Statistical mechanics of stochastic neural networks: Relationship between the self-consistent signal-to-noise analysis, Thouless-Anderson-Palmer equation, and replica symmetric calculation approaches

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We study the statistical mechanical aspects of stochastic analog neural network models for associative memory with correlation type learning. We take three approaches to derive the set of the order parameter equations for investigating statistical properties of retrieval states: the self-consistent signal-to-noise analysis (SCSNA), the Thouless-Anderson-Palmer (TAP) equation, and the replica symmetric calculation. On the basis of the cavity method the SCSNA can be generalized to deal with stochastic networks. We establish the close connection between the TAP equation and the SCSNA to elucidate the relationship between the Onsager reaction term of the TAP equation and the output proportional term of the SCSNA that appear in the expressions for the local fields.

I. INTRODUCTION

Statistical mechanical methods have turned out to be powerful for investigating neural network models of learning and memory [1–6]. The existence of certain energy functions plays an essential role for getting insights into behaviors of relevant macroscopic quantities, which are often called order parameters, by evaluating their minima based on the saddle point method. In the case of associative memory models the replica method [7] has been extensively employed to study statistical properties of retrieval states [1,2,5,8–12]. Amit, Gutfreund, and Sompolinsky [2,8,9] have applied the mean field model of spin glasses [7,13] to the Hopfield model [14] of Ising spin networks to obtain the storage capacity, which is given as a critical loading rate corresponding to the onset of a first kind phase transition, on the basis of evaluating the free energy. The statistical behavior of the Ising spin networks has alternatively been studied by means of the Thouless-Anderson-Palmer (TAP) equations [1,15–19], where the equilibrium states of the stochastic network can be described by deterministic equations. The transformation to the deterministic system can potentially save computational times required for numerically investigating equilibrium properties of the original stochastic systems of large size. The concept of the TAP equation has recently been gaining popularity among researchers working with communication theories from the viewpoint of engineering applications [20]. TAP equations are known to be derived by either a cavity method [1,19] or the Plefka method [16,18]. The TAP equation for the Hopfield model was first derived by means of a cavity method, but later turned out to be inconsistent with the result based on other methods [17–19].

In the case of neural network models where the energy concept does not make sense, however, the above-mentioned kind of statistical mechanical approaches cannot be applied. Neural networks in the real world have asymmetric synaptic connections that are incompatible with the energy concept. To cope with the difficulty in dealing with deterministic analog network models without the energy concept, Shiino and Fukai [21,22] devised a powerful method of the SCSNA, which is closely related to the cavity method. It has been applied to study the equilibrium properties of the associative memory of deterministic analog networks $x_i = F(\Sigma_{j \neq i} J_{ij} x_j)$, where J_{ij} represents synaptic coupling that may have certain types of asymmetric form [21,23,24] and the transfer function F is allowed to have an arbitrary shape [22–26]. Variants of the above analog networks such as oscillator networks based on phase oscillator models [27] have also been successfully studied using the self-consistent signal-to-noise analysis (SCSNA) to show that memory recall accompanied by synchronization of oscillators is of relevance in associative memory [28–33].

The SCSNA is a self-consistent method for properly renormalizing the so-called noise part due to interference of noncondensed patterns in the local field of a neuron $h_i = \Sigma_{j \neq i} J_{ij} x_j$: To extract pure noise obeying a Gaussian distribution one decomposes the local field in such a way that

$$\sum_{j \neq i} J_{ij} x_j = \xi_i^1 m + \sqrt{\alpha r} z_i + \Gamma_{\text{SCSNA}} x_i, \tag{1}$$

where $\xi_i^1 m$ represents a signal part involving the condensed pattern ξ_i^1, $\sqrt{\alpha r} z_i$ represents pure noise, and the last term, $\Gamma_{\text{SCSNA}} x_i$ represents the output proportional term, which are determined self-consistently. When the SCSNA is applied to a deterministic analog network with a monotonic transfer function, such as $\tanh \beta(\ )$ with β representing the analog gain, one obtains the same result as that by the replica method as are shown in the papers of Shiino and Fukai [10] and of Kuhn *et al.* [11,12]. Kuhn *et al.* studied stochastic analog networks with monotonic transfer functions using the replica calculations to deal with the deterministic limit.

We can apply the SCSNA to TAP equations of stochastic networks, which can be viewed as defining the equilibrium

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equations of certain deterministic analog networks, to obtain the set of order parameter equations. The TAP equation of the Ising spin network takes the following form with x_i ($i = 1, \ldots, N$) representing a thermal average of spins $\langle S_i \rangle$ [17,22]:

$$x_i = \tanh \beta \left(\sum_{j \neq i} J_{ij} x_j + \lambda_{\text{ONS}}^{(\text{ISING})} x_i \right), \qquad (2)$$

where J_{ij} is assumed to be given by the standard Hebb learning rule and

$$\lambda_{\text{ONS}}^{(\text{ISING})} = \frac{-\beta \alpha (1-q)}{1 - \beta(1-q)}. \qquad (3)$$

Shiino and Fukai [22] have shown that the application of the SCSNA to the above TAP equation leads to the same result as that of Amit, Geutfreund, and Sompolinsky (AGS). This procedure for obtaining the set of order parameter equations sheds light on the importance of the so-called Onsager reaction term that appears in the TAP equation (2): $\lambda_{\text{ONS}}^{(\text{ISING})} x_i$. The Onsager reaction term takes the form proportional to a term in the local field that would originate from the presence of self-couplings. The renormalized form of the local field (1) defined within the framework of the SCSNA also contains a similar term that is given by the output proportional term. In the case of the network of Eq. (2), the output proportional term $\Gamma_{\text{SCSNA}}^{(\text{ISING})} x_i$ has been found to equal the minus of the Onsager reaction term, so that they exactly cancel each other [22]:

$$\Gamma_{\text{SCSNA}}^{(\text{ISING})} = -\lambda_{\text{ONS}}^{(\text{ISING})}. \qquad (4)$$

This relation can be considered to give a statistical mechanical interpretation of the output proportional term derived from the SCSNA and also conversely the meaning of the Onsager reaction term from the viewpoint of a kind of signal-to-noise analysis. It will be worth noting that the distribution of the local fields of neurons of the Ising spin network $h_i = \sum_{j=1} J_{ij} x_j$ turns out to be non-Gaussian, while the distribution of the TAP local fields defined by $h_i^{\text{TAP}} = \sum_{j=i} J_{ij} x_j + \lambda_{\text{ONS}}^{(\text{ISING})} x_i$, which appears in Eq. (2), is indeed Gaussian owing to the above relation (4) together with Eq. (1) [22].

Effects of the output proportional term are pronounced in the case of deterministic analog networks with a nonmontonic transfer function [22–26,34,35] for which the existence of energy functions is not ensured. Use of nonmontonic transfer functions in associative memory neural networks has been shown to improve the network performances such that the storage capacity is increased beyond the well known value of 0.138 of the AGS under the correlation type learning rule [22–26,34,35]. In particular, when the degree of non-monotonicity of transfer functions is so large that the prescription of the Maxwell rule [10,22] is needed within the framework of the SCSNA, a phenomenon of super-retrieval has been shown to occur [22–26], where variance of pure noise vanishes and memory retrieval without errors is ensured for an extensive number of patterns. In this case the local field (1) consists only of a signal part and the output proportional term and the latter plays a crucial role for the occurrence of the super-retrieval [22,25]. Influences of sto-

chastic noise on the behavior of retrieval states of analog networks as well as of Ising spin networks that have non-monotonic transfer functions or transition probabilities are of interest from the viewpoint of studying the possibility of the occurrence of the super-retrieval phase. The study of such stochastic systems based on the assumption of Eq. (4), however, reveals that the super-retrieval state loses its stability [36,37].

A stochastic neural network of analog type was also studied previously for a coupled phase oscillator model using the SCSNA and the TAP equation approaches, and the relationship between the output proportional term and the Onsager reaction term in the stochastic local field was discussed [31] on the basis of the assumption for the relation analogous to Eq. (4).

The aims of the present paper are twofold. Firstly, we extend in a systematic manner the SCSNA that was devised for deterministic analog networks so as to cover stochastic analog network models to aim at obtaining its wider applicability. Secondly we want to study the relationship between the SCSNA, the TAP equation, and replica methods to get deep insights into the foundation of the SCSNA together with the treatment of the output proportional term as well as the Onsager reaction term for stochastic analog networks. We are particularly interested in investigating the problem of whether the relation analogous to Eq. (4) holds in more general situations.

