Perturbation-induced radiation by the Ablowitz-Ladik soliton

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An efficient formalism is elaborated to analytically describe dynamics of the Ablowitz-Ladik soliton in the presence of perturbations. This formalism is based on using the Riemann-Hilbert problem and provides the means of calculating evolution of the discrete soliton parameters, as well as shape distortion and perturbation-induced radiation effects. As an example, soliton characteristics are calculated for linear damping and quintic perturbations.

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I. INTRODUCTION

Dynamics of discrete solitons (intrinsic localized modes) in nonlinear lattices has become a topic of intense research summarized in a number of excellent reviews [1]. Propagation properties of waves arising as a result of the interplay of nonlinearity with lattice discreteness can be quite distinct from those inherent in continuous nonlinear systems and hold much promise for applications in various physical, biological, and technological problems. Examples are energy localization and transfer in systems of nonlinear oscillators [2], propagation of self-trapped beams in arrays of coupled nonlinear optical waveguides [3,4], nonlinear charge and excitation transport in biological macromolecules [5,6], local denaturation of DNA double helix [7], dynamics of localized excitations in arrays of coupled Josephson junctions [8], propagation of optical spatial solitons in nonlinear photonic crystals [9], and in diffraction-managed waveguide systems [10], creating discrete solitons in Bose-Einstein condensate [11]. Recently it was proposed [12] to use discrete solitons in two-dimensional networks of nonlinear waveguides to realize functional operations such as blocking, routing, logic functions, and time gating.

Most of the above phenomena are modeled by the discrete nonlinear Schrödinger (DNLS) equation or, in a more general setting, by the discrete self-trapping equation [2]. Recent developments in the study of the DNLS equation are reviewed in Refs. [13,14]. However, the standard DNLS equation is nonintegrable [15,16] and does not exhibit exact soliton solutions, though it can be derived as a discretization of the integrable continuous NLS equation. Hence, numerical methods are generally used to investigate nonlinear lattice dynamics on the basis of the DNLS equation.

On the other hand, there exists the integrable discretization of the NLS equation—the Ablowitz-Ladik (AL) equation [17] which has exact soliton solutions and admits the complete description in the framework of the inverse spectral method. Moreover, Konotop *et al.* [18] and Cai *et al.* [19] proved integrability of the inhomogeneous AL system in an external electric field of a particular form. Being unique from the mathematical point of view, the AL equation is less applicable in physics than the DNLS equation. Salerno [20] introduced an equation that interpolates between the DNLS and AL equations and permits studying (as a rule, numerically) the role of integrable and nonintegrable contributions to lattice properties [21]. The AL-DNLS system with an impurity was investigated by Hennig *et al.* [22]

A different point of view on the interrelation between the AL and DNLS equations was posed in Refs. [23–25]. In a definite region of parameters the DNLS equation can be treated as a perturbed version of the AL equation. When a perturbation is small, the discrete soliton perturbation theory can be successfully applied to analytically describe localized excitations in a system governed by the DNLS equation. Such an approach was developed in Refs. [23–25] in the framework of the adiabatic approximation, when a perturbation-induced radiation is ignored and a perturbation manifests itself as a slow evolution of initially constant AL soliton parameters. The evolution equations for the parameters were derived by Vakhnenko and Gaididei [23]. Stability aspects of Hamiltonian perturbations for the AL equation were discussed by Kapitula and Kevrekidis [26]. Recently the perturbative method to study the AL soliton dynamics was used in Ref. [27] in relation to energy transport in α-helical proteins and in Ref. [28] for the soliton in a random medium. Besides, Abdullaev *et al.* [29] proved the existence of discrete autosolitons in the AL model with linear and quintic damping, cubic amplification, and complex filtering (the discrete complex Ginzburg-Landau model). Exact solutions of this model for certain relations between parameters are given in Ref. [30].

It is well known that a perturbation of the soliton is also accompanied by radiation of small-amplitude dispersive linear waves (or shape distortion) [31], and a complete description of the perturbed soliton dynamics necessitates accounting for both the soliton parameter evolution and the radiation effects. Therefore, the main goal of the present paper is to develop a corresponding (relatively simple) formalism and to extend, as far as possible, the applicability of analytical methods in studying near-integrable nonlinear discrete systems. It should be noted in this connection that Konotop

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et al. [32] derived by means of the Gel'fand-Levitan-like summation equations the evolution equation for the reflection coefficient in the case of the inhomogeneous AL model but without using it for specific calculations. An estimation of radiative corrections to the AL soliton subjected to the stochastic perturbation was outlined in the important paper by Garnier [28] on the basis of conserved quantities.

Our approach utilizes the Riemann-Hilbert (RH) problem [33]. The application of the RH problem to perturbed nonlinear equations was initiated by Kivshar [34] on an example of the Landau-Lifshitz equation. A purely algebraical calculation of higher-order corrections to the perturbed NLS soliton and of radiation effects for a soliton in a doped fiber was performed on the basis of the RH problem in Ref. [35]. Such an approach has been proved to be efficient for a wide class of continuous perturbed nonlinear equations, including multicomponent ones [36].

This paper gives a self-contained exposition of the AL soliton perturbation theory. In Sec. II we fix preliminary facts concerning the AL spectral problem which are used in Sec. III to formulate the RH problem. In Sec. IV we describe a procedure to solve the RH problem with zeros and obtain immediately the AL soliton solution in Sec. V. We stress that calculations within the RH problem do not use discrete analogs of the Gel'fand-Levitan integral equations. Section VI is devoted to derivation of the evolution equations for the RH problem data associated with the AL soliton parameters. These equations exactly account for the perturbation and serve in the subsequent sections as the generating equations for the perturbative expansion. Section VII contains brief exposition of the adiabatic approximation, whereas Sec. VIII represents the main result of the paper—derivation of formulas for calculating radiative corrections from the continuous part of the RH problem data. In Sec. IX we illustrate the formalism by the examples of linear damping and quintic perturbations. Appendixes contain some technical details of the applications of the RH problem.

II. THE ABLOWITZ-LADIK SPECTRAL PROBLEM

A. Jost solutions and asymptotics

Integrable discretized NLS equation (AL equation)

$$iu_{nt}+u_{n+1}+u_{n-1}-2u_n+|u_n|^2(u_{n+1}+u_{n-1})=0 \tag{2.1}$$

for a scalar complex function u which depends on discrete variable n, $-\infty<n<\infty$, and time t admits the Lax representation with the AL spectral problem [17]

$$J(n+1)=(E+Q_n)J(n)E^{-1},$$
$$Q_n=\begin{pmatrix} 0 & u_n \\ -u_n^* & 0 \end{pmatrix}, \quad E=\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \tag{2.2}$$

and the evolutionary equation (subscript t means time derivative)

$$J_t(n)=V(n)J(n)-J(n)\Omega(z),$$
$$V(n)=i\begin{pmatrix} u_{n-1}^*u_n & zu_n-z^{-1}u_{n+1} \\ z^{-1}u_n^*-zu_{n-1}^* & -u_{n-1}u_n^* \end{pmatrix}+\Omega, \tag{2.3}$$
$$\Omega(z)=\frac{i}{2}(z-z^{-1})^2\sigma_3.$$

