Koch curves [23,24], Sierpinski gaskets [25,26], Sierpinacuteski carpets [27], and other fractal lattices [28,29], to our knowledge no result of the kinetic Glauber-Ising model has been reported so far for diamond-type hierarchical (DH) lattices.

In this paper, by performing the TDRG to the master equation, we study the kinetic Glauber-Ising model on a family of the DH lattices. As a physical problem, the DH fractal is explicitly considered by Kaufman and Griffiths in the 1980s [30–32]. Comparing with other fractal lattices, the DH lattices have the following two main features: (a) they are much more inhomogeneous, i.e., they have a much lower symmetry than other fractals, so they can provide insights into other low-symmetry problems, such as random magnets, surfaces, and the like, and (b) the fractal contains sites with different coordination numbers, and the coordination number q_i of site i is associated with the generating stage of the fractal. The latter leads to the difficulty of solving the RG recursion relations if we only choose the magnetic perturbation $\Phi(\mathbf{h},\sigma)$ as the conventional choice, i.e.,

$$\Phi(t)=\Phi(\mathbf{h},\sigma)=1+\sum_i h_{q_i}(t)\sigma_i, \tag{1}$$

where $h_{q_i}(t)$ is the reduced magnetic field associated with the coordination number q_i of site i. In order to solve this problem, we assume that $h_{q_i}(t)$ and $h_{q_j}(t)$ satisfy a certain relation. Based on our assumption we get the RG recursion relations, and further calculate the dynamic exponent for the DH lattices with different branch numbers. We find that the dynamic exponent increases with an increase of the number of the branch in a generator. We compare our results with those of previous studies for corresponding regular lattices.

The contents of the remainder of the paper is as follows. In Sec. II the formulation of the TDRG is introduced. Section III gives the procedure of the TDRG on a simple DH lattice and calculates the dynamical exponent. The results of some other DH lattices are given in Sec. IV. Section V is a brief summary and discussion. Some of the more tedious calculations of Sec. III are given in the Appendixes.

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II. FORMULATION OF THE TDRG METHOD

For convenience in Secs. III and IV, we first summarize the TDRG method of the kinetic Glauber-Ising model on a fractal lattice following Achiam [8,9,23]. Let $\sigma_i(=\pm 1)$ be the Ising spin variable located on the ith site of the fractal lattice. If we assume that there are only the nearest-neighbor (NN) spin interactions, the reduced Hamiltonian of the system is written as

$$-\beta H = K\sum_{\langle i,j\rangle} \sigma_i\sigma_j, \tag{2}$$

where $K=J/k_BT$ denotes the interaction between two NN spins σ_i and σ_j, J is the NN exchange integral of spins, k_B is the Boltzmann constant, and T the absolute temperature; the summation is taken over nearest neighbors. We assume that the system is in a constrained equilibrium state when time $t<0$. At time $t=0$ the constraint is removed and then the system relaxes towards the equilibrium via an interaction with a heat bath. Using $P(\sigma,t)$ denotes the time-dependent distribution function of the spin configuration $\sigma \equiv(\sigma_1,\sigma_2,\ldots,\sigma_i,\ldots,\sigma_N)$. After infinite time the system will approach the equilibrium state which is characterized by the equilibrium distribution function,

$$P_e(\sigma) = \frac{1}{Z}\exp(-\beta H), \tag{3}$$

where $Z=\Sigma_\sigma \exp(-\beta H)$ is the partition function of the system. To all appearances, one has $\lim_{t\to\infty} P(\sigma,t)=P_e(\sigma)$. The dynamical model studied here was proposed by Glauber in which only single spin transition is allowed each unit time [3]. Based on Glauber dynamics, the time-dependent distribution function $P(\sigma,t)$ satisfies the master equation [23]

$$\tau_0\frac{d}{dt}P(\sigma,t) = -\sum_i (1-p_i)W_i(\sigma_i)P(\sigma,t)$$

$$= -\sum_i \tilde{L}_i P(\sigma,t) = -\sum_i L_i\Phi(\sigma,t), \tag{4}$$

where τ_0 is a bare time scale characterizing the coupling to a heat bath, $\tilde{L}_i\equiv(1-p_i)W_i(\sigma_i)$ is the Liouville operator, $\Phi(\sigma,t)\equiv P(\sigma,t)/P_e(\sigma)$ is the deviation from equilibrium, p_i is a spin-flip operator which is defined by $p_i f(\sigma_1,\sigma_2,\ldots,\sigma_i,\ldots,\sigma_N)=f(\sigma_1,\sigma_2,\ldots,-\sigma_i,\ldots,\sigma_N)$, and $W_i(\sigma_i)$ is the transition probability, from spin σ_i to $-\sigma_i$, of spin σ_i. $W_i(\sigma_i)$ is subject to the detailed balance condition $\tilde{L}_i P_e(\sigma)=0$ which ensures the ergodicity of the system. The above relation cannot determine the form of $W_i(\sigma_i)$ uniquely. Traditionally, it is chosen as the form

$$W_i(\sigma_i) = \left[\frac{P_e(-\sigma_i)}{P_e(\sigma_i)}\right]^{1/2} = \exp\left(-K\sigma_i\sum_j \sigma_j\right), \tag{5}$$

where Σ_j denotes the summation for all nearest neighbors of site i.