This paper is organized as follows. In Sec. II we present a stochastic analog neural network model for associative memory together with a heuristic argument of the stochastic version of the original SCSNA and the TAP equations, which will be confirmed in later sections. In Sec. III we investigate the TAP equation of our model system by means of the cavity method, which requires two steps for determining the functional form of the TAP equation (pre-TAP equation) and the associated coefficient of the Onsager reaction term. In Sec. IV we systematically study the SCSNA of the stochastic version by making use of the results of Sec. III. We take slightly different two approaches for this purpose. First, applying the deterministic version of the SCSNA to the pre-TAP equation that can be viewed as a deterministic analog network yielding the same statistical properties as the stochastic network, we obtain the set of order parameter equations describing the retrieval states to confirm the validity of the treatment of Sec. II. The coefficient of the Onsager reaction term is shown to be determined in the course of this procedure to recover the result of Sec. III. In the second approach the same set of order parameter equations are derived from the full knowledge of the TAP equation. In Sec. V we present the replica symmetric analysis of our system to confirm again our theory of the SCSNA. Section VI is devoted to a summary and discussion.

II. STOCHASTIC ANALOG NETWORK AND THE SCSNA

A. Model

We consider a stochastic analog network of the form

$$\frac{dx_i}{dt} = -\frac{\partial \phi(x_i)}{\partial x_i} + \sum_{j \neq i} J_{ij} x_j + f_i(t), \quad i = 1, \ldots, N+1$$

$$\langle f_i(t) f_j(t') \rangle = 2D \delta(t - t') \delta_{ij} \qquad (5)$$

where $\phi(x_i)$ represents the potential and the synaptic coupling J_{ij} is assumed to be given by the standard Hebb learning rule

$$J_{ij}=\frac{1}{N}\sum_{\mu=1}^{p}\xi_i^{\mu}\xi_j^{\mu} \quad (i\neq j), \quad J_{ii}=0 \tag{6}$$

with ξ_i^{μ} $(\mu=1,\ldots,p)$ representing $p(=\alpha N)$ random memory patterns:

$$p_r(\xi_i^{\mu})=\tfrac{1}{2}\delta(\xi_i^{\mu}-1)+\tfrac{1}{2}\delta(\xi_i^{\mu}+1). \tag{7}$$

$D(\geq 0)$ represents the intensity of externally driven Langevin noise. The set of Langevin Eqs. (5) yields the Fokker-Planck equation of probability density $p(t,x_1,\ldots,x_{N+1})$:

$$\frac{\partial p}{\partial t}=-\sum_{i=1}^{N+1}\frac{\partial}{\partial x_i}\left[\left(-\frac{\partial\phi(x_i)}{\partial x_i}+\sum_{j\neq i}J_{ij}x_j\right)p\right]+D\sum_{i=1}^{N+1}\frac{\partial^2}{\partial x_i^2}p. \tag{8}$$

According to the *H*-theorem for the Fokker-Planck equation the probability density p is ensured to approach its equilibrium density $p_{\text{eq}}(t,x_1,\ldots,x_{N+1})$ for long times [38,39], which reads

$$p_{\text{eq}}=\frac{1}{Z_{N+1}}\exp\left[\frac{-1}{D}\left\{\sum_{i=1}^{N+1}\phi(x_i)-\tfrac{1}{2}\sum_{ij(i\neq j)}J_{ij}x_ix_j\right\}\right], \tag{9}$$

where Z_{N+1} is the partition function of the system.

We are interested in the equilibrium statistical behaviors of the order parameters relevant to associative memory retrieval under the assumption of synaptic couplings (6). When the potential $\phi(x)$ is of double-well type and just a single pattern is considered ($p=1$), the system turns out to be equivalent to the mean field model of ferromagnets, where D plays the role of temperature and the spontaneous symmetry breaking occurs below a certain critical value of D as a result of the onset of a pitchfork bifurcation in the limit of large N [40–43]. In such a case the method of the nonlinear Fokker-Planck equations [40–47], which belong to the class of nonlinear master equations [48], is known to be powerful for the analyses of equilibrium and nonequilibrium properties. In the absence of the Langevin noise ($D=0$) Eq. (5) takes the form of an analog neural network equation of associative memory, whose macroscopic equilibrium behavior for the retrieval states can be analyzed by the method of the SCSNA [21,22] when one considers an extensive number of patterns: $p(=\alpha N)$.

B. SCSNA: Heuristic derivation

We study the stochastic system given by Eq. (5) from the viewpoint of applying the SCSNA. Assuming $p=\alpha N$ and confining ourselves only to equilibrium or near equilibrium of the system, we formally apply the basic scheme (1) of the SCSNA to the stochastic quantity of the local field

$\Sigma_{j\neq i}J_{ij}x_j$. We then assume the following Ansatz that in the large N limit the stochastic local field can be split into three terms:

$$\sum_{j\neq i}J_{ij}x_j=\xi_i^1m^{(\text{II})}+\sqrt{\alpha r^{(\text{II})}}z_i+\widetilde{\Gamma}x_i, \tag{10}$$

where we suppose $m^{(\text{II})}$, $r^{(\text{II})}$, $\widetilde{\Gamma}$, and z_i to be nonstochastic quantities with respect to time t. The first term on the right-hand side (RHS) of Eq. (10) represents the signal part with $m^{(\text{II})}$ representing the overlap for the condensed pattern $\{\xi_i^1\}$,

$$m^{(\text{II})}=\frac{1}{N}\sum_{i=1}^{N}\xi_i^1x_i=\frac{1}{N}\sum_{i=1}^{N}\xi_i^1\langle x_i\rangle \tag{11}$$

and the second term represents the so-called noise part fluctuating over sites i that obeys a Gaussian distribution with mean 0 and variance $\alpha r^{(\text{II})}$. The third term $\widetilde{\Gamma}x_i$, which remains to be a stochastic variable, denotes the effective self-coupling one that arises from the renormalization of nonsignal part of the local field within the framework of the SCSNA. Noting that RHS of Eq. (10) involves only the variable of site i, we may rewrite Eq. (5) to have a single body Langevin equation for variable x_i,

$$\frac{dx_i}{dt}=-\frac{\partial\phi(x_i)}{\partial x_i}+\xi_i^1m^{(\text{II})}+\sqrt{\alpha r^{(\text{II})}}z_i+\widetilde{\Gamma}x_i+f_i(t), \tag{12}$$

where m, r, and $\widetilde{\Gamma}$ have to be determined self-consistently in the course of our analysis. The corresponding Fokker-Planck equation reads

$$\frac{\partial}{\partial t}p(x_i,t)=-\frac{\partial}{\partial x_i}[(-\phi'(x_i)+\xi_i^1m^{(\text{II})}+\sqrt{\alpha r^{(\text{II})}}z_i+\widetilde{\Gamma}x_i)p]$$
$$+D\frac{\partial^2 p}{\partial x_i^2}. \tag{13}$$

The equilibrium distribution is given by

$$p_{\text{eq}}(x_i)=C\exp\left(\frac{-\phi(x_i)+(\xi_i^1m^{(\text{II})}+\sqrt{\alpha r^{(\text{II})}}z_i)x_i+\dfrac{\widetilde{\Gamma}}{2}x_i^2}{D}\right). \tag{14}$$

The average is then given by

$$\langle x_i\rangle=\int x_ip_{\text{eq}}(x_i)dx_i, \tag{15}$$

which we consider to represent the renormalized output function $Y(z)$ in the SCSNA framework. In other words, the above equation may be viewed as resulting, via the SCSNA, from the equilibrium equation for a certain deterministic analog neural network model whose macroscopic statistical behavior is the same as that of the original stochastic network (5). We can indeed expect the TAP equation to serve as such an equilibrium equation, as is shown later.

