Non-normal and stochastic amplification of magnetic energy in the turbulent dynamo: Subcritical case

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Our attention focuses on the stochastic dynamo equation with non-normal operator that gives an insight into the role of stochastics and non-normality in magnetic field generation. The main point of this Brief Report is a discussion of the generation of a large-scale magnetic field that cannot be explained by traditional linear eigenvalue analysis. The main result is a discovery of nonlinear deterministic instability and growth of finite magnetic field fluctuations in $\alpha\beta$ dynamo theory. We present a simple stochastic model for the thin-disk axisymmetric $\alpha\Omega$ dynamo involving three factors: (a) non-normality generated by differential rotation, (b) nonlinearity reflecting how the magnetic field affects the turbulent dynamo coefficients, and (c) stochastic perturbations. We show that even for the *subcritical case* (all eigenvalues are negative), there are three possible mechanisms for the generation of magnetic field. The first mechanism is a deterministic one that describes an interplay between transient growth and nonlinear saturation of the turbulent α effect and diffusivity. It turns out that the trivial state is nonlinearly unstable to small but finite initial perturbations. The second and third are stochastic mechanisms that account for the interaction of non-normal effect generated by differential rotation with random additive and multiplicative fluctuations.

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The generation and maintenance of large-scale magnetic fields in stars and galaxies has attracted enormous attention in past years [1–4] (see also a recent review [5]). The main candidate to explain the process of conversion of the kinetic energy of turbulent flow into magnetic energy is the mean-field dynamo theory [2]. The standard dynamo equation for the large-scale magnetic field $\mathbf{B}(t,\mathbf{x})$ reads $\partial\mathbf{B}/\partial t=\mathrm{curl}(\alpha\mathbf{B})+\beta\Delta\mathbf{B}+\mathrm{curl}(\mathbf{u}\times\mathbf{B})$, where $\mathbf{u}$ is the mean velocity field, α is the coefficient of the α effect, and β is the turbulent magnetic diffusivity. This equation has been widely used for analyzing the generation of the large-scale magnetic field. Traditionally the mathematical procedure consists of looking for exponentially growing solutions of the dynamo equation with appropriate boundary conditions (*supercritical* case). While this approach has been quite successful in the prediction of large-scale magnetic field generation, it fails to predict the *subcritical* onset of a large-scale magnetic field for some turbulent flow. Although the trivial solution $\mathbf{B}=\mathbf{0}$ is linearly stable for the subcritical case (all eigenvalues are negative or the dynamo number is less than critical one), the non-normality due to differential rotation leads to the growth of initial finite perturbations [6]. It turns out that the nonlinear interactions (β suppression) amplify this transient growth further. Thus, instead of the generation of the large-scale magnetic field being a consequence of the linear instability of trivial state $\mathbf{B}=\mathbf{0}$, it results from the interaction of transient amplifications due to the non-normality with nonlinearities. Thus, the crucial idea behind subcritical transition is that the α effect might be weak, but the generation and maintenance of the large-scale magnetic field is still possible.

The importance of the transient growth of magnetic field for the induction equation has been discussed recently in Refs. [7,8]. The linear growth of magnetic field due to differential rotation has been discussed in Refs. [3,19]. Comprehensive reviews of subcritical transition in hydrodynamics due to the non-normality of the linearized Navier-Stokes equation can be found in Refs. [9,10]. It is known that non-normal dynamical systems have an extraordinary sensitivity to stochastic perturbations whuch leads to great amplifications of the average energy of the dynamical system [11]. Although the literature discussing the mean-field dynamo equation is massive, the effects of non-normality and random fluctuations are relatively unexplored. Several attempts have been made to understand the role of random fluctuations in magnetic field generation. Non-normality and small-scale fluctuations parametrized by stochastic additive forcing were the subject of recent research by Farrell and Ioannou [7]. The effect of random α fluctuations on the solution of the kinematic mean-field dynamo has been studied in Ref. [12]. However they did not discuss the non-normality of the dynamo equation and the possibility of stochastic transient growth of magnetic energy. Numerical simulations of magnetoconvection equations with noise and non-normal transient growth have been performed in Ref. [8].