It means that Eq. (2.1) arises as a compatibility condition for Eqs. (2.2) and (2.3). Here z is a constant spectral parameter and the star stands for the complex conjugation. The spectral problem in the form (2.2) differs slightly from the usual one [17] and permits introducing matrix Jost solutions $J_\pm(n)$ of Eq. (2.2) with the unit asymptotics, $J_\pm(n)\to\mathbb{1}$ as $n\to\pm\infty$. $J_\pm(n)$ solve Eq. (2.3) as well. The scattering matrix $S(z)$ defined by

$$J_-(n)=J_+(n)E^nS(z)E^{-n} \tag{2.4}$$

has the structure

$$S(z)=\begin{pmatrix} a_+ & -b_- \\ b_+ & a_- \end{pmatrix}.$$

The Jost solutions obey the conjugation condition

$$J_\pm^\dagger(n,\bar{z})=v_\pm(n)J_\pm^{-1}(n,z), \tag{2.5}$$

where $\bar{z}=1/z^*$, "†" means the Hermitian conjugation and

$$v_+(n)=\prod_{l=n}^{\infty}\rho_l^{-1}, \quad v_-(n)=\prod_{l=-\infty}^{n-1}\rho_l, \quad \rho_l=1+|u_l|^2.$$

We also obtain that $\det J_\pm(n,z)=v_\pm(n)$, $\det S=v$, where $v=\Pi_{n=-\infty}^{\infty}\rho_n$ and evidently $v_+(n)v=v_-(n)$. From Eqs. (2.4) and (2.5) we obtain $S^\dagger(\bar{z})=vS^{-1}(z)$ which gives $a_+^*(\bar{z})=a_-(z)$, $b_+^*(\bar{z})=b_-(z)$.

The AL spectral problem (2.2) obeys the important symmetry ("$\mathcal{P}$ parity"): if $J(n,z)$ is a solution, then

$$\mathcal{P}J(n,z)\equiv\sigma_3J(n,-z)\sigma_3 \tag{2.6}$$

is a solution, too. It follows from Eq. (2.6) that diagonal elements of $J(n,z)$ are even functions of z, while off-diagonal entries are odd functions. The same symmetry is valid for the Jost solutions and the matrix S, the latter means $a_\pm(z)=a_\pm(-z)$, $b_\pm(z)=-b_\pm(-z)$.

Now we consider asymptotic behavior of the solution $J(n,z)$ for $z\to\infty$. Let

$$J(n,z)=J^{(0)}(n)+z^{-1}J^{(1)}(n)+O(z^{-2}), \quad z\to\infty.$$

Inserting this expansion into Eq. (2.2) gives

$$J^{(0)}(n+1)=\begin{pmatrix} 1 & 0 \\ 0 & \rho_n \end{pmatrix}J^{(0)}(n), \tag{2.7}$$

while the potential u_n is retrieved as

$$u_n=-J_{12}^{(1)}/J_{22}^{(0)}. \tag{2.8}$$

Note that the asymptotics (2.7) is consistent with the $\mathcal{P}$-parity property (2.6). Similar results hold for $z\to 0$, when $J(n,z)=J_{(0)}(n)+zJ_{(1)}(n)+O(z^2)$:

$$J_{(0)}(n+1)=\begin{pmatrix}\rho_n & 0\\ 0 & 1\end{pmatrix}J_{(0)}(n).$$

B. Analyticity

Let $\mathcal{C}_\pm$ be the domains in the complex z plane lying outside $(+)$ and inside $(-)$ the unit circle $|z|=1$. It follows from the spectral problem (2.2) that the first column $J_-^{[1]}(n,z)$ of the Jost function J_- and the second one $J_+^{[2]}(n,z)$ of J_+ are analytical in $\mathcal{C}_+$ (and continuous for $z\to 1$). Hence, the matrix function

$$\Psi_+(n,z)=(J_-^{[1]},J_+^{[2]})(n,z)$$

is a solution of the spectral problem (2.2) and analytical as a whole in $\mathcal{C}_+$. We obtain from the conjugation formula, Eq. (2.5), that the rows $(J_-)^{-1}_{[1]}$ and $(J_+)^{-1}_{[2]}$ are analytical in $\mathcal{C}_-$. As a result, the matrix function

$$\Psi_-^{-1}(n,z)=\begin{pmatrix}(J_-)^{-1}_{[1]}\\ (J_+)^{-1}_{[2]}\end{pmatrix}(n,z)$$

is analytical as a whole in $\mathcal{C}_-$ and solves the adjoint spectral problem.

Analytical solutions can be expressed in terms of the Jost functions. Indeed,

$$\begin{aligned}\Psi_+&=(J_-^{[1]},J_+^{[2]})=(a_+J_+^{[1]}+z^{-2n}b_+J_+^{[2]},J_+^{[2]})\\ &=J_+E^nS_+E^{-n},\end{aligned}\tag{2.9}$$

$$S_+(z)=\begin{pmatrix}a_+ & 0\\ b_+ & 1\end{pmatrix},$$

as well as

$$\Psi_+=J_-E^nS_-E^{-n},\quad S_-=\begin{pmatrix}1 & b_-/v\\ 0 & a_+/v\end{pmatrix},$$

$$S_+=SS_-.$$

It follows from these formulas that

$$\det\Psi_+(n,z)=v_+(n)a_+(z).\tag{2.10}$$

In the same way we obtain

$$\Psi_-^{-1}=E^nT_+E^{-n}J_+^{-1}=E^nT_-E^{-n}J_-^{-1},$$

$$T_+=\begin{pmatrix}a_-/v & b_-/v\\ 0 & 1\end{pmatrix},\quad T_-=\begin{pmatrix}1 & 0\\ b_+ & a_-\end{pmatrix},\tag{2.11}$$

$$\det\Psi_-^{-1}=v_-^{-1}(n)a_-(z),\quad T_+S=T_-.$$

Asymptotic behavior of analytical solutions is derived directly from that of the Jost functions and Eqs. (2.9) and (2.11):

$$\begin{aligned}\Psi_+(n,z)&\to\begin{pmatrix}1 & 0\\ 0 & v_+(n)\end{pmatrix},\quad z\to\infty,\\ \Psi_-^{-1}(n,z)&\to\begin{pmatrix}v_-^{-1}(n) & 0\\ 0 & 1\end{pmatrix},\quad z\to 0.\end{aligned}\tag{2.12}$$

Hence, $\det\Psi_+\to v_+(n)$ as $z\to\infty$ which gives from Eq. (2.10) $a_+(z)\to 1$ as $z\to\infty$. Similarly, $a_-(z)\to 1$ as $z\to 0$. The conjugation formula for the analytical solutions follows from Eq. (2.5):

$$\Psi_+^\dagger(n,z)=B(n)\Psi_-^{-1}(n,\bar{z}),\quad B(n)=\begin{pmatrix}v_-(n) & 0\\ 0 & v_+(n)\end{pmatrix}.\tag{2.13}$$

III. MATRIX RIEMANN-HILBERT PROBLEM

Having matrix functions Ψ_+ and Ψ_-^{-1} which are analytical in two complementary domains $\mathcal{C}_+$ and $\mathcal{C}_-$ of the z plane and continuous on the contour $|z|=1$, we can pose the matrix RH problem

$$\Psi_-^{-1}(n,z)\Psi_+(n,z)=E^nG(z)E^{-n},\quad |z|=1\tag{3.1}$$

as a problem of analytical factorization of the matrix function $G(z)$ defined on the unit circle $|z|=1$. It follows from Eqs. (2.9) and (2.11) that the matrix G has the form

$$G=T_+S_+=T_-S_-=\begin{pmatrix}1 & b_-/v\\ b_+ & 1\end{pmatrix}.\tag{3.2}$$

The normalization of the RH problem is given by Eq. (2.12).