To study the critical slowing down, we limit that the system relaxes to the equilibrium by an infinitely small perturbation. In previous papers by other authors two forms of perturbations are studied: one is magneticlike and another is energylike. In this paper we only consider the case of the magneticlike perturbation from equilibrium. The deviation $\Phi(t)$ is chosen as the expression (1) so that it includes a set of operators which form an invariant subset in parameter space under TDRG transformation.

In order to get the dynamic exponent, we perform the time-dependent real-space RG transformation to the master equation (4), which consists of two steps. First, a real-space RG transformation by rescaling the space length, i.e., $x\to x'=bx$, is carried out. Based on the structure features of these fractals, we adopt the decimation RG transformation [33,34]. After an iteration of the RG transformation, some sites vanish and others are retained. This stage can be carried out by the following procedure: using the operator $T(\mu,\sigma)\equiv\Pi_i\delta(\mu_i-\sigma_i)$ to multiply the master equation, where μ_i is the new spin variable and i's are those sites retained after an iteration of the RG transformation, and then a summation for all spins σ is performed. This procedure can be described as

$$\mathcal{R}[f(\sigma)] = \sum_\sigma T(\mu,\sigma)f(\sigma) = f(\mu), \tag{6}$$

where $f(\sigma)$ is a function of $\{\sigma_i\}$. After the above process the master equation (4) becomes

$$\tau_0\frac{d}{dt}\mathcal{R}[P(\sigma,t)] = -\mathcal{R}\left[\sum_i L_i\Phi(\sigma,t)\right]. \tag{7}$$

Using the expression (1) it can be easily expressed as

$$\tau_0\frac{d}{dt}\mathcal{R}\left[\sum_i h_{q_i}(t)\sigma_i P_e(\sigma)\right] = -2\mathcal{R}\left[\sum_i h_{q_i}(t)P^{(i)}\sigma_i\right] \tag{8}$$

with $P^{(i)}\equiv W_i P_e(\sigma)$. From the expressions (5) and (3) we can see that $P^{(i)}$ is independent of spin σ_i. On the left-hand side, after the above transformation $P_e(\sigma)$ and $h_{q_i}(t)$ become $P'_e(\mu)$ and $h'_{q_i}(t)$, respectively. Obviously, it is the same as the standard static RG transformation. In the invariant subspace of the parameter space $(K,\mathbf{h})$, the transformation gives the recursion relations

$$K' = RK, \quad \mathbf{h}' = \mathbf{\Lambda}\cdot\mathbf{h}, \tag{9}$$

where K' is a new interaction,

$$\mathbf{h} = \begin{pmatrix} h_{q_1} \\ h_{q_2} \\ \vdots \end{pmatrix}, \quad \mathbf{h}' = \begin{pmatrix} h'_{q_1} \\ h'_{q_2} \\ \vdots \end{pmatrix},$$

and $\mathbf{\Lambda}$ is a transformation matrix. On the right-hand side, Eq. (8) transforms $P^{(i)}(\sigma)$ and $h_{q_i}(t)$ into $P'^{(i)}(\mu)$ and $h''_{q_i}(t)$, respectively, and gives the recursion relation

$$\mathbf{h}'' = \mathbf{\Omega}\cdot\mathbf{h}, \tag{10}$$

where

$$\mathbf{h}''=\begin{pmatrix} h''_{q_1} \\ h''_{q_2} \\ \vdots \end{pmatrix},$$

and $\mathbf{\Omega}$ is a transformation matrix as well.

The second step is the rescaling of the time. In this stage the invariant form of the master equation is restored by presenting $\mathbf{h}''$ in terms of $\mathbf{h}'$, and the time rescaling $\tau_0'=b^{-z}\tau_0$ is performed. The eigenvalues of $\mathbf{\Omega\Lambda}^{-1}$ are the time rescaling factor. Based on Ref. [35], if the matrices $\mathbf{\Lambda}$ and $\mathbf{\Omega}$ are commuted, one has

$$\frac{\lambda_{\max}}{\omega_{\max}}=b^z, \tag{11}$$

where $\omega_{\max}$ and $\lambda_{\max}$ are the largest eigenvalues of matrices $\mathbf{\Omega}$ and $\mathbf{\Lambda}$, respectively. Contrarily, the dynamic exponent is given by

$$\frac{\lambda_{\max}}{\omega_{\min}}=b^z, \tag{12}$$

where $\omega_{\min}$ is the smallest eigenvalues of matrix $\mathbf{\Omega}$, if the matrices $\mathbf{\Lambda}$ and $\mathbf{\Omega}$ are not commuted.

III. SIMPLE HIERARCHICAL LATTICE

We apply the above method to a simple hierarchical lattice which is a typical fractal and can be constructed by an iterative manner (see Fig. 1). The initiator is a two-point lattice joined by a single bond (construction stage $n=0$). Then the initiator is replaced with the generator which contains two branches of two bonds ($n=1$). Replacing every single bond on the generator itself, we get the second stage of the lattice. This procedure is infinitely repeated until an infinite lattice is formed. The most basic geometrical feature of fractals is the fractal dimensionality which is defined as

$$d_f=\frac{\ln N}{\ln b}. \tag{13}$$

Here N is the total mass (or the total volume) in the generator and b the rescaling factor. To all appearances, for the DH lattice considered here, there is $N=mb$, where the branch number in a generator is $m=2$ and the rescaling factor is $b=2$ (it equals the number of bonds in one branch). Thus, one gets the fractal dimensionality $d_f=\ln(mb)/\ln b=2$. Moreover, another parameter describing the geometrical feature is the order of ramification R, here $R=\infty$.