It is the purpose of this Brief Report to present a simple stochastic dynamo model for the thin-disk axisymmetric $\alpha\Omega$ dynamo involving three factors: non-normality, nonlinearity, and stochastic perturbations. The differential rotation (Ω effect) is a crucial factor for a non-normal behavior (we do not consider α^2 dynamo here). Recently it has been found [13] that the interactions of these factors lead to noise-induced phase transitions in a ''toy'' model mimicking a laminar-to-turbulent transition. In this Brief Report we discuss three possible mechanisms for the generation of a magnetic field that are not based on standard linear eigenvalue analysis of the dynamo equation. The first mechanism is a deterministic one that describes an interplay between transient growth and nonlinear saturation of both turbulent parameters: α and β. The second and third are stochastic mechanisms that account for the interaction of the non-normal effect generated by differential rotation with random additive and multiplicative fluctuations.

Here we study the non-normality and stochastic perturbation effects on the growth of magnetic field by using a Moss's no-z model for galaxies [14]. Despite its simplicity the no-z model proves to be very robust and gives reasonable results compared with real observations. We consider a thin turbulent disk of conducting fluid of uniform thickness $2h$ and radius R ($R \gg h$), which rotates with angular velocity $\Omega(r)$ [3,4]. We consider the case of $\alpha\Omega$ dynamo for which the differential rotation dominates over the α effect. We leave out the issue of spatial distribution of magnetic field along the radius r and the height of the disk. Our main purpose here is to concentrate on the studies of the influence of random fluctuations and non-normality on the dynamo process. One can write then the following stochastic equations for the azimuthal $B_\varphi(t)$ and radial $B_r(t)$ components of the axisymmetric magnetic field:

$$\frac{dB_r}{dt} = -\frac{\alpha(|\mathbf{B}|,\xi_\alpha(t))}{h}B_\varphi - \frac{\pi^2\beta(|\mathbf{B}|)}{4h^2}B_r + \xi_f(t),$$

$$\frac{dB_\varphi}{dt} = g_\omega B_r - \frac{\pi^2\beta(|\mathbf{B}|)}{4h^2}B_\varphi, \tag{1}$$

where $\alpha(|\mathbf{B}|,\xi_\alpha(t))$ is the random nonlinear function describing the α effect, $\beta(|\mathbf{B}|)$ is the turbulent magnetic diffusivity, and $g_\omega = r\,d\Omega/dr$ is the measure of differential rotation (usually $r\,d\Omega/dr < 0$). Here we have used a phenomenological mesoscopic approach [17,18] in which the coefficients in the classical dynamo equations are considered to be random functions of time plus random additive noise. Of course, the present paper addresses the oversimplified case of magnetic field generation. Nonetheless, we present this work as a precise illustration of the influence the random fluctuations and non-normality may play in the generation process, and which therefore should be accounted for in complicated dynamo modeling. Nonlinearity of the functions $\alpha(|\mathbf{B}|,\xi_\alpha(t))$ and $\beta(|\mathbf{B}|)$ reflects how the growing magnetic field $\mathbf{B}$ affects the turbulent dynamo coefficients. There is an uncertainty about how the dynamo coefficients are suppressed by the mean field, and current theories seem to disagree about the exact form of this suppression [20]. Here we describe the dynamo saturation by using the simplified forms [5]

$$\alpha(|\mathbf{B}|,\xi_\alpha(t) = (\alpha_0 + \xi_\alpha(t))\varphi_\alpha(|\mathbf{B}|),$$

$$\beta(|\mathbf{B}|) = \beta_0\varphi_\beta(|\mathbf{B}|), \tag{2}$$

where $\varphi_{\alpha,\beta}(|\mathbf{B}|)$ is a decaying function such that $\varphi_{\alpha,\beta}(0) = 1$. In what follows we use [5]

$$\varphi_\alpha(|\mathbf{B}|) = [1 + k_\alpha(B_\varphi/B_{eq})^2]^{-1},$$

$$\varphi_\beta(|\mathbf{B}|) = \left(1 + \frac{k_\beta}{1 + (B_{eq}/B_\varphi)^2}\right)^{-1}, \tag{3}$$

where k_α and k_β are constants of order one, and B_{eq} is the equipartition strength. It should be noted that for the $\alpha\Omega$ dynamo the azimuthal component $B_\varphi(t)$ is much larger than the radial field $B_r(t)$, therefore, $\mathbf{B}^2 \simeq B_\varphi^2$. We did not include the strong dependence of α and β on the magnetic Reynolds number R_m.