The RH problem (3.1) has a noncanonical normalization depending on the potential u_n. It has been proved [37] that it is possible to reformulate the AL spectral problem (2.2) so as to arrive at the RH problem with the canonical normalization and to give a Hamiltonian formulation with the canonical Poisson brackets. However, the above canonicity is achieved at the cost of *nonlinear* dependence of the spectral problem on the potential. Being useful for treating nonperturbative AL equation and its integrable generalizations, such an approach seems to be of less value for the case of perturbations.

In general, the matrices Ψ_+ and Ψ_-^{-1} have zeros in some points z_j and $\overline{z_k}$ in their regions of analyticity, i.e., $\det\Psi_+(z_j)=0$, $z_j\in\mathcal{C}_+$, $j=1,2,\ldots,N_+$, and $\det\Psi_-^{-1}(\overline{z_k})=0$, $\overline{z_k}\in\mathcal{C}_-$, $k=1,2,\ldots,N_-$. We suppose that all zeros are simple and of finite number with $N_+=N_-\equiv N$ (in other words, we have zero-index RH problem). In virtue of the $\mathcal{P}$-parity, zeros appear in pairs as $\pm z_j$ and $\pm\overline{z_k}$. Taking into account Eqs. (2.10) and (2.11), we conclude that zeros of Ψ_+ and Ψ_-^{-1} are given by zeros of the scattering matrix elements: $a_+(\pm z_j)=0$ and $a_-(\pm\overline{z_k})=0$.

IV. REGULARIZATION OF THE RIEMANN-HILBERT PROBLEM

We will solve the RH problem (3.1) with zeros by means of its regularization. This procedure consists in extracting from $\Psi_\pm$ rational factors which are responsible for the existence of zeros. Indeed, near the point z_j we have $\det\Psi_+(z)\sim(z-z_j)$. Let us introduce a rational matrix function $\Xi_j^{-1}=\mathbb{1}+(z_j-\overline{z_j})(z-z_j)^{-1}P_j$, where P_j is a rank 1 projector, $P_j^2=P_j$. Because $P_j=\mathrm{diag}(1,0)$ in an appropriate basis, we obtain $\det\Xi_j^{-1}=(z-\overline{z_j})(z-z_j)^{-1}$. Therefore, the product $\Psi_+(z)\Xi_j^{-1}(z)$ is regular in the point z_j (its determinant is nonzero in this point). Regularization of the zero $-z_j$ is given by a rational function $\Xi_{-j}^{-1}=\mathbb{1}-(z_j-\overline{z_j})(z+z_j)^{-1}P_{-j}$. As a result, the matrix function $\psi_+(n,z)=\Psi_+(n,z)\Xi_j^{-1}\Xi_{-j}^{-1}$ is regular in the points $\pm z_j$. In the same manner we regularize the matrix $\Psi_-^{-1}(n,z)$ with zeros in $\pm\overline{z_k}$. Namely, the matrix

$$\psi_-^{-1}(n,z)=\Xi_{-k}\Xi_k\Psi_-^{-1}(n,z)$$

is regular in the points $\pm\overline{z_k}$ and

$$\Xi_k=\mathbb{1}-\frac{z_k-\overline{z_k}}{z-\overline{z_k}}P_k,\quad \Xi_{-k}=\mathbb{1}+\frac{z_k-\overline{z_k}}{z+\overline{z_k}}P_{-k}.$$

Regularizing all $4N$ zeros of the RH problem (3.1), we represent the functions $\Psi_\pm$ as a product

$$\Psi_\pm=\psi_\pm\Gamma \tag{4.1}$$

of the rational matrix function

$$\Gamma(n,z)=\Xi_{-N}\Xi_N\Xi_{-(N-1)}\Xi_{N-1}\cdots\Xi_{-1}\Xi_1 \tag{4.2}$$

and the holomorphic matrix functions $\psi_\pm(n,z)$ which solve the regular RH problem (i.e., without zeros):

$$\psi_-^{-1}(n,z)\psi_+(n,z)=\Gamma(n,z)E^nG(z)E^{-n}\Gamma^{-1}(n,z). \tag{4.3}$$

The appearance of a simple zero z_j of the matrix Ψ_+ means that there exists an eigenvector $|j\rangle$ with zero eigenvalue,

$$\Psi_+(n,z_j)|j\rangle=0. \tag{4.4}$$

Taking the Hermitean conjugation of this equality with account of the conjugation property (2.13), we obtain $\langle j|B\Psi_-^{-1}(n,\overline{z_j})=0$ with $\langle j|=|j\rangle^\dagger$. Therefore, the projector P_j can be naturally defined as

$$P_j=\frac{|j\rangle\langle j|B}{\langle j|B|j\rangle}. \tag{4.5}$$

For the zero $-z_j$ we have $\Psi_+(n,-z_j)|-j\rangle=0$. In virtue of the $\mathcal{P}$ parity, both vectors $|j\rangle$ and $|-j\rangle$ are interrelated, $|-j\rangle=\sigma_3|j\rangle$, and therefore $P_{-j}=\sigma_3P_j\sigma_3$.

For practical purposes, it is more convenient to decompose the products (4.2) into simple fractions. Following Refs. [38,39], we obtain

$$\Gamma(n,z)=\mathbb{1}-\sum_{j,k=1}^{2N}\frac{1}{z-\overline{z_k}}|y_j\rangle(D^{-1})_{jk}\langle y_k|B, \tag{4.6}$$

$$\Gamma^{-1}(n,z)=\mathbb{1}+\sum_{j,k=1}^{2N}\frac{1}{z-z_j}|y_j\rangle(D^{-1})_{jk}\langle y_k|B$$

with new vectors $|y_j\rangle$, where zeros are enumerated as z_1, $-z_1,z_2,-z_2,\ldots,z_N,-z_N$ (and similarly for $\pm\overline{z_k}$), whereas matrix elements D_{kj} are given by

$$D_{kj}=\langle y_k|\frac{B}{z_j-\overline{z_k}}|y_j\rangle. \tag{4.7}$$

It is seen from Eq. (4.6) that the asymptotic expansion for $\Gamma(n,z)$ has the form

$$\Gamma(n,z)=\mathbb{1}+z^{-1}\Gamma^{(1)}(n)+O(z^{-2}).$$

Because $\Psi_+=\Psi_+^{(0)}+z^{-1}\Psi_+^{(1)}+O(z^{-2})=\psi_+(\mathbb{1}+z^{-1}\Gamma^{(1)}+O(z^{-2}))$, we can choose a z-independent function ψ_+ as a solution of the regular RH problem (4.3):

$$\psi_+(n)=\Psi_+^{(0)}(n)=\begin{pmatrix}1 & 0\\ 0 & v_+(n)\end{pmatrix}, \tag{4.8}$$

where the last equality follows from Eq. (2.12). Therefore, in accordance with Eqs. (2.8) and (4.1), the solution $u_n(t)$ of the AL equation can be retrieved from the solution of the RH problem as

$$u_n(t)=-\lim_{z\to\infty}\frac{(z\Psi_+)_{12}}{\Psi_{+22}}=-\frac{\Psi_{+12}^{(1)}}{\Psi_{+22}^{(0)}}=-\frac{\Gamma_{12}^{(1)}}{v_+(n)}. \tag{4.9}$$