Then, using the SCSNA, we formally obtain the order-parameter equations for $m^{(II)}$, $q^{(II)}$, $r^{(II)}$, $U^{(II)}$, and $\Gamma_{SCSNA}^{(II)}$ [21,22]:

$$m^{(II)} = \left\langle \xi \int Dz \langle x \rangle \right\rangle_\xi , \tag{16a}$$

$$q^{(II)} = \left\langle \int Dz \langle x \rangle^2 \right\rangle_\xi , \tag{16b}$$

$$\sqrt{\alpha r^{(II)}} U^{(II)} = \left\langle \int Dz\, z \langle x \rangle \right\rangle_\xi , \tag{16c}$$

$$\Gamma_{SCSNA}^{(II)} = \frac{\alpha U^{(II)}}{1 - U^{(II)}} , \tag{16d}$$

$$r^{(II)} = \frac{q^{(II)}}{(1 - U^{(II)})^2} . \tag{16e}$$

Furthermore, it will be reasonable to assume that

$$\tilde{\Gamma} = \Gamma_{SCSNA}^{(II)} , \tag{17}$$

since we may expect from Eq. (10) that

$$\sum_{j \neq i} J_{ij} \langle x_j \rangle = \xi_i^1 m^{(II)} + \sqrt{\alpha r^{(II)}} z_i + \tilde{\Gamma} \langle x_i \rangle \tag{18}$$

and we are applying the SCSNA prescription (1) to the RHS of Eq. (18).

From Eqs. (16) and (17) together with Eq. (15), which we expect to constitute the set of order parameter equations for the SCSNA of stochastic version, one can determine the storage capacity as the marginal value of the storage ratio α for the existence of the retrieval solution with $m^{(II)} \neq 0$. We will confirm the validity of the above procedure based on the Ansatz (10) in later sections.

It should be noted that in the deterministic limit $D \to 0$ the average in Eq. (15) turns out to be given by the saddle point evaluation of the integral of Eq. (15), which reads

$$-\frac{\partial \phi(x_i)}{\partial x_i} + \xi_i^1 m^{(II)} + \sqrt{\alpha r^{(II)}} z_i + \tilde{\Gamma} x_i = 0. \tag{19}$$

This equation is the same as what is obtained from the application of the SCSNA to the deterministic analog network equation that follows from setting $D = 0$ and $dx_i/dt = 0$ in Eq. (5).

It is also worth noting that we can propose the explicit form of the TAP equation for our system that is consistent with the analog of Eq. (4). Noting Eq. (15) together with Eq. (18), we suppose the TAP equation to be of the form

$$\langle x_i \rangle = G\left(\sum_{j \neq i} J_{ij} \langle x_j \rangle + \lambda_{ONS}^{(II)} \langle x_i \rangle \right), \tag{20}$$

with

$$G(y) = \int x_i C \exp\left(\frac{-\phi(x_i) + y x_i + \dfrac{\tilde{\Gamma}}{2} x_i^2}{D} \right) dx_i \tag{21}$$

where we have introduced the coefficient of the Onsager reaction term λ_{ONS}, which has to satisfy

$$\lambda_{ONS}^{(II)} = -\tilde{\Gamma}. \tag{22}$$

Then noting Eq. (17), we see that the analog of relation (4) holds

$$\lambda_{ONS}^{(II)} = -\Gamma_{SCSNA}^{(II)}. \tag{23}$$

We can apply the deterministic version of the SCSNA to the TAP equation (20). This is the very situation where we write down Eqs. (16) under Eq. (15).

III. TAP EQUATION

In this section we systematically derive the TAP equation for our stochastic system (5) to justify Eqs. (20), (21), and (23), using the cavity method [1,19]. In view of the equilibrium density given by Eq. (9) we consider an $(N+1)$-body system with coordinates $(x_0, x_1, \ldots, x_N)$, whose Hamiltonian is given by

$$H^{(N+1)} = \sum_{i=0}^{N} \phi(x_i) - \frac{1}{2} \sum_{ij(i \neq j)} J_{ij} x_i x_j , \tag{24}$$

where

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu , \quad i, j = 0, \ldots, N. \tag{25}$$

Then the Hamiltonian can be rewritten as

$$H^{(N+1)} = H^{(N)} + \phi(x_0) - h_0 x_0 \tag{26}$$

with

$$h_0 = \sum_{j=1}^{N} J_{0j} x_j , \tag{27}$$

$$H^{(N)} = \sum_{i=1}^{N} \phi(x_i) - \frac{1}{2} \sum_{ij(i \neq j)} J_{ij} x_i x_j , \tag{28}$$

where $H^{(N)}$ represents the Hamiltonian of the N-body subsystem with coordinates $(x_1, \ldots, x_N)$. We set

$$\beta = \frac{1}{D} \tag{29}$$

in what follows. We consider the probability density

$$P_{N+1}(x_0, h_0) = \frac{1}{Z_{N+1}} \int \delta\left(h_0 - \sum_{j=1}^{N} J_{0j} x_j \right)$$

$$\times e^{-\beta H^{(N+1)}} dx_1 \cdots dx_N , \tag{30}$$

where Z_{N+1} is the partition function of the system with $H^{(N+1)}$. We can rewrite Eq. (30) as

$$P_{N+1}(x_0,h_0) = \frac{Z_N}{Z_{N+1}} \frac{1}{Z_N} \int \delta\left(h_0 - \sum_{j=1}^{N} J_{0j} x_j\right)$$

$$\times e^{-\beta H^{(N)}} e^{\beta[h_0 x_0 - \phi(x_0)]} dx_1 \cdots dx_N$$

$$= \frac{Z_N}{Z_{N+1}} e^{\beta(h_0 x_0 - \phi(x_0))} P_N(h_0), \qquad (31)$$

where

$$P_N(h_0) = \frac{1}{Z_N} \int \delta\left(h_0 - \sum_{j=1}^{N} J_{0j} x_j\right) e^{-\beta H^{(N)}} dx_1 \cdots dx_N. \qquad (32)$$

We have

$$\frac{Z_{N+1}}{Z_N} = \frac{1}{Z_N} \int e^{-\beta H^{(N+1)}} dx_0 dx_1 \cdots dx_N$$

$$= \int Z_0(h_0) P_N(h_0) dh_0$$

$$\equiv \langle Z_0(h_0) \rangle_N, \qquad (33)$$

where we have defined

$$Z_0(h_0) = \int e^{-\beta(\phi(x_0) - h_0 x_0)} dx_0 \qquad (34)$$

and $\langle \ \rangle_N$ stands for the average over the probability density $P_N(h_0)$. Using Eqs. (31) and (33), we obtain the average of $\langle x_0 \rangle_{N+1}$ in the $(N+1)$-body system as

$$\langle x_0 \rangle_{N+1} = \int x_0 P_{N+1}(x_0,h_0) dx_0 dh_0$$

$$= \frac{\langle A(h_0) \rangle_N}{\langle Z_0(h_0) \rangle_N}, \qquad (35)$$

where

$$A(h_0) = \int x_0 e^{-\beta(\phi(x_0) - h_0 x_0)} dx_0, \qquad (36)$$

$$\langle A(h_0) \rangle_N = \int A(h_0) P_N(h_0) dh_0. \qquad (37)$$

We also obtain the average of local field h_0 in the $(N+1)$-body system as

$$\langle h_0 \rangle_{N+1} = \int h_0 P_{N+1}(x_0,h_0) dx_0 dh_0 = \frac{\langle h_0 Z_0(h_0) \rangle_N}{\langle Z_0(h_0) \rangle_N}. \qquad (38)$$

Since h_0 in Eq. (27) can be considered to be a sum of independent random variables, it obeys a Gaussian distribution. Its mean and variance are given, respectively, by

$$\langle h_0 \rangle_N = \sum_{j=1}^{N} J_{0j} \langle x_j \rangle_N, \qquad (39)$$

$$\langle (\delta h_0)^2 \rangle_N = \sum_{ij=1}^{N} J_{0i} J_{0j} \langle \delta x_i \delta x_j \rangle_N = \sum_{\mu\nu}^{p} \xi_0^\mu \xi_0^\nu \langle \delta m_\mu \delta m_\nu \rangle_N$$

$$= \sum_{\mu}^{p} \langle (\delta m_\mu)^2 \rangle_N \equiv r_N \qquad (40)$$

with

$$m_\mu = \frac{1}{N} \sum_{j}^{N} \xi_j^\mu x_j, \qquad (41)$$

where $\delta h_0 = h_0 - \langle h_0 \rangle_N$, and we have noted that the off-diagonal terms in the sum over μ and ν have only a negligible contribution for sufficient large N under the condition that $p = \alpha N$.