We focus on the n-stage lattice, which can be treated as the group of many basic cells (generators). According to the expression (2) the reduced Hamiltonian of this system can be presented as the summation of the Hamiltonians of the basic cells, i.e.,

$$\mathcal{H}=\sum_{\alpha}\mathcal{H}_{cell}^{\alpha}, \tag{14}$$

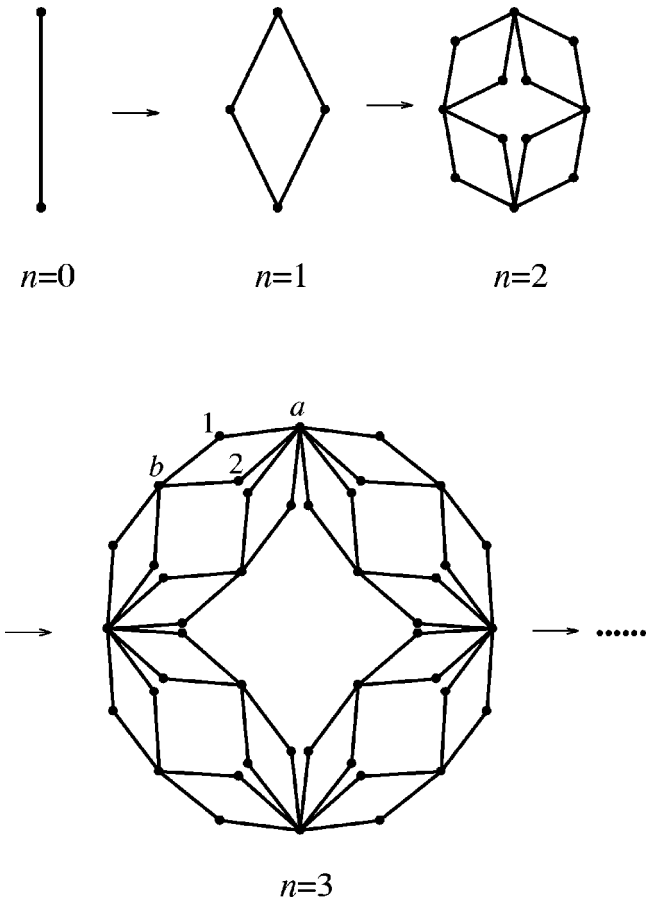

FIG. 1. The procedure of the construction of a simple hierarchical lattice.

where $\mathcal{H}_{cell}^{\alpha}=K(\sigma_a^{\alpha}+\sigma_b^{\alpha})(\sigma_1^{\alpha}+\sigma_2^{\alpha})$, α denotes the αth cell with sites a, b, 1, and 2 in the lattice (see Fig. 2). Then the equilibrium distribution function of the system is expressed as

$$P_e(\sigma)=\frac{1}{Z}\prod_{\alpha}\exp[K(\sigma_1^{\alpha}+\sigma_2^{\alpha})(\sigma_a^{\alpha}+\sigma_b^{\alpha})], \tag{15}$$

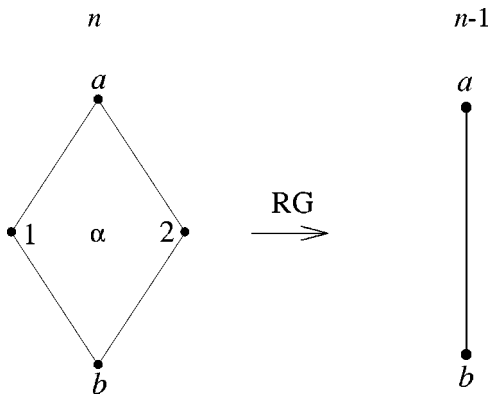

FIG. 2. The procedure of the RG transformation for the cell (or generator) α belonging to the n-stage lattice. The coordination numbers of the cell are $q_a=2^n$, $q_b=4$, and $q_1=q_2=2$. After a step of the RG transformation the cell is transformed to the bond ab with $q_a'=2^{n-1}$ and $q_b'=2$.

and the perturbation from equilibrium can be specifically written as in the form

$$\Phi(t)=1+\sum_{\alpha}\left(\frac{h_{q_a}}{2^{n-1}}\sigma_a^{\alpha}+\frac{h_{q_b}}{2}\sigma_b^{\alpha}+h_2(\sigma_1^{\alpha}+\sigma_2^{\alpha})\right), \tag{16}$$

where the factors $1/2^{n-1}$ and $1/2$ in terms $h_{q_a}\sigma_a^{\alpha}$ and $h_{q_b}\sigma_b^{\alpha}$ come from the fact that sites a and b, respectively, are in the 2^{n-1} cells and 2 cells.