The matrix Γ is mainly determined by the vector $|y_j\rangle$. Now we derive a coordinate dependence of the vector. It follows from the spectral problem that

$$\Psi_+(n+1,z_j)|y_j,n+1,t\rangle=0$$
$$=[E(z_j)+Q_n]\Psi_+(n,z_j)\times E^{-1}(z_j)|y_j,n+1,t\rangle.$$

Hence, we can pose $E^{-1}(z_j)|y_j,n+1,t\rangle=|y_j,n,t\rangle$, or

$$|y_j,n,t\rangle=E^n(z_j)|y_j,t\rangle, \tag{4.10}$$

where the vector $|y_j,t\rangle$ does not depend on n. Similarly, it follows from Eqs. (2.3) and (4.4) that $|y_j,n,t\rangle_t=\Omega(z_j)|y_j,n,t\rangle$. Therefore, the coordinate dependence of $|y_j,n,t\rangle$ is given as

$$|y_j,n,t\rangle=E^n(z_j)e^{\Omega(z_j)t}|p\rangle,\quad |p\rangle=\mathrm{const}. \tag{4.11}$$

Finally, we find from the identity $\det\Psi_+(z_j,n,t)=0$ that zeros z_j do not depend on n and t. Zeros $\pm z_j$, $\pm\overline{z_j}$ and vectors $|y_j,n,t\rangle$ comprise the discrete part of the RH problem data, while the functions $b_\pm(z)$ entering the matrix G (3.2) are responsible for the continuous data with the dependence on t of the form

$$G_t=[\Omega,G]. \tag{4.12}$$

V. SOLITON SOLUTION

In what follows we will not dwell on evident formulas for the general case of $4N$ zeros. Instead we will give a detailed account of the case of four zeros corresponding to a soliton

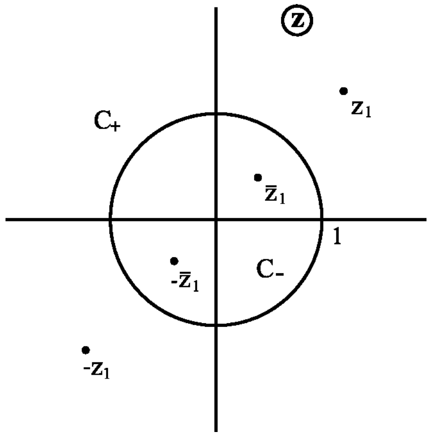

FIG. 1. Typical arrangement of zeros corresponding to a single soliton.

(Fig. 1). Hence, after the regularization of the matrix RH problem with zeros $\pm z_1$ and $\pm\overline{z_1}$, we arrive at the regular RH problem (4.3). Solitons are associated with the discrete part of the RH data, while the continuous data are now trivial ($G=\mathbb{1}$). Hence, the solutions of the regular RH problem are written in accordance with Eq. (4.8) as

$$\psi_+(n)=\psi_-(n)=\begin{pmatrix} 1 & 0 \\ 0 & v_+(n) \end{pmatrix}. \tag{5.1}$$

It is possible to express $v_+(n)$ [and $v_-(n)$] through $\Gamma(n,z=0)$. Indeed, because now $\Psi_+=\Psi_-$, we find from Eq. (2.12) $\Psi_+\to\mathrm{diag}(v_-(n),1)$ as $z\to 0$ which gives the following from $\Psi_+=\psi_+\Gamma$ and Eqs. (4.11) and (2.13):

$$\Gamma(n,0)=\mathrm{diag}(v_-(n),v_+^{-1}(n)), \tag{5.2}$$

$$B=\mathrm{diag}(\Gamma_{11}(n,0),\Gamma_{22}(n,0)^{-1}).$$

Thus, the reconstruction formula (4.9) for solitons is written more conveniently as

$$u_n(t)=-\Gamma_{12}^{(1)}(n)\Gamma_{22}(n,0). \tag{5.3}$$

Hence, it is the matrix $\Gamma(n,z)$ that determines the soliton solution. For simplicity, we will hereafter denote the vector $|y_1,n,t\rangle$ as $|n\rangle$ with $|y_{-1},n,t\rangle=\sigma_3|n\rangle$. Denoting $z_1=\exp[(1/2)(\mu+ik)]$ and $(p_1/p_2)=\exp(a+i\varphi)$, where p_1 and p_2 are components of the constant vector $|p\rangle$, we find from Eq. (4.11) the vector $|n\rangle$ explicitly:

$$|n\rangle=e^{(1/2)(a+i\varphi)}\begin{pmatrix} e^{(1/2)(x_n+i\varphi_n)} \\ e^{-(1/2)(x_n+i\varphi_n)} \end{pmatrix}. \tag{5.4}$$

Here $x_n=\mu n-2t\sinh\mu\sin k+a$, $\varphi_n=kn+2t(\cosh\mu\cos k-1)+\varphi$.

As regards the matrix Γ, it follows from Eq. (4.6) with $N=1$, $z_2=-z_1$, and $\overline{z_2}=-\overline{z_1}$ that

$$\begin{aligned}\Gamma(n,z)=\mathbb{1}&-\frac{1}{z-\overline{z_1}}[|n\rangle(D^{-1})_{11}\langle n|B+\sigma_3|n\rangle(D^{-1})_{21}\langle n|B]\\ &-\frac{1}{z+\overline{z_1}}[|n\rangle(D^{-1})_{12}\langle n|B\sigma_3+\sigma_3|n\rangle\\ &\times(D^{-1})_{22}\langle n|B\sigma_3].\end{aligned} \tag{5.5}$$