It can also be assumed that in the large N limit

$$r_N \to \tilde{R}. \qquad (42)$$

Then we have the probability density of h_0

$$P_N(h_0) = \frac{1}{\sqrt{2\pi \tilde{R}}} \exp\left[-\frac{(h_0 - \langle h_0 \rangle_N)^2}{2\tilde{R}}\right] \qquad (43)$$

for sufficient large N.

Using this we can compute the averages in the N-body system,

$$\langle Z_0(h_0) \rangle_N = \int \int dx_0 dh_0 \frac{1}{\sqrt{2\pi \tilde{R}}}$$

$$\times \exp\left[-\beta(\phi(x_0) - h_0 x_0) - \frac{(h_0 - \langle h_0 \rangle_N)^2}{2\tilde{R}}\right]$$

$$= \int \exp\left[-\beta(\phi(x_0) - \langle h_0 \rangle_N x_0) + \frac{1}{2}\tilde{R}\beta^2 x_0^2\right] dx_0, \qquad (44)$$

$$\langle A(h_0) \rangle_N = \int x_0 \exp\{-\beta[\phi(x_0) - \langle h_0 \rangle_N x_0] + \tfrac{1}{2}\tilde{R}\beta^2 x_0^2\} dx_0. \qquad (45)$$

Furthermore, noting that

$$\langle (h_0 - \langle h_0 \rangle_N) Z_0(h_0) \rangle_N = \int \int \frac{du\, dx_0}{\sqrt{2\pi\tilde{R}}} u \exp\left[-\beta\left(\phi(x_0) - \langle h_0 \rangle_N x_0 - \frac{\tilde{R}\beta}{2} x_0^2\right) - \frac{(u - \tilde{R}\beta x_0)^2}{2\tilde{R}}\right]$$

$$= \int \tilde{R}\beta x_0 \exp\left[-\beta\left(\phi(x_0) - \langle h_0 \rangle_N x_0 - \frac{\tilde{R}\beta}{2} x_0^2\right)\right] dx_0 = \tilde{R}\beta \langle A(h_0) \rangle_N, \tag{46}$$

we have

$$\langle h_0 Z_0(h_0) \rangle_N = \langle (h_0 - \langle h_0 \rangle_N) Z_0(h_0) \rangle_N + \langle h_0 \rangle_N \langle Z_0(h_0) \rangle_N$$

$$= \tilde{R}\beta \langle A(h_0) \rangle_N + \langle h_0 \rangle_N \langle Z_0(h_0) \rangle_N. \tag{47}$$

It follows from Eqs. (35), (44), and (45) that

$$\langle x_0 \rangle_{N+1} = \frac{\displaystyle\int x_0 \exp\left[-\beta\left(\phi(x_0) - \langle h_0 \rangle_N x_0 - \frac{\tilde{R}\beta}{2} x_0^2\right)\right] dx_0}{\displaystyle\int \exp\left[-\beta\left(\phi(x_0) - \langle h_0 \rangle_N x_0 - \frac{\tilde{R}\beta}{2} x_0^2\right)\right] dx_0}. \tag{48}$$

It also follows from Eqs. (38), (44), and (47) that

$$\langle h_0 \rangle_{N+1} = \langle h_0 \rangle_N + \tilde{R}\beta \langle x_0 \rangle_{N+1}. \tag{49}$$

Substituting this into Eq. (48), one has

$$\langle x_0 \rangle_{N+1} = F\left(\sum_{j=1}^{N} J_{0j} \langle x_j \rangle_{N+1} - \tilde{R}\beta \langle x_0 \rangle_{N+1}\right) \tag{50}$$

with

$$F(h) = \frac{\displaystyle\int x_0 \exp\left[-\beta\left(\phi(x_0) - h x_0 - \frac{\tilde{R}\beta}{2} x_0^2\right)\right] dx_0}{\displaystyle\int \exp\left[-\beta\left(\phi(x_0) - h x_0 - \frac{\tilde{R}\beta}{2} x_0^2\right)\right] dx_0}, \tag{51}$$

where we have noted that

$$\langle h_0 \rangle_{N+1} = \sum_{j=1}^{N} J_{0j} \langle x_j \rangle_{N+1}. \tag{52}$$

Equation (50) takes the form of the TAP equation. However, we still have to determine $\tilde{R}$ in Eq. (50). For this reason, to be precise we refer to Eq. (50) as the pre-TAP equation in the present paper. For the purpose of obtaining $\tilde{R}$ we proceed to the second step of the cavity method.

Letting the Hamiltonian of the network with p stored random patterns be

$$H_p = \sum_{i=1}^{N} \phi(x_i) - \frac{1}{2} \sum_{ij(i\neq j)} \left(\frac{1}{N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu\right) x_i x_j \tag{53}$$

we consider the Hamiltonian of the network that is obtained by adding a new pattern ξ_i^0 $(i = 1, \ldots, N)$ independent of ξ_i^μ $(i = 1, \ldots, N, \ \mu = 1, \ldots, p)$ to the above-mentioned system,

$$H_{p+1} = H_p - \frac{1}{2N} \sum_{ij(i\neq j)}^{N} \xi_i^0 \xi_j^0 x_i x_j = H_p - \frac{N}{2} m_0^2 + \frac{1}{2N} \sum_{i}^{N} x_i^2 \tag{54}$$

with

$$m_0 = \frac{1}{N} \sum_{i}^{N} \xi_i^0 x_i. \tag{55}$$

We define the probability density of the m_0 that is considered as the random variable in the $p + 1$ pattern system as

$$P_{p+1}(m_0)$$

$$= \frac{1}{Z_{p+1}} \int \delta\left(m_0 - \frac{1}{N} \sum_{i}^{N} \xi_i^0 x_i\right)$$

$$\times e^{-\beta H_{p+1}} dx_1 \cdots dx_N = \frac{1}{Z_{p+1}} \int \delta\left(m_0 - \frac{1}{N} \sum_{i}^{N} \xi_i^0 x_i\right)$$

$$\times \exp\left[-\beta\left(H_p - \frac{N}{2} m_0^2 + \frac{1}{2N} \sum_{i}^{N} x_i^2\right)\right] dx_1 \cdots dx_N. \tag{56}$$

Noting that $(1/2N) x_i^2$ can be neglected in comparison with $\phi(x_i)$ for large N, we have

$$P_{p+1}(m_0) = \frac{Z_p}{Z_{p+1}} e^{(1/2)\beta N m_0^2} P_p(m_0). \tag{57}$$

To obtain the probability density $P_p(m_0)$ in the p pattern system we note

$$\langle m_0 \rangle_p = 0 \tag{58}$$

and compute the variance of m_0,

$$\langle m_0^2 \rangle_p = \langle (\delta m_0)^2 \rangle_p = \frac{1}{N^2} \sum_{ij}^{N} \xi_i^0 \xi_j^0 \langle \delta x_i \delta x_j \rangle_p$$

$$= \frac{1}{N^2} \sum_i^N \langle (\delta x_i)^2 \rangle_p = \frac{1}{N^2} \sum_i^N (\langle x_i^2 \rangle_p - \langle x_i \rangle_p^2)$$

$$= \frac{1}{N} (q_3 - q), \tag{59}$$

where we have neglected the contribution from the off-diagonal terms in the summation and introduced

$$q_3 = \frac{1}{N} \sum_i^N \langle x_i^2 \rangle_p, \quad q = \frac{1}{N} \sum_i^N \langle x_i \rangle_p^2. \tag{60}$$

Hence the sum m_0 of independent random variables in the p pattern system obeys the Gaussian distribution

$$P_p(m_0) = \sqrt{\frac{N}{2\pi(q_3 - q)}} \exp\left[-\frac{N m_0^2}{2(q_3 - q)} \right]. \tag{61}$$

Using Eq. (57) together with Eq. (61), one obtains the variance of m_0 in the $p+1$ pattern system

$$\langle (\delta m_0)^2 \rangle_{p+1} = \frac{q_3 - q}{N[1 - \beta(q_3 - q)]}. \tag{62}$$

Substituting this into Eq. (40) one has

$$\tilde{R} = \frac{\alpha(q_3 - q)}{1 - \beta(q_3 - q)}. \tag{63}$$

Equation (50) together with Eq. (63) constitutes the TAP equation. The fact that the TAP equation obtained here is the same as the one proposed in Sec. II will be shown in the next section.