Now let us carry out the RG transformation to the Eq. (8) for this lattice. First, we analyze the RG transformation of the magnetic fields h_i's. For the generator α in Fig. 2 we can easily see $q_a=2^n$, $q_b=4$, and $q_1=q_2=2$. Under a step of the RG transformation the lattice is transformed from the n stage to the $n-1$ stage by tracing over the two internal spins 1 and 2 of the generator, and the number of spins becomes N' from N. Noting that in the above RG transformation the coordination numbers of the sites a and b have changed, i.e., they become $q_a'=2^{n-1}$ and $q_b'=2$, respectively. Thus we can obtain the RG transformation relations as the form

$$h'_{2^{n-1}}=f(h_{2^n},h_4,h_2),\quad h_2'=g(h_{2^n},h_4,h_2).$$

Unfortunately, from the relations we cannot obtained the fixed point of the RG transformation. We find that h_{q_a}, h_{q_b}, and h_{q_1} (or h_{q_2}) cannot compose an invariant subspace. In order to overcome the difficulty, we assume that the magnetic field h_{q_i} on site i is proportional to the coordination number q_i, i.e., h_{q_i} and h_{q_j} satisfy

$$\frac{h_{q_i}}{h_{q_j}}=\frac{q_i}{q_j}. \tag{17}$$

Using this assumption the perturbation from equilibrium (16) can be reduced as

$$\Phi(t)=1+h_2(t)\sum_{\alpha}(\sigma_a^{\alpha}+\sigma_b^{\alpha}+\sigma_1^{\alpha}+\sigma_2^{\alpha}), \tag{18}$$

and further we can calculate the dynamics exponent z by performing the TDRG to Eq. (8).

The decimation RG transformation of the left-hand side of Eq. (8) can be demonstrated as

$$\begin{aligned}\frac{d}{dt}\mathcal{R}P(\sigma,t)&=\frac{d}{dt}\mathcal{R}\left[\sum_i h_{q_i}(t)\sigma_i P_e(\sigma)\right]\\&=\frac{d}{dt}h_2(t)\mathcal{R}\left[P_e(\sigma)\sum_{\alpha}(\sigma_a^{\alpha}+\sigma_b^{\alpha}+\sigma_1^{\alpha}+\sigma_2^{\alpha})\right].\end{aligned} \tag{19}$$

In this case the operator $\mathcal{R}$ can be written as in the form

$$\begin{aligned}\mathcal{R}&=\sum_{\sigma}\prod_i\delta(\mu_i-\sigma_i)\\&=\prod_{\beta}\sum_{\sigma_1^{\beta},\sigma_2^{\beta}}\sum_{\sigma_a^{\beta},\sigma_b^{\beta}}\delta(\mu_a^{\beta}-\sigma_a^{\beta})\delta(\mu_b^{\beta}-\sigma_b^{\beta}).\end{aligned} \tag{20}$$

By the decimation RG transformation to the equilibrium distribution function $P_e(\sigma)$, we can easily get

$$\mathcal{R}P_e(\sigma)=P_e'(\mu), \tag{21}$$

where $P_e'(\mu)$ is the equilibrium distribution function of the $(n-1)$-stage system,

$$P_e'(\mu)=\frac{1}{Z'}\exp\left[\sum_{\beta}(K'\mu_a^{\beta}\mu_b^{\beta})\right]. \tag{22}$$

Z' is the corresponding partition function of the system,

$$Z'=Z/A^{N'},\quad A=2^2\frac{1+\tanh^2 K}{1-\tanh^2 K}, \tag{23}$$

and the new parameter K' satisfies the relation

$$\tanh\left(\frac{K'}{2}\right)=\tanh^2 K, \tag{24}$$

or $K'=\ln\cosh 2K$, which has been derived by Yang in the previous article [36]. Making use of the expression (20) we get the RG transformations of $P_e(\sigma)\sigma_{a,b}^{\alpha}$ and $P_e(\sigma)\sigma_{1,2}^{\alpha}$ as (see Appendix A)

$$\mathcal{R}[P_e(\sigma)\sigma_{a,b}^{\alpha}]=P_e'(\mu)\mu_{a,b}^{\alpha}, \tag{25}$$

and

$$\begin{aligned}\mathcal{R}[P_e(\sigma)\sigma_{1,2}^{\alpha}]=&\frac{2^2(1+\tanh^2 K)\cosh^4 K\tanh K}{A\cosh K'(1+\tanh K')}\\&\times(\mu_a^{\alpha}+\mu_b^{\alpha})P_e'(\mu),\end{aligned} \tag{26}$$

where $\sigma_{i,j}^{\alpha}$ denotes σ_i^{α} or σ_j^{α}. Substituting (25) and (26) into (19) and noting the recursion relation (24), one gets the left-hand side of the master equation (8)

$$\begin{aligned}\frac{d}{dt}\mathcal{R}P(\sigma,t)=&\frac{d}{dt}h_2(t)\sum_{\alpha}\left(1+\frac{2\tanh K}{1+\tanh^2 K}\right)\\&\times(\mu_a^{\alpha}+\mu_b^{\alpha})P_e'(\mu).\end{aligned} \tag{27}$$