The multiplicative noise $\xi_\alpha(t)$ describes the effect of rapid random fluctuations of α. We assume that they are more important than the random fluctuations of the turbulent magnetic diffusivity β [12]. The additive noise $\xi_f(t)$ represents the stochastic forcing of unresolved scales [7]. Both noises are independent Gaussian random processes with zero means $\langle \xi_\alpha(t)\rangle = 0$, $\langle \xi_f(t)\rangle = 0$ and correlations

$$\langle \xi_\alpha(t)\xi_\alpha(s)\rangle = 2D_\alpha\delta(t-s), \quad \langle \xi_f(t)\xi_f(s)\rangle = 2D_f\delta(t-s). \tag{4}$$

The intensity of the noises is measured by the parameters D_α and D_f. One can show [13] that the additive noise in the second equation in Eq. (1) is less important.

The governing equations (1) can be nondimensionalized by using an equipartition field strength B_{eq}, a length h, and a time Ω_0^{-1}, where Ω_0 is the typical value of angular velocity. By using the dimensionless parameters

$$g = \frac{|g_\omega|}{\Omega_0}, \quad \delta = \frac{R_\alpha}{R_\omega}, \quad \varepsilon = \frac{\pi^2}{4R_\omega},$$

$$R_\alpha = \frac{\alpha_0 h}{\beta}, \quad R_\omega = \frac{\Omega_0 h^2}{\beta}, \tag{5}$$

we can write the stochastic dynamo equations in the form of SDE's

$$dB_r = -[\delta\varphi_\alpha(B_\varphi)B_\varphi + \varepsilon\varphi_\beta(B_\varphi)B_r]dt$$
$$- \sqrt{2\sigma_1}\varphi_\alpha(B_\varphi)B_\varphi\, dW_1 + \sqrt{2\sigma_2}\, dW_2,$$

$$dB_\varphi = -(gB_r + \varepsilon\varphi_\beta(B_\varphi)B_\varphi)dt, \tag{6}$$

where W_1 and W_2 are independent standard Wiener processes. The dynamical system (6) is subjected to the multiplicative and additive noises with the following corresponding intensities:

$$\sigma_1 = D_\alpha/(h^2\Omega_0), \quad \sigma_2 = D_f/(B_{eq}^2\Omega_0). \tag{7}$$

It is well known that the presence of noise can dramatically change the properties of a dynamical system [18]. Since the differential rotation dominates over the α effect ($R_\alpha \ll R_\omega$), system (6) involves two small parameters $\delta = R_\alpha/R_\omega$ and $\varepsilon = \pi^2/4R_\omega$ whose typical values are 0.01–0.1 ($R_\omega = 10$–100, $R_\alpha = 0.1$–1). These parameters play very important roles in what follows. For small values of δ and ε, the linear operator in Eq. (6) is a highly non-normal one ($g \sim 1$). This can lead to a large transient growth of the azimuthal component $B_\varphi(t)$ in a subcritical case. Similar deterministic low-dimensional models have been proposed to explain the subcritical transition in the Navier-Stokes equations (see, for example, Refs. [15,16]). The probability density function $p(t,B_r,B_\varphi)$ obeys the Fokker-Planck equation associated with Eq. (6) [17]. Using this equation in the linear

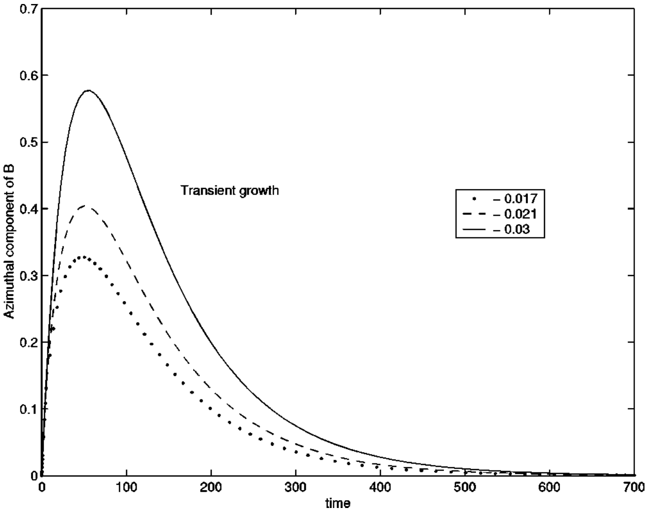

FIG. 1. Linear case: the azimuthal component B_φ as a function of time $[B_\varphi(0)=0]$ for $g=1$, $\delta=10^{-4}$, and $\varepsilon=2\times10^{-2}$ and different initial values of B_r, $-0.017, -0.021$, and -0.03.