Calculating then matrix elements D_{kj} (4.7) with $\det\Gamma(n,0)=\exp(2\mu)$, we obtain from Eq. (5.5)

$$\begin{aligned}\Gamma(n,z)&=\mathbb{1}-\frac{\sinh\mu}{2(z-\overline{z_1})}\tilde{F}_-(n)-\frac{\sinh\mu}{2(z+\overline{z_1})}\tilde{F}_+(n),\\ \Gamma^{-1}(n,z)&=\mathbb{1}+\frac{\sinh\mu}{2(z-z_1)}F_-(n)+\frac{\sinh\mu}{2(z+z_1)}F_+(n),\end{aligned} \tag{5.6}$$

where

$$\tilde{F}_-(n)=\begin{pmatrix} \dfrac{\exp\left[\mu(n-\frac{1}{2}-x+\frac{i}{2}k\right]}{\cosh\mu(n-1-x)} & \dfrac{\exp[ik(n-x)+i\alpha-\mu]}{\cosh\mu(n-1-x)} \\ \dfrac{\exp[-ik(n-1-x)-i\alpha+\mu]}{\cosh\mu(n-x)} & \dfrac{\exp\left[-\mu(n-\frac{1}{2}-x)+\frac{i}{2}k\right]}{\cosh\mu(n-x)} \end{pmatrix}$$

$$F_-(n)=\begin{pmatrix} \dfrac{\exp\left[\mu(n-\frac{1}{2}-x+\frac{i}{2}k\right]}{\cosh\mu(n-x)} & \dfrac{\exp[ik(n-x)+i\alpha-\mu]}{\cosh\mu(n-1-x)} \\ \dfrac{\exp[-ik(n-1-x)-i\alpha+\mu]}{\cosh\mu(n-x)} & \dfrac{\exp\left[-\mu(n-\frac{1}{2}-x)+\frac{i}{2}k\right]}{\cosh\mu(n-1-x)} \end{pmatrix}$$

$$\tilde{F}_+(n)=-\sigma_3\tilde{F}_-(n)\sigma_3,\quad F_+(n)=-\sigma_3F_-(n)\sigma_3. \tag{5.7}$$

Here

$$x(t)=2t\frac{\sinh\mu}{\mu}\sin k+x_0, \quad x_0=-\frac{a}{\mu}-\frac{3}{2},$$
$$\alpha(t)=2t\left(\cosh\mu\cos k+\frac{k}{\mu}\sinh\mu\sin k-1\right)+\alpha_0, \tag{5.8}$$
$$\alpha_0=\varphi-\frac{ak}{\mu}-k.$$

As a result, we obtain from Eq. (5.3) the AL soliton solution [17]

$$u_n(t)=\exp[ik(n-x)+i\alpha]\frac{\sinh\mu}{\cosh\mu(n-x)}. \tag{5.9}$$

Here and in what follows we write for simplicity $\cosh[\mu(n-x)]$ as $\cosh\mu(n-x)$. The AL soliton depends on four constant parameters μ, k, x_0, and α_0 which determine soliton mass 2μ, its group velocity $v_{gr}=2[(\sinh\mu)/\mu]\sin k$, soliton maximum position $x(t)$ and phase $\alpha(t)$.

It should be noted for later use that in the presence of a perturbation Eqs. (5.8) are modified due to possible perturbation-induced evolution of the soliton parameters:

$$x(t)=\frac{2}{\mu}\int^t\sinh\mu(t')\sin k(t')dt'+x_0(t),$$
$$\alpha(t)=2\int^t[\cosh\mu(t')\cos k(t')-1]dt' \tag{5.10}$$
$$+2\frac{k}{\mu}\int^t\sinh\mu(t')\sin k(t')dt'+\alpha_0(t).$$

VI. PERTURBATION-INDUCED EVOLUTION OF THE RH DATA: EXACT RESULTS

Having formulated the basic ingredients of the RH approach to the AL system, we now proceed to the consideration of the perturbed AL equation

$$iu_{nt}+u_{n+1}+u_{n-1}-2u_n+|u_n|^2(u_{n+1}+u_{n-1})=\epsilon r_n. \tag{6.1}$$

The small parameter ϵ characterizes the perturbation amplitude and r_n describes the functional form of the perturbation. To find corrections to the soliton caused by a perturbation, we first derive the corresponding evolution of the RH data. In order to distinguish between the "integrable" and "perturbative" contributions to the evolution equations, we will assign the variational derivative $\delta/\delta t$ to the latter. For example, we write $i\delta u_n/\delta t=\epsilon r_n$, as follows from Eq. (6.1), or, in matrix form,

$$i\frac{\delta Q_n}{\delta t}=\epsilon\hat{R}_n, \quad \hat{R}_n=\begin{pmatrix}0 & r_n\\ r_n^* & 0\end{pmatrix}. \tag{6.2}$$

A. Continuous data

Consider the spectral problem (2.2). The perturbation causes a variation δQ_n of the potential which in turn leads to a variation of the Jost solutions. It follows from Eq. (2.2) that these variations are written in the form

$$E^{-n}J_-^{-1}(n)\delta J_-(n)E^n=\sum_{l=-\infty}^{n-1}E^{-(l+1)}J_-^{-1}\times(l+1)\delta Q_l J_-(l)E^l, \tag{6.3}$$
$$E^{-n}J_+^{-1}(n)\delta J_+(n)E^n=-\sum_{l=n}^{\infty}E^{-(l+1)}J_+^{-1}\times(l+1)\delta Q_l J_+(l)E^l,$$

where $\delta Q_l=(\delta Q_l/\delta t)\delta t$ and we have used $\delta J_\pm(n)\to 0$ as $n\to\pm\infty$. Hence, due to the definition (2.4), we obtain from Eq. (6.3) a variation of the scattering matrix:

$$\frac{\delta S}{\delta t}=-i\epsilon S_+\Upsilon_+(z)S_-^{-1}=-i\epsilon T_+^{-1}\Upsilon_-(z)T_-. \tag{6.4}$$

Here the matrices $S_\pm$ and $T_\pm$ are defined in Eqs. (2.9) and (2.11) and we introduce the matrix function

$$\Upsilon_\pm(N_a,N_b)=\sum_{l=N_a}^{N_b}E^{-(l+1)}\Psi_\pm^{-1}(l+1)\hat{R}_l\Psi_\pm(l)E^l,$$
$$\Upsilon_\pm(z)=\Upsilon_\pm(-\infty,\infty). \tag{6.5}$$

Note that they are the analytical solutions $\Psi_\pm$ that enter naturally the matrices $\Upsilon_\pm$.

Now we derive a variation of Ψ_+. We have from Eq. (2.9) that $\delta\Psi_+=\delta J_+E^nS_+E^{-n}+J_+E^n\delta S_+E^{-n}$. The first term on the right-hand side is transformed by means of Eq. (6.3) to $i\epsilon\Psi_+(n)E^n\Upsilon_+(n,\infty)E^{-n}\delta t$, while the second term, due to Eq. (6.4) and a trick $\delta S_+=\delta SM_{11}$, $M_{11}=\mathrm{diag}(1,0)$, is written as $-i\epsilon\Psi_+(n)E^n\Upsilon_+(z)M_{11}E^{-n}\delta t$. Therefore, the variation of $\Psi_+(n)$ takes the form

$$\frac{\delta\Psi_+(n,z)}{\delta t}=-i\epsilon\Psi_+(n,z)E^n\Pi_+(n,z)E^{-n}, \tag{6.6}$$

where Π_+ is the evolution functional [36] defined here by

$$\Pi_+(n,z)=\begin{pmatrix}\Upsilon_{+11}(-\infty,n-1) & -\Upsilon_{+12}(n,\infty)\\ \Upsilon_{+21}(-\infty,n-1) & -\Upsilon_{+22}(n,\infty)\end{pmatrix}. \tag{6.7}$$