Based on the idea of the decimation RG transformation, it can be also written as in the form

$$\begin{aligned}\frac{d}{dt}\mathcal{R}P(\sigma,t)&=\frac{d}{dt}P_e'(\mu)\sum_{\alpha}\left(\frac{h'_{q'_{\alpha}}}{2^{n-1}}\mu_a^{\alpha}+\frac{h'_{q'_b}}{2}\mu_b^{\alpha}\right)\\&=\frac{d}{dt}\frac{h_2'(t)}{2}P_e'(\mu)\sum_{\alpha}(\mu_a^{\alpha}+\mu_b^{\alpha}).\end{aligned} \tag{28}$$

Comparing expressions (27) with (28) one gets the recursion relation of $h_2(t)$ as

$$h_2'(t)=2(1+\tanh 2K)h_2(t). \tag{29}$$

From the above we see that under the assumption (17) the matrix Λ only contains a single element $2(1+\tanh 2K)$. Thus, at the critical point the eigenvalue of the transformation matrix Λ of $h_2(t)$ is

$$\lambda=2(1+\tanh 2K^*). \tag{30}$$

The right-hand side of the master equation (8) is the RG transformation of the summation of the terms $h_{q_i}P^{(i)}\sigma_i$. For this system the summation $\Sigma_i h_{q_i}P^{(i)}\sigma_i$ can be written as the summation of $\mathcal{G}_\alpha$ for all the cells, i.e., $\Sigma_i h_{q_i}P^{(i)}\sigma_i=\Sigma_\alpha\mathcal{G}_\alpha$, where

$$\mathcal{G}_\alpha=\frac{h_{q_a}}{2^{n-1}}P_\alpha^{(a)}\sigma_a^\alpha+\frac{h_{qb}}{2}P_\alpha^{(b)}\sigma_b^\alpha+h_2(\sigma_1^\alpha P_\alpha^{(1)}+\sigma_2^\alpha P_\alpha^{(2)}). \tag{31}$$

Noting that $P^{(i)}\equiv W_iP_e(\sigma)$ is independent of spin σ_i, then we have $\mathcal{R}[\sigma_1^\alpha P_\alpha^{(1)}]=\mathcal{R}[\sigma_2^\alpha P_\alpha^{(2)}]=0$. Otherwise, by the calculation we get the RG transformations of $P_\alpha^{(a)}\sigma_a^\alpha$ and $P_\alpha^{(b)}\sigma_b^\alpha$ as follows (see Appendix B):

$$2\mathcal{R}[P_\alpha^{(a)}\sigma_a^\alpha]=P_\alpha'^{(a)}\mu_a^\alpha A^{-2n-1}(2\cosh K)^{2n}, \tag{32}$$

$$2\mathcal{R}[P_\alpha^{(b)}\sigma_b^\alpha]=P_\alpha'^{(b)}\mu_b^\alpha A^{-2}(2\cosh K)^4, \tag{33}$$

where $P_\alpha'^{(a)}=W_aP_e'(\mu)$ and $P_\alpha'^{(b)}=W_bP_e'(\mu)$. Therefore, Eqs. (31), (32), and (33) give the following result of the RG transformation:

$$\mathcal{R}\left[\frac{h_{q_a}(t)}{2^{n-1}}P_\alpha^{(a)}\sigma_a^\alpha+\frac{h_{q_b}(t)}{2}P_\alpha^{(b)}\sigma_b^\alpha\right]=\frac{h''_{q_a}(t)}{2^{n-1}}\mu_a^\alpha P_\alpha'^{(a)}+\frac{h''_{q_b}(t)}{2}\mu_b^\alpha P_\alpha'^{(b)}, \tag{34}$$

where

$$h''_{q_a}(t)=\frac{2(2\cosh K)^{2^n}}{A^{2^{n-1}}}h_{q_a},\quad h''_{q_b}(t)=\frac{2(2\cosh K)^4}{A^2}h_{q_b}. \tag{35}$$

The relations can also be written as the matrix form (10), where the matrices $\mathbf{h}$, $\mathbf{h}''$, and $\mathbf{\Omega}$ are, respectively,

$$\mathbf{h}=\begin{pmatrix}h_{q_a}\\ h_{q_b}\end{pmatrix},\mathbf{h}''=\begin{pmatrix}h''_{q_a}\\ h''_{q_b}\end{pmatrix},$$

and

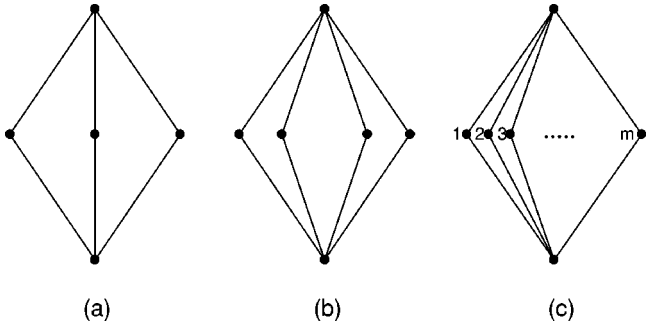

FIG. 3. Some examples of the basic cell of a family of the DH lattices. (a) $m=2$; (b) $m=4$; (c) m is arbitrary.