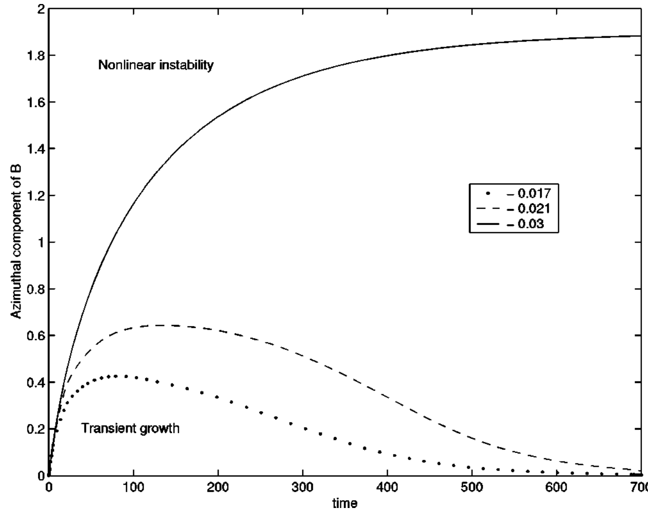

FIG. 2. Nonlinear case: B_φ as a function of time $[B_\varphi(0)=0]$ for $g=1$, $\delta=10^{-4}$ and $\varepsilon=2\times10^{-2}$ and different initial values of B_r, $-0.017, -0.021, -0.03$.

case one can find a closed system of ordinary differential equations for the moments $\langle B_r^2\rangle$, $\langle B_r B_\varphi\rangle$, and $\langle B_\varphi^2\rangle$:

$$\frac{d}{dt}\begin{pmatrix}\langle B_r^2\rangle\\ \langle B_r B_\varphi\rangle\\ \langle B_\varphi^2\rangle\end{pmatrix}=\begin{pmatrix}-2\varepsilon & -2\delta & \sigma_1\\ -g & -2\varepsilon & -\delta\\ 0 & -2g & -2\varepsilon\end{pmatrix}\begin{pmatrix}\langle B_r^2\rangle\\ \langle B_r B_\varphi\rangle\\ \langle B_\varphi^2\rangle\end{pmatrix}+\begin{pmatrix}\sigma_2\\ 0\\ 0\end{pmatrix}. \tag{8}$$

Now we are in a position to discuss three possible scenarios for the subcritical generation of galactic magnetic field.

Deterministic subcritical generation. Let us examine the deterministic transient growth of the magnetic field in the subcritical case. To illustrate the non-normality effect consider first the linear case without noise terms. The dynamical system (6) takes the form

$$\frac{d}{dt}\begin{pmatrix}B_r\\ B_\varphi\end{pmatrix}=\begin{pmatrix}-\varepsilon & -\delta\\ -g & -\varepsilon\end{pmatrix}\begin{pmatrix}B_r\\ B_\varphi\end{pmatrix}. \tag{9}$$

Since $\delta\ll1$, $\varepsilon\ll1$, and $g\sim1$, this system involves a highly non-normal matrix with two eigenvalues $\gamma_{1,2}=-\varepsilon\pm\sqrt{\delta g}$ (the corresponding eigenvectors are almost parallel). The supercritical excitation condition $\gamma_1>0$ can be written as $\sqrt{\delta g}>\varepsilon$ or $\sqrt{R_\alpha R_\omega g}>\pi^2/4$ [4]. Consider the subcritical case when $0<\delta<\varepsilon^2/g$. The solution of system (9) with the initial conditions $B_r(0)=-2c\sqrt{\delta/g}$ and $B_\varphi(0)=0$ is $B_\varphi(t)=c(e^{\gamma_1 t}-e^{\gamma_2 t})$. Thus $B_\varphi(t)$ exhibits large transient growth over a time scale of order $1/\varepsilon$ before decaying exponentially. In Fig. 1 we plot the azimuthal component B_φ as a function of time for $g=1$, $\delta=10^{-4}$, and $\varepsilon=2\times10^{-2}$ and different initial values of B_r $[B_\varphi(0)=0]$.