IV. SCSNA OF THE STOCHASTIC VERSION: SYSTEMATIC DERIVATION

We develop in a systematic manner the SCSNA that becomes applicable to the stochastic network (5) to show the validity of the anzatz of Sec. II. We make use of the cavity method to lay its foundation and pay attention to the first step of the previous section in deriving the TAP equation. To this end, we pick up an arbitrary element x_i among $N+1$ ones of the network to view it as x_0 and to make an appropriate renumbering of the variables for applying the basic equation (48) to the stochastic network (5).

Confining ourselves only to equilibrium states of the network, we can repeat the argument of the previous section leading to Eq. (48). Then using Eqs. (48) and (51) we note that the average $\langle x_0 \rangle_{N+1}$ is given by

$$\langle x_0 \rangle_{N+1} = F(\langle h_0 \rangle_N). \tag{64}$$

Then the local field $\langle h_0 \rangle_N$ is rewritten, in terms of the pattern overlaps defined by Eq. (41), as

$$\langle h_0 \rangle_N = \sum_{j=1}^{N} \frac{1}{N} \sum_{\mu=1}^{p} \xi_0^\mu \xi_j^\mu \langle x_j \rangle_N$$

$$= \xi_0^1 \langle m_1 \rangle_N + \sum_{\mu=2}^{p} \xi_0^\mu \langle m_\mu \rangle_N. \tag{65}$$

We assume here that the pattern overlap order parameter corresponding to the condensed pattern takes a finite value m of $O(1)$ in the limit $N \to \infty$:

$$\langle m_1 \rangle_N \to m \tag{66}$$

whereas for the pattern overlaps for noncondensed patterns

$$\langle m_\mu \rangle_N \simeq O\left(\frac{1}{\sqrt{N}} \right). \tag{67}$$

The second term of the last line of Eq. (65) representing the noise part of the local field is a sum of independent random variables. Accordingly it should obey a Gaussian distribution. This implies that the size of the noise part is distributed according to a Gaussian distribution, when site 0 runs over the whole network, and also that the site average can be replaced by the average over the Gaussian distribution as well as over that of the condensed pattern. It follows that

$$\left\langle \sum_{\mu=2}^{p} \xi_0^\mu \langle m_\mu \rangle_N \right\rangle_{\text{site}} = 0, \tag{68}$$

$$\left\langle \left(\sum_{\mu=2}^{p} \xi_0^\mu \langle m_\mu \rangle_N \right)^2 \right\rangle_{\text{site}} = \sigma^2, \tag{69}$$

where $\langle\ \rangle_{\text{site}}$ represents the site average that is given by taking average with respect to random patterns ξ_i^μ ($i = 0, \ldots, N$, $\mu = 2, \ldots, p$). The expression for the constant σ^2 will be given in the course of the analysis. Evaluation of the value for σ^2 at this stage of the analysis, however, can be straightforwardly made and presented in Appendix A.

Then in the large N limit we can rewrite Eq. (64) in terms of a Gaussian random variable $\tilde{z}$ with mean 0 and variance σ^2 as

$$\langle x_0 \rangle = F(\xi_0^1 m + \tilde{z}) \tag{70}$$

where we have introduced $\tilde{z} = \sum_{\mu=2}^{p} \xi_0^\mu \langle m_\mu \rangle_N$ and $\langle x_0 \rangle_{N+1} = \langle x_0 \rangle$. The overlap m can be rewritten as

$$m = \left\langle \xi_0^1 \int F(\xi_0^1 m + \tilde{z}) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-\tilde{z}^2}{2\sigma^2} \right) d\tilde{z} \right\rangle_{\xi_0^1}$$

$$\equiv \langle \xi_0^1 F \rangle_{\tilde{z}, \xi_0^1} \tag{71}$$

where $\langle\ \rangle_{\xi_0^1}$ represents average over the condensed pattern ξ_0^1 and $\langle\ \rangle_{\tilde{z}, \xi_0^1}$ average over ξ_0^1 and the Gaussian random variable $\tilde{z}$. When we note that the pre-TAP equation (50) holds with respect to equilibrium states with subscript 0 capable of representing every site, we can view such an equation as defining a deterministic analog network equation corresponding to the stochastic network (5):

$$u_i = F\left(\sum_{j \neq i}^{N+1} J_{ij} u_j - \tilde{R}\beta u_i\right), \quad i = 1, \ldots, N+1 \quad (72)$$

with u_i representing $\langle x_i \rangle_{N+1}$. Note that the coefficient of the Onsager reaction term λ_{ONS} is expressed as

$$\lambda_{\text{ONS}} = -\tilde{R}\beta.$$

We will apply the SCSNA to the above analog network. For the moment we can proceed without knowing the explicit expression (63) for $\tilde{R}$. Then we have the renormalized output equation that is obtained as a result of the renormalizing noise part of the local field [21,22],

$$u_i = F[\xi_i^1 m + \tilde{\tilde{z}}_i + (\Gamma_{\text{SCSNA}} - \tilde{R}\beta) u_i], \quad (73)$$

where $\tilde{\tilde{z}}$ represents a Gaussian random variable and the constant Γ_{SCSNA} represents the coefficient of the output proportional term characteristic to the SCSNA, which is determined just below.

Comparing with Eq. (70), we obtain an important result

$$\Gamma_{\text{SCSNA}} = \tilde{R}\beta, \quad (74)$$

which verifies Eq. (23) and simplifies the matter considerably, together with $\tilde{z} = \tilde{\tilde{z}}_i$. Relevant quantities necessary for determining retrieval states U, Γ_{SCSNA}, σ^2, and q are given within the framework of the SCSNA as follows [21,22]:

$$U = \left\langle \frac{dF}{d\tilde{z}} \right\rangle_{\tilde{z}, \xi_0^1}, \quad (75)$$

$$\Gamma_{\text{SCSNA}} = \frac{\alpha U}{1 - U}, \quad (76)$$

$$\sigma^2 = \frac{\alpha q}{(1 - U)^2}, \quad (77)$$

$$q = \langle F^2 \rangle_{\tilde{z}, \xi_0^1}. \quad (78)$$

Here we note that use of the same notation q as in Eq. (60) is made for Eq. (78). Differentiating $F(h)$ in Eq. (51) with respect to h yields

$$\frac{dF}{dh} = \beta(\langle x_0^2 \rangle_{\text{eq}} - \langle x_0 \rangle_{\text{eq}}^2), \quad (79)$$

where $\langle \ \rangle_{\text{eq}}$ denotes the "thermal average" given by

$$\langle g(x_0) \rangle_{\text{eq}} = \frac{\displaystyle\int g(x_0)\exp\left[-\beta\left(\phi(x_0) - hx_0 - \frac{\tilde{R}\beta}{2}x_0^2\right)\right] dx_0}{\displaystyle\int \exp\left[-\beta\left(\phi(x_0) - hx_0 - \frac{\tilde{R}\beta}{2}x_0^2\right)\right] dx_0}. \quad (80)$$

Then we have from Eqs. (75) and (60)

$$U = \beta(\langle\langle x_0^2 \rangle_{\text{eq}}\rangle_{\tilde{z}, \xi_0^1} - \langle\langle x_0 \rangle_{\text{eq}}^2\rangle_{\tilde{z}, \xi_0^1}) = \beta(q_3 - q). \quad (81)$$

Equations (74) and (76) give $\tilde{R}$ as a function of U,

$$\tilde{R} = \frac{\alpha U}{\beta(1 - U)}, \quad (82)$$

which is necessary to determine the functional form F of the TAP equation. Substituting Eq. (81) into Eq. (82) we recover Eq. (63), which has previously been obtained for the Onsager reaction term coefficient $-\tilde{R}\beta$ of the TAP equation by means of the second step of the cavity method. The TAP equation

$$\langle x_i \rangle = F\left(\sum_{j \neq i} J_{ij}\langle x_j \rangle - \frac{\alpha U}{1 - U}\langle x_i \rangle\right), \quad i = 1, \ldots, N$$

with F given by Eq. (51) also turns out to recover the one given by Eq. (20) with Eq. (23) we have proposed in Sec. II by means of a heuristic argument.