$$\mathbf{\Omega}=\begin{pmatrix}\dfrac{1}{(1+\tanh^2 K)^{2^{n-1}}} & 0\\ 0 & \dfrac{1}{(1+\tanh^2 K)^2}\end{pmatrix}.$$

Noting $1+\tanh^2 K>1$ and $2^{n-1}>2$ (for $n>2$), we can easily see that at the critical point the largest eigenvalue of the matrix $\mathbf{\Omega}$ is

$$\omega_{\max}=\frac{1}{(1+\tanh^2 K^*)^2}. \tag{36}$$

Thus, based on the expressions (11), (30), and (36), we derive the dynamical exponent of the system as

$$z=\frac{\ln 2}{\ln b}+\frac{\ln[(1+\tanh^2 K^*)(1+\tanh K^*)^2]}{\ln b}, \tag{37}$$

where $b=2$ is the length rescaling factor associated with the RG transformation. From Eq. (24) we can obtain $K^*=0.609$. Therefore we get $z=2.626$.

IV. RESULTS OF OTHER HIERARCHICAL LATTICES

In this section we give the results of the kinetic Glauber-Ising model on some m-branch DH lattices, where m is an arbitrary natural number. These lattices and the simplest DH lattice considered in Sec. III are similar on generating method. The only difference is here the generator contains m branches, not only two branches. The number of bonds in one branch equals 2 as well. Examples of the basic cell of these lattices are plotted in Fig. 3. According to the definition (13) the fractal dimensionality of this family fractals is $d_f=1+\ln m/\ln 2$, and the order of ramification is $R=\infty$ as well. $m=1$ and $m=2$, respectively, correspond to the one-dimensional lattice and the simplest DH lattice studied in the above section. As in Sec. III, the Hamiltonian of the m-branch DH system can be expressed as in the form

$$-\beta H=K\sum_\alpha\sum_{i=1}^m\sigma_i^\alpha(\sigma_a^\alpha+\sigma_b^\alpha), \tag{38}$$

where σ_i^α $(i=1,2,...,m)$ are the spins of the internal sites, 1, 2,..., and m, of the cell α in the lattice, the summation Σ_α

TABLE I. The values of K at the critical point and the dynamical exponents z vs values of the fractal dimension d_f and the branch number m.

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
d_f	1	2	2.585	3	3.322	3.585	3.807	4	4.170	4.322	4.459	4.585	4.700	4.807	4.907
K^*	∞	0.609	0.361	0.261	0.206	0.170	0.145	0.126	0.112	0.101	0.091	0.084	0.077	0.072	0.067
z	2	2.626	2.769	2.930	3.087	3.235	3.372	3.500	3.618	3.728	3.831	3.927	4.017	4.103	4.183

goes over all cells of the lattice. The equilibrium distribution function of this system is written as

$$P_e(\sigma)=\frac{1}{Z}\prod_{\alpha}\ \exp\left[K\sum_{i=1}^{m}\ \sigma_i^{\alpha}(\sigma_a^{\alpha}+\sigma_b^{\alpha})\right],$$

and the perturbation from equilibrium is expressed as in the form

$$\Phi(t)=1+\sum_{\alpha}\ \left(\frac{h_{q_a}(t)}{m^{n-1}}\sigma_a^{\alpha}+\frac{h_{q_b}(t)}{2}\sigma_b^{\alpha}+h_2(t)\sum_{i=1}^{m}\ \sigma_i^{\alpha}\right).$$

Still consider the n-stage lattice. With the same method as in Sec. III, we can carry out the decimation RG transformation to Eq. (8). We should notice that in a cell of the n-stage lattice the coordination numbers of the sites a and b are $q_a=m^n$ and $q_b=2m$, respectively, and after a step of the renormalization transformation they become $q_a=m^{n-1}$ and $q_b=2$, respectively.

Using relation (17), from the left-hand side of Eq. (8) we can get $h_2'(t)=\lambda h_2(t)$ with

$$\lambda=2m\left[\frac{1}{2}+F(K,K',m)\right], \tag{39}$$

$$F(K,K',m)=\frac{\tanh K}{(1+\tanh K')\cosh K'}\ \frac{(1+\tanh^2 K)^{(m/2)-1}}{(1-\tanh^2 K)^{m/2}}. \tag{39}$$

For the right-hand side of Eq. (8), we have

$$\mathcal{R}\left[\sum_i\ (1-p_i)W_iP(\sigma,t)\right]$$
$$=2\sum_{\alpha}\ \left[\frac{h''_{q_a}(t)}{m^{n-1}}\mu_a^{\alpha}P'^{(a)}_{\alpha}+\frac{h''_{q_b}(t)}{2}\mu_b^{\alpha}P'^{(b)}_{\alpha}\right],$$

where $h''_{q_a}(t)=A^{-m^{n-1}}(2\cosh K)^{m^n}h_{q_a}$, $h''_{q_b}(t)=A^{-2}(2\cosh K)^{2m}h_{q_b}$. The two relations can be written as in the matrix form (10), but here

$$\mathbf{\Omega}=\begin{pmatrix}\frac{1}{(1+\tanh^2 K)^{m^{n-1}}} & 0\\ 0 & \frac{1}{(1+\tanh^2 K)^{m}}\end{pmatrix}.$$