Of course without nonlinear terms any initial perturbation decays. However if we take into account the back reaction suppressing the effective dissipation, one can expect an entirely different global behavior. In the deterministic case there can be several stationary solutions to Eq. (6) [21]. In Fig. 2 we illustrate the role of transient growth and nonlinearity in the transition to a nontrivial state using Eq. (3) with $k_\alpha=0.5$ and $k_\beta=3$. We plot the azimuthal component B_φ as a function of time with the initial condition $B_\varphi(0)=0$. We use the same values of parameters g, δ, and ε and three initial values of $B_r(0)$ as in Fig. 1. One can see from Fig. 2 that the trivial solution $B_\varphi=B_r=0$ is nonlinearly unstable to small but finite initial perturbations of B_r, such as, $B_r(0)=-0.03$. For fixed values of the parameters in nonlinear system (6), there exists a threshold amplitude for the initial perturbation, above which $B_\varphi(t)$ grows and below which it eventually decays.

Stochastic subcritical generation due to additive noise. This scenario has been already discussed in the literature [7] (see also Ref. [11] for hydrodynamics). The physical idea is that the average magnetic energy is maintained by additive Gaussian random forcing representing unresolved scales. It is clear that the nonzero additive noise $(\sigma_2\neq0)$ ensures the stationary solution to Eq. (8). If we assume for simplicity $\sigma_1=0$ and $\delta=0$ then the dominant stationary moment is

$$\langle B_\varphi^2\rangle_{st}=(g^2\sigma_2)/(4\varepsilon^3). \tag{10}$$

We can see that due to the non-normality of system (9) the average stationary magnetic energy $E_{st}\sim\langle B_\varphi^2\rangle_{st}$ exhibits a high degree of sensitivity with respect to the small parameter ε: $E_{st}\sim\varepsilon^{-3}$ [11,13].

Stochastic subcritical generation due to multiplicative noise. Here we discuss the divergence of the average magnetic energy $E(t)\sim\langle B_\varphi^2\rangle$ with time t due to the random fluctuations of the α parameter. Although the first moments tend to zero in the subcritical case, the average energy $E(t)$ grows as $e^{\lambda t}$ when the intensity of noise σ_1 exceeds a critical value. The growth rate λ is the positive real root of the characteristic equation for system (8)

$$(\lambda+2\varepsilon)^3-4\delta g(\lambda+2\varepsilon)-2\sigma_1 g^2=0. \qquad (11)$$

For $\delta=0$, the growth rate is $\lambda_0=-2\varepsilon+(2\sigma_1 g^2)^{1/3}$ as long as it is positive, and the excitation condition can be written as $\sigma_1>\sigma_{cr}=2\varepsilon^3/g^2$. It means that the generation of average magnetic energy occurs for $\alpha_0=0$! It is interesting to compare this criterion with the classical supercritical excitation condition: $\delta g>\varepsilon^2$ [4]. To assess the significance of this parametric instability it is useful to estimate the magnitude of the critical noise intensity σ_{cr}. First let us estimate the parameter $\varepsilon=\pi^2\beta/(4\Omega_0 h^2)$. The turbulent magnetic diffusivity is given by $\beta\simeq lv/3$, where v is the typical velocity of turbulent eddy $v\simeq 10\ \mathrm{km\,s^{-1}}$, and l is the turbulent scale, $l\simeq 100$ pc. For spiral galaxies, the typical values of the thickness h and the angular velocity Ω_0 are $h\simeq 400$ pc and $\Omega_0\simeq 10^{-15}\ \mathrm{s^{-1}}$; $g\simeq 1$ [4]. It gives an estimate of $\varepsilon\simeq 0.128$, that is, $\sigma_{cr}\simeq 8.4\times 10^{-3}$. In general $\lambda(\delta)=\lambda_0+[4/3g(2\sigma_1 g)^{-1/3}]\delta+o(\delta)$. This analysis predicts an amplification of the average magnetic energy in system (6) where no such amplification is observed in the absence of noise. The value of the critical noise intensity parameter σ_{cr}, above which the instability occurs, is proportional to ε^3, that is, very small indeed. In summary, we have discussed galactic magnetic field generation that cannot be explained by traditional linear eigenvalue analysis of dynamo equation. We have presented a simple stochastic model for the $\alpha\Omega$ dynamo involving three factors: (a) non-normality due to differential rotation, (b) nonlinearity of the turbulent dynamo α effect and diffusivity β, and (c) additive and multiplicative noises. We have shown that even for the subcritical case, there are three possible scenarios for the generation of large-scale magnetic field. The first mechanism is a deterministic one that describes an interplay between transient growth and nonlinear saturation of the turbulent α effect and diffusivity. We have shown that the trivial state $\mathbf{B}=0$ can be nonlinearly unstable with respect to small but finite initial perturbations. The second and third are stochastic mechanisms that account for the interaction of non-normal effect generated by differential rotation with random additive and multiplicative fluctuations. We have shown that multiplicative noise associated with the α effect leads to exponential growth of the average magnetic energy even in the subcritical case.