Noting $\sigma^2 = \alpha r$ we see that Eqs. (51), (71), (75), (77), (78), and (82) constitute the set of order parameter equations for our stochastic system (5), giving the framework of the SCSNA of the stochastic version, which ensures the validity of the result described in Sec. II under the correspondence $m^{(\text{II})} = m$, $q^{(\text{II})} = q$, $r^{(\text{II})} = r$, $U^{(\text{II})} = U$, and $\Gamma_{\text{SCSNA}}^{(\text{II})} = \Gamma_{\text{SCSNA}}$. In particular, we note that the crucial ansatz (10) in the heuristic derivation of the SCSNA has been justified.

We can take an alternative approach to obtain the above set of order parameter equations without using the deterministic version of the SCSNA by assuming, this time, the TAP equation with the coefficient of the Onsager reaction term given in terms of $\tilde{R}$ which is a function of q and q_3 [Eq. (63)]. We note in this case that noise variance σ^2 in Eq. (77) can be given by a direct evaluation as shown in Appendix A leading to Eq. (77) together with Eqs. (75) and (78). We have Eq. (71) as before. Since combining Eqs. (63) and (81) gives the expression (82) for $\tilde{R}$, which in turn determines the function F in Eq. (51), we conclude our alternative procedure for obtaining the stochastic version of the SCSNA.

V. REPLICA SYMMETRIC CALCULATION

The result obtained in the preceding section can be confirmed by the standard method of statistical mechanics. In other words, the set of order parameter equations are recovered by the method of replica symmetric calculation [2,9–12], with which we are concerned in this section.

The partition function in Eq. (9) reads

$$Z = \int \prod_i dx_i \exp\left[-\beta\left\{\sum_i \left(\phi(x_i) + \frac{1}{2}\alpha x_i^2\right)\right.\right.$$
$$\left.\left. - \frac{1}{2N}\sum_\mu \left(\sum_i \xi_i^\mu x_i\right)^2\right\}\right]. \quad (83)$$

The standard replica symmetric calculation can be applied to this partition function to obtain the free energy. A brief derivation is given in Appendix B. The resultant free energy takes the form

$$f = \frac{\beta}{2}m^2 + \frac{\alpha}{2}\left\{\ln(1-\beta Q) + \frac{\beta(Q-q_1)}{1-\beta Q}\right\} + \alpha\beta^2(r_2'Q + q_1 R)$$

$$-\left\langle \frac{1}{\sqrt{2\pi}}\int \exp\left(-\frac{z^2}{2}\right)\ln[T_r \exp(\beta\sqrt{2\alpha r_2'}xz + \alpha\beta^2 Rx^2 \right.$$

$$\left. - \beta\tilde{\phi}(x) + \beta\xi xm)]dz\right\rangle. \tag{84}$$

Differentiating f with respect to m, q_1, Q, R, r_2' yields a set of order parameter equations,

$$\frac{\partial f}{\partial m}=0: \quad m=\left\langle\left\langle \int Dz\,\xi\langle x\rangle \right\rangle\right\rangle_\xi, \tag{85}$$

$$\frac{\partial f}{\partial q_1}=0: \quad \frac{-1}{2(1-\beta Q)}+\beta R=0, \tag{86}$$

$$\frac{\partial f}{\partial Q}=0: \quad 2r_2'=\frac{q_1-Q}{(1-\beta Q)^2}, \tag{87}$$

$$\frac{\partial f}{\partial R}=0: \quad q_1=\left\langle\left\langle \int Dz\langle x^2\rangle \right\rangle\right\rangle, \tag{88}$$

$$\frac{\partial f}{\partial r_2'}=0: \quad \beta Q=\left\langle\left\langle \int Dz\,z\,\frac{1}{\sqrt{2\alpha r_2'}}\langle x\rangle \right\rangle\right\rangle, \tag{89}$$

where

$$\langle g(x)\rangle=\frac{T_r g(x)\exp[S]}{T_r \exp[S]}, \tag{90}$$

$$S=\beta\sqrt{2\alpha r_2'}xz + \alpha\beta^2 Rx^2 - \beta\tilde{\phi}(x) + \beta\xi xm, \tag{91}$$

$$\int Dz\cdots=\int dz\,\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\cdots.$$

Note that differentiating $\langle x\rangle$ with respect to $\sqrt{2\alpha r_2'}z$ yields

$$\frac{d\langle x\rangle}{\sqrt{2\alpha r_2'}dz}=\beta(\langle x^2\rangle - \langle x\rangle^2). \tag{92}$$

Accordingly, Eq. (89) can be rewritten as

$$Q=q_1-\left\langle\left\langle \int Dz\langle x\rangle^2 \right\rangle\right\rangle. \tag{93}$$

Then one has

$$q_2\equiv q_1-Q=\left\langle\left\langle \int Dz\langle x\rangle^2 \right\rangle\right\rangle. \tag{94}$$

With setting

$$q=q_2, \quad U=\beta Q, \quad \Gamma_{\text{SCSNA}}=2\alpha\beta R-\alpha, \quad r=2r_2' \tag{95}$$

together with

$$Y(z)=\langle x\rangle, \tag{96}$$

Eqs. (85), (94), (89), (87), and (86) constitute the order parameter equations that recover those given by the SCSNA of Secs. II and IV. A similar set of order parameter equations was also obtained by Kuhn *et al.* [11,12] using the replica calculation for a different type of stochastic analog network with multiplicative noise.

Our present result shows that the SCSNA is on the same level of analysis as the replica symmetric calculation, as far as a stochastic system satisfies the energy condition.

VI. SUMMARY AND DISCUSSION

We have studied a stochastic analog neural network model for associative memory to elucidate statistical mechanical aspects of the mean field model of a certain type of random systems to which the present model with random patterns stored belongs. We have taken three approaches, i.e., TAP equation, SCSNA, and replica symmetric calculation approaches to compare them and obtained the same set of the order parameter equations for investigating statistical properties of retrieval states.

The main results are the following: (1) We have obtained the TAP equation for the stochastic analog networks satisfying the energy condition using the cavity method. (2) We have formulated the SCSNA of the stochastic version to obtain the set of order parameter equations for memory retrieval states. (3) We have established the connection between the Onsager reaction term in the TAP equation and the output proportional term in the local field of the SCSNA.

The advantage of conducting the comparative study is to get deep insights into the structure of the TAP equation and its relation to the SCSNA. Indeed, the SCSNA method, which was originally developed for deterministic analog networks has been extended so as to be applicable to the case of stochastic networks and the relationship between the TAP equation and the SCSNA has been made clear. The key idea bridging over the two approaches is the concept of cavity method together with the treatment of the Onsager reaction term. To be more specific, we have employed the cavity method for deriving the pre-TAP equation and applied the SCSNA to such a deterministic equation satisfied by thermal average to obtain the set of the order parameter equations. This process constitutes the stochastic version of the SCSNA and reveals an important relation as shown by Eqs. (4) or (23), that is, the Onsager reaction term that appears in the TAP equation and the output proportional term in the local field that occurs within the framework of the SCSNA cancel each other. As a result of this, the Onsager reaction term turns out to be explicitly determined without resorting to the second step that is usually used in the cavity approach. On the other hand, when we first assume the full knowledge of

the TAP equation with the Onsager reaction term given by Eq. (63), we have shown that it is not necessary to use the deterministic version of the SCSNA for deriving the stochastic version of the SCSNA.

Our systematic approach in Sec. IV of the present paper to develop the stochastic version of the SCSNA is based on the model equation (5) where the existence of an energy function is assumed for simplicity. Extending present analysis to other network models with the energy concept will be straightforward. Our recent study indeed shows that the SCSNA, TAP, and replica approaches are consistent with each other also for an oscillator neural network based on the phase oscillator model that has an energy function, and the relation (4) is proven [49]. The outline of the stochastic SCSNA for this oscillator neural network is given in Appendix C. The Ansatz or heuristic derivation of the SCSNA described in Sec. II, whose validity has been confirmed for the model Equation (5), is quite naive and seems free from such an energy condition. We may expect its validity to still hold for more general cases without the energy condition by observing preliminary results based on numerical simulations for such systems, although it is hard to prove the validity by following the procedure for obtaining the marginal distribution of x_0 as given in Sec. III for lack of the Gibbs-type equilibrium densities. The problem of finding the TAP equation for general systems, if any, will be of value and interest. There is also an open problem of whether the SCSNA can deal with the instability of the replica symmetric solutions. Such issues will be studied elsewhere.