Obviously, the largest eigenvalue of the matrix is

$$\omega_{\max}=\frac{1}{(1+\tanh^2 K)^m}. \tag{40}$$

For calculating the dynamical exponent we also need to know the critical point of the systems. Carrying out the decimation RG transformation to $P_e(\sigma)$, one can obtain the recursion relation

$$\tanh(K'/m)=\tanh^2 K \tag{41}$$

or $K'=m\ln\sqrt{\cosh 2K}$ [36]. By mean of Eqs. (41), (12), (39), and (40), the values of K at the critical point and the dynamical exponents for any value of m can be calculated, in principle. The numerical results for some values of m or the fractal dimension d_f of the lattice are listed as in Table I. It is clear that the dynamic exponent z increases with an increase of m (or fractal dimension d_f), which can be interpreted as that the relaxation time τ depends on the structure of the system, i.e., the more complicated the structure of the system is, the longer it takes to reach the equilibrium. We notice that the result is different from that of the kinetic Gaussian model, which gives the result $z=2$ for the family of the DH lattices [37].

V. SUMMARY AND DISCUSSION

The kinetic Glauber-Ising model has been investigated on a family of the DH lattices with different m, the branch number in a cell (or generator). In order to study the relaxation of the spins of these systems, we considered the magneticlike perturbation which results in a small deviation from equilibrium. For overcoming the difficulty of inhomogeneity in the structure of the lattices, it is assumed that the magnetic field h_{q_i} in the magneticlike perturbation is proportional to the coordination number q_i of site i. Using the assumption we have performed the TDRG transformation to the master equations, and have obtained the dynamic exponent z of these systems. We found that the dynamic exponent depends on the branch number m (or various fractal dimension d_f) in a generator, i.e., it increases with an increase of m.

We now compare our results with those of previous studies for regular lattices. For the lattice with $m=1$ ($d_f=d=1$), i.e., a one-dimensional chain, the result $z=2$ obtained here is the same as that, an exact result, of Glauber [3]. For the DH lattice with $m=2$ ($d_f=d=2$), our result is $z=2.626$. For a two-dimensional regular lattice, the results from other methods vary from $z=1.4$ to 2.7 [38]. Some results are given as follows.

(a) $z\simeq 2.7$ by the block-site TDRG method in the first

order cumulant expansion, on a two-dimensional regular lattice [8].

(b) $z=2.16$ TDRG method in the second order cumulant approximation (triangular lattice) [39].

(c) $z=2.16$ by Linke [40], $z=2.13$ by Williams [41], using the Monte Carlo simulation.

(d) $z=1.4$ the RG Monte Carlo method by Ma [21], and $z=2.126$ by Bausch [42].

We notice that our result of the DH lattice with $m=2$ is higher than those of the two-dimensional regular lattice or the triangular lattice, except $z\simeq 2.7$ by the TDRG method in the first order cumulant expansion. Noting the distinction of the $d_f=2$ DH lattice with a self-similarity structure (scaling invariance) and the two-dimensional regular lattice with translational symmetry, thus the difference of values of the dynamic exponent for the two kinds of lattices is reasonable. In fact, in the limit of thermodynamics there are some sites with infinite coordination numbers in the DH lattice, but the coordination numbers are all 4 in the two-dimensional regular lattice or 6 in the triangular lattice. Moreover, the DH lattice even may be inserted into a three-dimensional space, not only a two-dimensional space. The result also shows that the dynamic exponent depends on not only the dimension but also on the structure of the lattice.

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APPENDIX A: CALCULATING THE EXPRESSIONS (25) AND (26)

We calculate $\mathcal{R}[P_e(\sigma)\sigma_{a,b}^{\alpha}]$ and $\mathcal{R}[P_e(\sigma)\sigma_{1,2}^{\alpha}]$. Noting the expressions (15) and (20), $\mathcal{R}[P_e(\sigma)\sigma_{a,b}^{\alpha}]$ is calculated as follows:

$$\begin{aligned}
&\mathcal{R}[P_e(\sigma)\sigma_{a,b}^{\alpha}] \\
&\quad=\frac{1}{Z}\prod_{\beta}\left[\sum_{\sigma_1^{\beta},\sigma_2^{\beta}}\sum_{\sigma_a^{\beta},\sigma_b^{\beta}}\delta(\mu_a^{\beta}-\sigma_a^{\beta})\delta(\mu_b^{\beta}-\sigma_b^{\beta})\right] \\
&\qquad\times P_e(\sigma)\sigma_{a,b}^{\alpha} \\
&\quad=\frac{1}{Z}\prod_{\beta}\sum_{\sigma_1^{\beta},\sigma_2^{\beta}}\exp[K(\sigma_1^{\beta}+\sigma_2^{\beta})(\mu_a^{\beta}+\mu_b^{\beta})]\mu_{a,b}^{\alpha} \\
&\quad=\frac{1}{Z}A^{N'}\mu_{a,b}^{\alpha}\prod_{\beta}\exp(K'\mu_a^{\beta}\mu_b^{\beta}) \\
&\quad=\mu_{a,b}^{\alpha}P_e'(\mu),
\end{aligned}\tag{A1}$$

where $P_e'(\mu)$ and A have been given in expressions (22) and (23). $\mathcal{R}[P_e(\sigma)\sigma_1]$ can be calculated as