Therefore, in the case of perturbations the evolutionary equation for Ψ_+ gains the additional term responsible for the perturbation:

$$\Psi_{+t}=V\Psi_+-\Psi_+\Omega-i\epsilon\Psi_+E^n\Pi_+E^{-n}. \tag{6.8}$$

Similarly,

$$\frac{\delta\Psi_-^{-1}}{\delta t}=i\epsilon E^n\Pi_-E^{-n}\Psi_-^{-1}, \tag{6.9}$$

with

$$\Pi_{-}(n,z)=\begin{pmatrix} \Upsilon_{-11}(-\infty,n-1) & \Upsilon_{-12}(-\infty,n-1) \\ -\Upsilon_{-21}(n,\infty) & -\Upsilon_{-22}(n,\infty) \end{pmatrix} \tag{6.10}$$

and

$$\Psi_{-t}^{-1}=-\Psi_{-}^{-1}V+\Omega\Psi_{-}^{-1}+i\epsilon E^{n}\Pi_{-}E^{-n}\Psi_{-}^{-1}. \tag{6.11}$$

Remarkably, the functions $\Upsilon_{\pm}$ are interrelated by the matrix G entering the RH problem:

$$\Upsilon_{-}=G\Upsilon_{+}G^{-1}. \tag{6.12}$$

The evolution functionals $\Pi_{\pm}$ play the key role in the analysis of a perturbation because they contain all needed information about it [36]. It is seen from the definitions (6.5), (6.7), and (6.10) that the matrices $\Pi_{\pm}$ are meromorphic (and $E^{n}\Pi_{\pm}E^{-n}$ are bounded) in $\mathcal{C}_{\pm}$ having simple poles at zeros of $\det\Psi_{+}(z)$ and $\det\Psi_{-}^{-1}$, respectively. Further, the evolution equation for G follows easily from Eqs. (3.1), (6.8), and (6.11) and takes the form

$$G_{t}=[\Omega,G]-i\epsilon(G\Pi_{+}-\Pi_{-}G), \tag{6.13}$$

or, for $\tilde{G}=\exp(-\Omega t)G\exp(\Omega t)$,

$$\tilde{G}_{t}=-i\epsilon(\tilde{G}e^{-\Omega t}\Pi_{+}e^{\Omega t}-e^{-\Omega t}\Pi_{-}e^{\Omega t}\tilde{G}). \tag{6.14}$$

B. Discrete data

In the point z_1

$$\Psi_{+}(n,z_{1})|n\rangle=0 \tag{6.15}$$

and near this point

$$\Pi_{+}(z)=\Pi_{+}^{(\mathrm{reg})}(z)+\frac{1}{z-z_{1}}\mathrm{Res}_{z=z_{1}}\Pi_{+}(z), \tag{6.16}$$

where $\Pi_{+}^{(\mathrm{reg})}$ is the regular part of Π_{+} in the point z_1 and $\mathrm{Res}_{z=z_1}$ stands for the residue at $z=z_1$. It is shown in Appendix A that the evolution equation for the eigenvector takes the form

$$|n\rangle_{t}=\Omega(z_{1})|n\rangle+i\epsilon E^{n}(z_{1})\Pi_{+}^{(\mathrm{reg})}(z_{1})E^{-n}(z_{1})|n\rangle. \tag{6.17}$$

Remember that the n dependence of $|n\rangle$ is given by Eq. (4.10), $|n\rangle=E^{n}(z_{1})|\tilde{p}\rangle$, with the n-independent vector $|\tilde{p}\rangle$. Therefore, the perturbation-induced evolution of the vector $|p\rangle=\exp[-\Omega(z_1)t]|\tilde{p}\rangle$ is governed by the equation

$$|p\rangle_{t}=i\epsilon e^{-\Omega(z_{1})t}\Pi_{+}^{(\mathrm{reg})}(z_{1})e^{\Omega(z_{1})t}|p\rangle. \tag{6.18}$$

For notational convenience, the exponent implies hereafter the integration with respect to time for the time-dependent discrete RH data. In the absence of perturbation, the vector $|p\rangle$ in Eq. (6.18) coincides with that in Eq. (4.11).

Evolution of zero z_1 is derived by taking the total time derivative of $\det\Psi_{+}(z_1)=0$. We obtain

$$z_{1t}=-\left[\frac{(\partial/\partial t)\det\Psi_{+}(z)}{(\partial/\partial z)\det\Psi_{+}(z)}\right]_{z_{1}}.$$

Because $(\partial/\partial z)\det\Psi_{+}=-i\epsilon\,\mathrm{tr}\Pi_{+}\det\Psi_{+}$, $\det\Psi_{+}=v_{+}(n)a_{+}(z)$ [see Eq. (2.10)] and $a_{+}(z)=(z^{2}-z_{1}^{2})(z^{2}-\bar{z}_{1}^{-2})^{-1}$, the latter formula following from $a_{+}(z)=a_{+}(-z)$, $\lim a(z)\to 1$ as $z\to\infty$ and $a_{+}(\pm z_{1})=0$, we obtain a simple equation

$$z_{1t}=i\epsilon\,\mathrm{Res}_{z=z_{1}}\mathrm{tr}\Pi_{+}(n,z). \tag{6.19}$$

It is important that left-hand sides of Eqs. (6.18) and (6.19) do not depend on n. Therefore, we can consider these equations for $n\to+\infty$ where

$$\Pi_{+}(z)=\begin{pmatrix} \Upsilon_{+11}(z) & 0 \\ \Upsilon_{+21}(z) & 0 \end{pmatrix}.$$

As a result, the evolution equations for the discrete RH data are finally written as

$$z_{1t}=i\epsilon\,\mathrm{Res}_{z=z_{1}}\Upsilon_{+11}(z), \tag{6.20}$$

$$|p\rangle_{t}=i\epsilon e^{-\Omega(z_{1})t}\begin{pmatrix} \Upsilon_{+11}^{(\mathrm{reg})}(z_{1}) & 0 \\ \Upsilon_{+21}^{(\mathrm{reg})}(z_{1}) & 0 \end{pmatrix}e^{\Omega(z_{1})t}|p\rangle. \tag{6.21}$$

It should be noted that Eqs. (6.14), (6.20), and (6.21) are exact. However, they cannot be directly applied because the matrices $\Pi_{\pm}$ and $\Upsilon_{\pm}$ depend on unknown solutions $\Psi_{\pm}$ of the spectral problem with the perturbed potential. In the following sections we will describe for sufficiently small ϵ the iterative RH problem-based procedure to consecutively account for two main approximations: the leading-order adiabatic approximation and the next-order (the first-order) one.