APPENDIX A: DERIVATION OF EQ. (77) USING A DIRECT COMPUTATION BASED ON THE RENORMALIZATION OF NOISE [21,22]

We directly derive the expression for noise variance in Eq. (77) from Eq. (69). We first evaluate $\langle m_\mu \rangle$ in Eq. (69). Noting Eq. (64) with N replaced by $N-1$, we have

$$\langle m_\mu \rangle_N = \frac{1}{N} \sum_{j}^{N} \xi_j^\mu \langle x_j \rangle_N = \frac{1}{N} \sum_{j}^{N} \xi_j^\mu F(\langle h_j \rangle_{N-1}) \quad \text{(A1)}$$

with

$$\langle h_j \rangle_{N-1} = \sum_{k \neq j} J_{jk} \langle x_k \rangle_{N-1}$$

$$= \frac{1}{N} \sum_{k \neq j} \sum_\nu \xi_j^\nu \xi_k^\nu \langle x_k \rangle_{N-1}$$

$$= \frac{1}{N} \sum_{k \neq j} \xi_j^\mu \xi_k^\mu \langle x_k \rangle_{N-1} + \frac{1}{N} \sum_{k \neq j} \sum_{\nu \neq \mu} \xi_j^\nu \xi_k^\nu \langle x_k \rangle_{N-1}.$$
$$\text{(A2)}$$

Then it follows that

$$F(\langle h_j \rangle_{N-1}) = F(\langle h_j \rangle_{N-1}^{(\mu)}) + \xi_j^\mu \langle m_\mu \rangle_{N-1} F'(\langle h_j \rangle_{N-1}^{(\mu)})$$

$$+ o\left(\frac{1}{\sqrt{N}}\right) \quad \text{(A3)}$$

where noting Eq. (67) we have performed the Taylor expansion and defined

$$\langle h_j \rangle_{N-1}^{(\mu)} = \frac{1}{N} \sum_{k \neq j} \sum_{\nu \neq \mu} \xi_j^\nu \xi_k^\nu \langle x_k \rangle_{N-1}. \quad \text{(A4)}$$

Substituting Eq. (A3) into Eq. (A1) leads to

$$\langle m_\mu \rangle_N = \frac{1}{N} \sum_{j} \xi_j^\mu F(\langle h_j \rangle_{N-1}^{(\mu)})$$

$$+ \frac{1}{N} \sum_{j} \langle m_\mu \rangle_{N-1} F'(\langle h_j \rangle_{N-1}^{(\mu)}). \quad \text{(A5)}$$

Assuming N so large that $\langle m_\mu \rangle_N = \langle m_\mu \rangle_{N-1}$, one then has

$$\langle m_\mu \rangle_N = \frac{\frac{1}{N} \sum_j \xi_j^\mu F(\langle h_j \rangle_{N-1}^{(\mu)})}{1 - \langle F' \rangle_{\bar{z}, \xi}}, \quad \text{(A6)}$$

where $\langle F' \rangle_{\bar{z}, \xi}$ represents the site average of F' as given in text. Accordingly it follows that

$$\sum_{\mu=2}^{p} \langle m_\mu \rangle_N^2 = \frac{1}{(1-U)^2} \frac{1}{N^2}$$

$$\times \sum_\mu \sum_j \sum_k \xi_j^\mu \xi_k^\mu F(\langle h_j \rangle_{N-1}^{(\mu)}) F(\langle h_k \rangle_{N-1}^{(\mu)})$$

$$= \frac{1}{(1-U)^2} \frac{\alpha}{N} \sum_j [F(\langle h_j \rangle_{N-1})]^2$$

$$= \frac{\alpha}{(1-U)^2} \langle F^2 \rangle_{\bar{z}, \xi}, \quad \text{(A7)}$$

where we have put $U = \langle F' \rangle_{\bar{z}, \xi}$ as in Eq. (75) and noted that one can safely replace $\langle h_j \rangle_{N-1}^{(\mu)}$ with $\langle h_j \rangle_{N-1}$ in this stage of the analysis. Equations (69) and (A7) give Eq. (77).

APPENDIX B: DERIVATION OF THE FREE ENERGY (84) BASED ON THE REPLICA SYMMETRIC CALCULATION

Using the partition function (83) corresponding to the equilibrium probability density (9), one computes the free energy by means of

$$f = \lim_{N \to \infty} \lim_{n \to 0} \frac{\langle Z^n \rangle - 1}{nN}. \quad \text{(B1)}$$

Introducing n replicas and using the Hubbard-Stratonovich transformation, one obtains Z^n as

$$Z^n = \int \prod_{i,a} dx_{ia} \exp\left[-\beta \sum_a \left\{ \sum_i \tilde{\phi}(x_{ia}) \right.\right.$$

$$\left.\left. -\frac{1}{2N} \sum_\mu \left(\sum_i \xi_i^\mu x_{ia} \right)^2 \right\} \right]$$

$$= \int \prod_{i,a} dx_{ia} \int \prod_{\mu,a} dm_{\mu a} \left(\frac{\beta N}{2\pi} \right)^{pn/2} \exp\left[-\beta \sum_{ia} \tilde{\phi}(x_{ia}) \right.$$

$$\left. -\frac{\beta N}{2} \sum_{\mu a} m_{\mu a}^2 + \beta \sum_{\mu a} \sum_i \xi_i^\mu x_{ia} m_{\mu a} \right], \qquad \text{(B2)}$$

where

$$\tilde{\phi}(x_{ia}) = \phi(x_{ia}) + \tfrac{1}{2}\alpha x_{ia}^2. \qquad \text{(B3)}$$

After some manipulations we have

$$\langle Z^n \rangle = \langle Z^n \rangle_{\mu \geq 2} = \left(\frac{\beta N}{2\pi} \right)^{pn/2} \left(\frac{\beta N}{2\pi} \right)^{-(p-1)n/2} \left(\frac{-\alpha\beta^2 N}{2\pi i} \right)^{n^2}$$

$$\times \int \prod_a dm_a \prod_{(ab)} dq_{ab} \prod_{(ab)} dr_{ab} \exp\Omega \qquad \text{(B4)}$$

with

$$\Omega = -\frac{\beta N}{2} \sum_a m_a^2 - \frac{p-1}{2} \ln[\det B(\{q_{ab}\})]$$

$$-N \sum_{(ab)} \alpha\beta^2 r_{ab} q_{ab} + N \langle \ln(T_{rx_a}\{\exp[-\beta H_\xi(\{m\})]\}) \rangle_\xi, \qquad \text{(B5)}$$

where the trace stands for

$$T_{r\{x_{ia}\}}[\ ] \equiv \int \prod_{(ia)} dx_{ia}[\ ] \qquad \text{(B6)}$$

with the Hamiltonian

$$H_\xi(\{m_1\}) \equiv -\left(\alpha\beta \sum_{(ab)} r_{ab} x_a x_b - \sum_a \tilde{\phi}(x_a) + \sum_a \xi x_a m_{1a} \right) \qquad \text{(B7)}$$

and the matrix B is defined by

$$B = I - \beta Q \qquad \text{(B8)}$$

with

$$Q_{ab} = \sum_i \frac{1}{N} x_{ia} x_{ib}. \qquad \text{(B9)}$$

In the large N limit the above integral in Eq. (B4) can be evaluated using the saddle point method where we assume the replica symmetry

$$m_a = m, \quad q_{ab} = q_2 \quad (a \neq b), \quad q_{aa} = q_1, \quad r_{ab} = r_2$$

$$(a \neq b), \quad r_{aa} = r_1. \qquad \text{(B10)}$$

Then it follows that

$$f = \lim_{N \to \infty} \lim_{n \to 0} \frac{\langle Z^n \rangle - 1}{nN} = \frac{\beta}{2} m^2 + \alpha\beta^2 r_1 q_1$$

$$-\alpha\beta^2 \frac{r_2}{2} q_2 + \lim_{n \to 0} \frac{\alpha}{2n} \ln(\det B)$$

$$-\lim_{n \to 0} \frac{1}{n} \langle \ln(T_{rx_a}\{\exp[-\beta H_\xi(m)]\}) \rangle_\xi, \qquad \text{(B11)}$$

where one can easily evaluate

$$\lim_{n \to 0} \frac{1}{n} \ln(\det B) = \ln(1 - \beta q_1 + \beta q_2) - \frac{\beta q_2}{1 - \beta q_1 + \beta q_2}. \qquad \text{(B12)}$$