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[1] H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, New York, 1978).

[2] F. Krause and K. H. Rädler, *Mean-Field Magnetohydrodynamics and Dynamo Theory* (Academic-Verlag, Berlin, 1980).

[3] Ya. B. Zeldovich *et al.*, *Magnetic Fields in Astrophysics* (Gordon and Breach Science Publishers, New York, 1983).

[4] A. A. Ruzmaikin *et al.*, *Magnetic Fields in Galaxies* (Kluwer Academic Publishers, Dordrecht, 1988).

[5] L. Widrow, Rew. Mod. Phys. **74**, 775 (2002).

[6] An operator is said to be non-normal if it does not commute with its adjoint in the corresponding scalar product. Typical example of transient growth can be represented by the function $\exp(-t)-\exp(-2t)$ for $t\geqslant 0$.

[7] B. Farrel and P. Ioannou, Astrophys. J. **522**, 1088 (1999).

[8] J.R. Gog, I. Opera, M. R. E. Proctor, and A.M. Rucklidge, Proc. R. Soc. London, Ser. A **455**, 4205 (1999).

[9] S. Grossmann, Rev. Mod. Phys. **72**, 603 (2000).

[10] P. J. Schmid and D. S. Henningson, *Stability and Transition in Shear Flows* (Springer, Berlin, 2001).

[11] B. F. Farrell and P. J. Ioannou, Phys. Fluids A **5**, 2600 (1993); Phys. Rev. Lett. **72**, 1188 (1994); J. Atmos. Sci. **53**, 2025 (1996).

[12] P. Hoyng, D. Schmitt, and L. J. W. Teuben, Astron. Astrophys. **289**, 265 (1994); E. Vishniac and A. Brandenburg, Astrophys. J. **475**, 263 (1997).

[13] S. Fedotov *et al.*, Phys. Rev. E **66**, 066310 (2002).

[14] D. Moss, Mon. Not. R. Astron. Soc. **275**, 191 (1995); R. Beck *et al.*, Annu. Rev. Astron. Astrophys. **34**, 155 (1996).

[15] L. N. Trefethen *et al.*, Science **261**, 578 (1993).

[16] T. Gebhardt and S. Grossmann, Phys. Rev. E **50**, 3705 (1994); J. S. Baggett *et al.*, Phys. Fluids **7**, 833 (1995); J. S. Baggett and L. N. Trefethen, *ibid.* **9**, 1043 (1997).

[17] C. W. Gardiner, *Handbook of Stochastic Methods*, 2nd ed. (Springer, New York, 1996).

[18] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer, Berlin, 1984).

[19] E. Blackman, Astrophys. J. **496**, L17 (1998).

[20] S. I. Vainshtein and F. Cattaneo, Astrophys. J. **393**, 165 (1992); F. Cattaneo, and D. W. Hughes, Phys. Rev. E **54**, R4532 (1996); A. V. Gruzinov and P. H. Diamond, Phys. Rev. Lett. **72**, 1651 (1994); A. Brandenburg, Astrophys. J. **550**, 824 (2001); I. Rogachevskii and N. Kleeorin, Phys. Rev. E **64**, 056307 (2001); E. G. Blackman and G.B. Field, Phys. Rev. Lett. **89**, 265007 (2002).

[21] S. M. Tobias, Astrophys. J. Lett. **467**, 870 (1996).