VII. ADIABATIC APPROXIMATION

Within the adiabatic approximation, we ignore radiation effects and assume that the soliton adjusts its hyperbolic secant shape to perturbation at the cost of slow evolution of the parameters. Evolution equations for the soliton parameters in the adiabatic approximation have the form

$$\mu_{t}=\epsilon\sinh\mu\sum_{n=-\infty}^{\infty}\frac{\mathrm{Im}(R_{n})\cosh\mu(n-x)}{\cosh\mu(n+1-x)\cosh\mu(n-1-x)}, \tag{7.1}$$

$$k_{t}=-\epsilon\sinh\mu\sum_{n=-\infty}^{\infty}\frac{\mathrm{Re}(R_{n})\sinh\mu(n-x)}{\cosh\mu(n+1-x)\cosh\mu(n-1-x)}, \tag{7.2}$$

$$x_{t}=\frac{2}{\mu}\sinh\mu\sin k+\frac{\epsilon}{\mu}\sinh\mu \times\sum_{n=-\infty}^{\infty}\frac{(n-x)\mathrm{Im}(R_{n})\cosh\mu(n-x)}{\cosh\mu(n+1-x)\cosh\mu(n-1-x)}, \tag{7.3}$$

$$\alpha_t=2\left(\cosh\mu\cos k+\frac{k}{\mu}\sinh\mu\sin k-1\right)$$
$$+\epsilon\sum_{n=-\infty}^{\infty}\Big\{[(n-x)\sinh\mu\sinh\mu(n-x)$$
$$-\cosh\mu\cosh\mu(n-x)]\mathrm{Re}(R_n)+\frac{k}{\mu}(n-x)$$
$$\times\sinh\mu\cosh\mu(n-x)\mathrm{Im}(R_n)\Big\}$$
$$\times\mathrm{sech}\,\mu(n+1-x)\mathrm{sech}\,\mu(n-1-x). \tag{7.4}$$

Here $R_n=r_n\exp[-ik(n-x)-i\alpha]$ and r_n is constructed by means of the AL soliton solution (5.9) . Equations (7.1)–(7.3) have been obtained for the first time in Ref. [23]. The derivation of Eqs. (7.1)–(7.4) within the RH problem approach is given in the Appendix B.

VIII. RADIATION EFFECTS

The continuous part of the RH data describes a distortion of the soliton shape and emission of small-amplitude dispersive waves by soliton. To account for the continuous data, we should abandon the condition $G=\mathbb{1}$ and admit a z dependence of the regular RH problem solutions $\psi_\pm$. In other words, we pose

$$G=\mathbb{1}+\epsilon g(z),\quad \psi_+(n,z)=\psi_+^0(n)[\mathbb{1}+\epsilon\phi(n,z)], \tag{8.1}$$

where ψ_+^0 stands for the solution (5.1) of the regular RH problem (4.3) in the adiabatic approximation, whereas the off-diagonal matrices $g(z)$ and $\phi(z)$ describe first-order corrections. Therefore, the reconstruction formula (4.9) takes now the form

$$u_n=-\lim_{z\to\infty}\frac{[z\psi_+^0(\mathbb{1}+\epsilon\phi)\Gamma]_{12}}{[\psi_+^0(\mathbb{1}+\epsilon\phi)\Gamma]_{22}}$$
$$=-\Gamma_{12}^{(1)}(n)\Gamma_{22}(n,0)-\epsilon\phi_{12}^{(1)}(n)\Gamma_{22}(n,0). \tag{8.2}$$

The first term on the right-hand side of Eq. (8.2) represents the familiar soliton solution in the adiabatic approximation and the second one is responsible for radiation (soliton shape distortion). For the derivation of Eq. (8.2) we employ the fact that the off-diagonal matrix ϕ satisfies the asymptotic condition $\phi\to z^{-1}\phi^{(1)}+O(z^{-2})$ with

$$\phi^{(1)}=\begin{pmatrix}0 & \phi_{12}^{(1)}\\ \phi_{21}^{(1)} & 0\end{pmatrix}.$$

Evaluation of $\phi_{12}^{(1)}$ and hence of radiation corrections to soliton solution reduces to solving the regular RH problem (4.3) with G as in Eq. (8.1). Indeed, we have $\psi_-^{-1}\psi_+=\mathbb{1}+\epsilon\Gamma E^n g(z)E^{-n}\Gamma^{-1}$ and the jump of the piecewise holomorphic function $\psi(z)=\{\psi_+(z),z\in\mathcal{C}_+;\psi_-(z),z\in\mathcal{C}_-\}$ across the contour $|z|=1$ is written as

$$\psi_+-\psi_-=\epsilon\psi_+^0\Gamma E^n gE^{-n}\Gamma^{-1}. \tag{8.3}$$

Here we omit terms with higher order of ϵ and invoke the equality $\psi_-^0=\psi_+^0$ [see Eq. (5.1)] valid in the adiabatic approximation. The Plemelj formula gives the following for $z\in\mathcal{C}_+$:

$$\psi_+(z)=\psi_+^0\left[\mathbb{1}+\frac{\epsilon}{2\pi i}\oint_{|z|=1}\frac{dz'}{z'-z}(\Gamma E^n gE^{-n}\Gamma^{-1})(z')\right].$$

Inserting here ψ_+ from Eq. (8.1) and performing the asymptotic expansion at $z\to\infty$, we obtain the expansion coefficient

$$\phi^{(1)}(n)=-\frac{1}{2\pi i}\oint_{|z|=1}dz(\Gamma E^n gE^{-n}\Gamma^{-1})(z) \tag{8.4}$$

determining the radiation correction (8.2). Therefore, our next step is concerned with finding the matrix g.

To this end, we turn to the evolution equation (6.14) for the matrix $\tilde{G}$ which is evidently related to g:

$$\tilde{G}=\mathbb{1}+\epsilon\tilde{g},\quad \tilde{g}=e^{-\Omega t}ge^{\Omega t}. \tag{8.5}$$

Substituting this equation into Eq. (6.14), we obtain in the first order of ϵ

$$i\tilde{g}_t=e^{-\Omega t}(\Pi_+-\Pi_-)e^{\Omega t}. \tag{8.6}$$

Because $\tilde{g}$ does not depend on n, we can put $n\to\infty$ in Eq. (8.6) which gives

$$\Pi_+(n\to\infty)-\Pi_-(n\to\infty)=\begin{pmatrix}\Upsilon_{+11}-\Upsilon_{-11} & -\Upsilon_{-12}\\ \Upsilon_{+21} & 0\end{pmatrix}.$$

Moreover, it follows from Eqs. (6.5) and (8.1) that $\Upsilon_-=\Upsilon_+$ in the first order of ϵ. As a result,

$$i\tilde{g}_t=e^{-\Omega t}\begin{pmatrix}0 & -\Upsilon_{+12}\\ \Upsilon_{+21} & 0\end{pmatrix}e^{\Omega t}$$

and the equation for $\tilde{g}_{12}$ takes the form

$$\tilde{g}_{12t}=i\exp[-i(z-z^{-1})^2t]\Upsilon_{+12}. \tag{8.7}$$

It is important to stress that because Υ_{+12} corresponds to the first-order correction, we can replace in the definition (6.5) of Υ_+ unknown solution ψ_+ of the regular RH problem (8.3) by the known one ψ_+^0. Integrating then Eq. (8.7), we can find the matrix g (8.5).