The trace part of Eq. (B11) can be calculated using the Hubbard-Stratonovich transformation as

$$\lim_{n \to 0} \frac{1}{n} \langle \ln(T_{rx_a}\{\exp[-\beta H_\xi(m)]\}) \rangle_\xi$$

$$= \left\langle \int dz \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} \right) \ln[T_{rx} \exp(\beta \hat{H}_\xi)] \right\rangle_\xi, \qquad \text{(B13)}$$

where

$$\hat{H}_\xi = \sqrt{2\alpha r_2'} zx + \alpha\beta(r_1 - r_2')x^2 - \tilde{\phi}(x) + \xi xm. \qquad \text{(B14)}$$

Accordingly we have

$$f = \frac{\beta}{2} m^2 + \alpha\beta^2 r_1 q_1 - \alpha\beta^2 r_2' q_2$$

$$+ \frac{\alpha}{2}\left[\ln(1 - \beta q_1 + \beta q_2) - \frac{\beta q_2}{1 - \beta q_1 + \beta q_2} \right]$$

$$- \left\langle \int dz \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} \right) \ln[T_{rx} \exp(\beta \hat{H}_\xi)] \right\rangle_\xi. \qquad \text{(B15)}$$

Setting

$$Q = q_1 - q_2,$$

$$R = r_1 - r_2', \qquad \text{(B16)}$$

we finally arrive at the expression for the free energy (84)

$$f = \frac{\beta}{2} m^2 + \alpha\beta^2 (r_2' Q + q_1 R) + \frac{\alpha}{2}\left[\ln(1 - \beta Q) + \frac{\beta(Q - q_1)}{1 - \beta Q} \right]$$

$$- \left\langle \int dz \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} \right) \ln[T_{rx} \exp(\beta \hat{H}_\xi)] \right\rangle_\xi. \qquad \text{(B17)}$$

APPENDIX C: STOCHASTIC SCSNA FOR THE OSCILLATOR NETWORK

The network dynamics is assumed to be given by

$$\frac{d\theta_i}{dt} = -\sum_j J_{ij} \sin(\theta_i - \theta_j) + f_i(t), \quad i = 1, \ldots, N \tag{C1}$$

with white noise $f_i(t)$ and synaptic coupling J_{ij} being given as in Eqs. (5), (6), and (7). We are concerned with the equilibrium state. Noting that the system has rotational symmetry, one can choose a gauge such that one of the overlap order parameters that will be given below vanishes: $m_s^1 = 0$.

In this case the stochastic SCSNA claims the following ansatz of the local fields of the oscillator neurons:

$$h_i^c = \sum_{j \neq i} J_{ij} \cos \theta_j = \xi_i^1 m_c^1 + \tilde{z}_c + \Gamma_{\text{SCSNA}}^{(c)} \cos \theta_i,$$

$$h_i^s = \sum_{j \neq i} J_{ij} \sin \theta_j = \xi_i^1 m_s^1 + \tilde{z}_s + \Gamma_{\text{SCSNA}}^{(s)} \sin \theta_i = \tilde{z}_s$$

$$+ \Gamma_{\text{SCSNA}}^{(s)} \sin \theta_i, \tag{C2}$$

where

$$m_c^1 = \frac{1}{N} \sum_j \xi_j^1 \langle \cos \theta_j \rangle_{\text{eq}}, \quad m_s^1 = \frac{1}{N} \sum_j \xi_j^1 \langle \sin \theta_j \rangle_{\text{eq}}, \tag{C3}$$

where $\langle \ \rangle_{\text{eq}}$ denotes the average over the marginal equilibrium probability density $P_{\text{eq}}(\theta_i)$ that is self-consistently determined below. $\Gamma_{\text{SCSNA}}^{(c)}$ and $\Gamma_{\text{SCSNA}}^{(s)}$, analogs of Eq. (16d) are given, within the framework of the SCSNA, by

$$\Gamma_{\text{SCSNA}}^{(c)} = \frac{\alpha U_c}{1 - U_c}, \quad \Gamma_{\text{SCSNA}}^{(s)} = \frac{\alpha U_s}{1 - U_s}, \tag{C4}$$

with U_c and U_s representing

$$U_c = \left\langle \frac{\partial}{\partial \tilde{z}_c} \langle \cos \theta \rangle_{\text{eq}} \right\rangle_{\text{site}}, \quad U_s = \left\langle \frac{\partial}{\partial \tilde{z}_s} \langle \sin \theta \rangle_{\text{eq}} \right\rangle_{\text{site}}, \tag{C5}$$

where the site average $\langle \ \rangle_{\text{site}}$ denotes the average over the pattern $\{\xi_i^1\}$ and the two Gaussian noises. The Gaussian noises $\tilde{z}_c$ and $\tilde{z}_s$ with mean 0 are independent of each other and have variances

$$\langle \tilde{z}_c^2 \rangle_{\text{site}} = \frac{\alpha \langle [\langle \cos \theta \rangle_{\text{eq}}]^2 \rangle_{\text{site}}}{(1 - U_c)^2}, \quad \langle \tilde{z}_s^2 \rangle_{\text{site}} = \frac{\alpha \langle [\langle \sin \theta \rangle_{\text{eq}}]^2 \rangle_{\text{site}}}{(1 - U_s)^2}. \tag{C6}$$

The Fokker-Planck equation corresponding to the single-body Langevin equation that is reduced from Eq. (C1) using Eq. (C2) yields the marginal equilibrium probability density $P_{\text{eq}}(\theta_i)$, which is the analog of Eq. (14),

$$P_{\text{eq}}(\theta_i; \xi_i^1 m_c^1 + \tilde{z}_c, \tilde{z}_s) = C \exp\left(\frac{-H(\theta_i)}{D}\right) \tag{C7}$$

with C representing the normalization constant and

$$H(\theta_i) = -\cos \theta_i (\xi_i^1 m_c^1 + \tilde{z}_c) - \tilde{z}_s \sin \theta_i$$

$$- \frac{\alpha(U_c - U_s)}{4(1 - U_c)(1 - U_s)} \cos 2\theta_i. \tag{C8}$$

Equations (C4)–(C6) together with those for the overlap m_c^1 [Eq. (C3)], Edwards-Anderson-like order parameters $\langle [\langle \cos \theta \rangle_{\text{eq}}]^2 \rangle_{\text{site}}$ and $\langle [\langle \sin \theta \rangle_{\text{eq}}]^2 \rangle_{\text{site}}$ constitute a set of order parameter equations for the oscillator network (C1), which can be justified by applying the method presented in Sec. IV. The deterministic limit $D \to 0$ recovers the previous result [30,49].

The TAP equation under the gauge with $m_s^1 = 0$ for the oscillator network (C1) can be straightforwardly obtained as in deriving Eq. (20) of Sec. II by noting Eq. (C2):

$$\langle \cos \theta_i \rangle_{\text{eq}} = \int_0^{2\pi} d\theta_i \cos \theta_i P_{\text{eq}}(\theta_i; H_i^{\text{TAP}(c)}, H_i^{\text{TAP}(s)})$$

$$\langle \sin \theta_i \rangle_{\text{eq}} = \int_0^{2\pi} d\theta_i \sin \theta_i P_{\text{eq}}(\theta_i; H_i^{\text{TAP}(c)}, H_i^{\text{TAP}(s)}), \tag{C9}$$

where

$$H_i^{\text{TAP}(c)} = \sum_{j \neq i} J_{ij} \langle \cos \theta_j \rangle_{\text{eq}} - \Gamma_{\text{SCSNA}}^{(c)} \langle \cos \theta_i \rangle_{\text{eq}},$$

$$H_i^{\text{TAP}(s)} = \sum_{j \neq i} J_{ij} \langle \sin \theta_j \rangle_{\text{eq}} - \Gamma_{\text{SCSNA}}^{(s)} \langle \sin \theta_i \rangle_{\text{eq}}. \tag{C10}$$

We see that the analog of relation (4) also holds in this case. The above results can be confirmed by a more systematic approach based on the procedure presented in Secs. III and IV. The TAP equation under a more general situation without using the gauge $m_s^1 = 0$ can also easily be obtained, ensuring the relationship between the output proportional term and the Onsager reaction term as claimed in the present paper. More details will be published elsewhere [49].

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