$$\begin{aligned}
\mathcal{R}[P_e(\sigma)\sigma_1^{\alpha}]&=\frac{1}{Z}\prod_{\beta}\sum_{\sigma_1^{\beta},\sigma_2^{\beta}}\exp[K(\mu_a^{\beta}+\mu_b^{\beta})(\sigma_1^{\beta}+\sigma_2^{\beta})]\sigma_1^{\alpha} \\
&=\frac{1}{Z}\prod_{\beta(\neq\alpha)}\left[\sum_{\sigma_1^{\beta},\sigma_2^{\beta}}\exp[K(\mu_a^{\beta}+\mu_b^{\beta})(\sigma_1^{\beta}+\sigma_2^{\beta})]\right]\sum_{\sigma_1^{\alpha},\sigma_2^{\alpha}}\exp[K(\mu_a^{\alpha}+\mu_b^{\alpha})(\sigma_1^{\alpha}+\sigma_2^{\alpha})]\sigma_1^{\alpha} \\
&=P'(\mu)A^{-1}\exp(-K'\mu_a^{\alpha}\mu_b^{\alpha})2^2(\mu_a^{\alpha}+\mu_b^{\alpha})(1+\mu_a^{\alpha}\mu_b^{\alpha}\tanh^2 K)\cosh^4 K\tanh K \\
&=2^2(\mu_a^{\alpha}+\mu_b^{\alpha})(1+\mu_a^{\alpha}\mu_b^{\alpha}\tanh^2 K)\cosh^4 K\tanh K\frac{P'(\mu)}{A\cosh K'(1+\mu_a^{\alpha}\mu_b^{\alpha}\tanh K')} \\
&=\frac{2^2(1+\tanh^2 K)\cosh^4 K\tanh K}{A\cosh K'(1+\tanh K')}(\mu_a^{\alpha}+\mu_b^{\alpha})P'(\mu).
\end{aligned}\tag{A2}$$

In the last step of the above calculation we have considered that it equals zero when $\mu_a^{\alpha}=-\mu_b^{\alpha}$, and not zero when $\mu_a^{\alpha}=\mu_b^{\alpha}$ $(\mu_a^{\alpha}\mu_b^{\alpha}=1)$.

APPENDIX B: CALCULATION OF THE EXPRESSIONS (32) AND (33)

Noting the form of the expression (5) of the transition probability W_i, thus $P^{(i)}\equiv W_iP_e(\sigma)$ is independent of spin σ_i. $\mathcal{R}[P_{\alpha}^{(a)}\sigma_a^{\alpha}]$ can be calculated as follows:

$$\mathcal{R}[P_\alpha^{(a)}\sigma_a^\alpha]=\prod_\beta\left[\sum_{\sigma_1^\beta,\sigma_2^\beta}\sum_{\sigma_a^\beta,\sigma_b^\beta}\delta(\mu_a^\beta-\sigma_a^\beta)\delta(\mu_b^\beta-\sigma_b^\beta)\right]P_\alpha^{(a)}\sigma_a^\alpha$$

$$=\prod_\beta\left[\sum_{\sigma_1^\beta,\sigma_2^\beta}\sum_{\sigma_a^\beta,\sigma_b^\beta}\delta(\mu_a^\beta-\sigma_a^\beta)\delta(\mu_b^\beta-\sigma_b^\beta)\right]\times\frac{1}{Z}\sigma_a^\alpha\prod_\beta\ \exp[K(\sigma_a^\beta+\sigma_b^\beta)(\sigma_1^\beta+\sigma_2^\beta)]\exp\left[-K\sigma_a^\alpha\sum_j\ \sigma_j\right]$$

$$=\frac{1}{Z}\mu_a^\alpha\prod_\beta\left[\sum_{\sigma_1^\beta,\sigma_2^\beta}\exp[K(\mu_a^\beta+\mu_b^\beta)(\sigma_1^\beta+\sigma_2^\beta)]\right]\exp\left[-K\mu_a^\alpha\sum_j\ \sigma_j\right],$$

where Σ_j denotes the summation for all nearest neighbors of site i. Therefore, we have

$$\mathcal{R}[P_\alpha^{(a)}\sigma_a^\alpha]=\frac{1}{Z}\mu_a^\alpha\prod_{\beta(\neq\alpha,\gamma,\ldots)}\left[\sum_{\sigma_1^\beta,\sigma_2^\beta}\exp[K(\mu_a^\beta+\mu_b^\beta)(\sigma_1^\beta+\sigma_2^\beta)]\right]\prod_{\alpha,\gamma,\ldots}\sum_{\sigma_1^\alpha,\sigma_2^\alpha}\exp[K\mu_a^\alpha(\sigma_1^\alpha+\sigma_2^\alpha)]$$

$$=\frac{1}{Z}\mu_a^\alpha\prod_{\beta(\neq\alpha,\gamma,\ldots)}[A\exp(K',\mu_a^\beta\mu_b^\beta)](2\cosh K)^{2n}$$

where the product does not contain α, γ,... (they denote 2^{n-1} cells). Substituting the value of A in the expression (23) into above expression we can easily get the result (32). In the same manner we can obtain the result (33).

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