The further stage is to consider the integrand in Eq. (8.4). It can be shown from $(\Gamma E^n gE^{-n}\Gamma^{-1})_{12}=z^{-2n}\Gamma_{12}(\Gamma^{-1})_{12}g_{21}+z^{2n}\Gamma_{11}(\Gamma^{-1})_{22}g_{12}$ and explicit expressions (5.7) for Γ that the term with g_{21} is multiplied by $\mathrm{sech}^2\mu(n-x-1)$ and hence vanishes at $n\to\pm\infty$. As a result, we are left with

$$I_{12}\equiv(\Gamma E^{n}gE^{-n}\Gamma^{-1})_{12}=\begin{cases}\dfrac{z^2-z_1^2}{z^2-\bar{z}_1^2}z^{2n}g_{12}, & n\to+\infty\\ \dfrac{z^2-\bar{z}_1^2}{z^2-z_1^2}z^{2n}g_{12}, & n\to-\infty.\end{cases} \tag{8.8}$$

Let us summarize the main steps in calculating the radiation correction for a given perturbation r_n. First, we should explicitly find the function $\Upsilon_{+12}(z)$ from the definition (6.5) with $\Psi_+=\psi_+^0\Gamma$, ψ_+^0 and Γ being given in Eqs. (5.1) and (5.7), respectively. Then we integrate Eq. (8.7) and obtain the matrix g given in Eqs. (8.5) and (8.1). For the known function $g_{12}(z)$ we obtain the integrand (8.8). Finally, after calculating the integral (8.4) we arrive at the needed result.

In the following section we illustrate the proposed formalism on an example of calculating the radiation corrections to the AL soliton in the case of some model perturbations.

IX. EXAMPLES

Here we apply our formalism to describe the perturbed AL soliton dynamics for the typical representatives of dissipative and conservative perturbations—linear damping $r_n=-iu_n$ and quintic perturbation $r_n=|u_n|^4u_n$. The interplay between the dissipative and conservative perturbations for the AL model is considered in the adiabatic approximation by Abdullaev *et al.* [29] and numerically by Soto-Crespo *et al.* [41].

A. Linear damping

In this case $\mathrm{Re}R_n=0$, $\mathrm{Im}R_n=-\sinh\mu\,\mathrm{sech}\mu(n-x)$, and we have in the adiabatic approximation

$$k=\text{const},\quad \sinh\mu=\sinh(\mu_0)e^{-2\epsilon t},\quad \mu_0=\mu(t=0),$$

$$x_t=v_{gr}-\frac{2\pi\epsilon}{\mu^2}\frac{\tanh\mu}{\sinh(\pi^2/\mu)}\sin 2\pi x,\qquad \delta\alpha/\delta t=kx_t.$$

In the process of obtaining the equation for x_t we use the Poisson summation formula [40]

$$\sum_{n=-\infty}^{\infty}f(n\mu)=\frac{1}{\mu}\int_{-\infty}^{\infty}dy\,f(y)\left[1+2\sum_{s=1}^{\infty}\cos\frac{2\pi sy}{\mu}\right] \tag{9.1}$$

and, following Ref. [24], we restrict ourselves to the linear harmonic term ($s=1$) only. Higher harmonics contain the factor $\exp(-\pi^2 s/\mu)$ which for $\mu\approx1$ is evidently small. Hence, mass of the soliton decreases exponentially, its group velocity acquires a constant value ($=2\sin k$) after some transient period (Fig. 2), while its phase is governed by the evolution of the soliton position $x(t)$.

Now we embark on a calculation of radiation effects. Following the prescriptions of Sec. VIII we find at first the matrix function Υ_+ written in accordance with Eq. (6.5) as

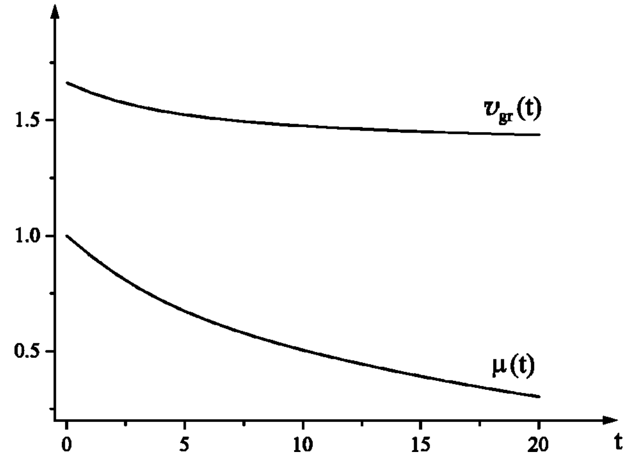

FIG. 2. Evolution of the soliton mass ($m=2\mu$) and group velocity v_{gr} in the case of linear damping for $k=\pi/4$, $\epsilon=0.03$, and $\mu(t=0)=1$.

$$\Upsilon_+(z)=\sum_{n=-\infty}^{\infty}E^{-(n+1)}(n+1)\Gamma^{-1}(n+1)\mathcal{R}_n\Gamma(n)E^{n}. \tag{9.2}$$

Here

$$\mathcal{R}_n=(\psi_+^0)^{-1}(n+1)\hat{R}\psi_+^0(n)=\begin{pmatrix}0 & r_n\Gamma_{22}(n,0)^{-1}\\ r_n^*\Gamma_{22}(n+1,0) & 0\end{pmatrix}$$

and

$$\Gamma_{22}(n,0)=e^{\mu}\frac{\cosh\mu(n-x-1)}{\cosh\mu(n-x)}.$$

Substituting Γ and Γ^{-1} (5.6) into Eq. (9.2), we arrive at

$$\Upsilon_{+12}=z^{-1}\frac{e^{-\mu+i\alpha-ikx}}{\cosh\mu-\cos(k-2\theta)}\times\{[1-\cosh\mu\cos(k-2\theta)]\times S_1+i\sinh\mu\sin(k-2\theta)S_2\},$$

where

$$S_{\{1,2\}}=\sum_{n=-\infty}^{\infty}\frac{e^{ikn}z^{-2n}\begin{Bmatrix}\cosh\mu(n-x)\\ \sinh\mu(n-x)\end{Bmatrix}}{\cosh\mu(n-x-1)\cosh\mu(n-x+1)}.$$

Calculating these sums by means of the Poisson formula (9.1), we obtain a simple expression

$$\Upsilon_{+12}=-\frac{\pi}{\mu z}\frac{\exp(-\mu+i\alpha-2i\theta x)}{\cosh\mu\cosh[\pi(k-2\theta)/2\mu]}. \tag{9.3}$$

Here we pose $z=\exp(i\theta)$ bearing in mind subsequent integration along the contour $|z|=1$. What is more, because the radiation correction (8.2) is multiplied by ϵ, we restrict our-

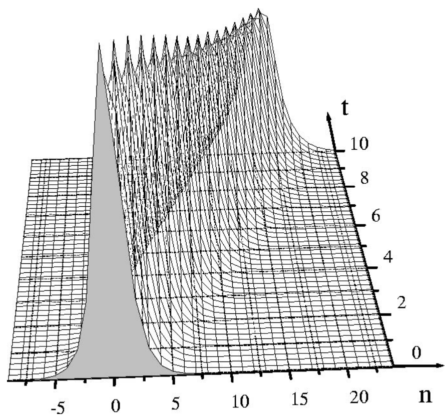

FIG. 3. Evolution of the perturbed AL soliton u_n (8.2) with account of the first-order correction. The soliton parameters are the same as in Fig. 2.

selves to the leading term in each sum. Integrating then Eq. (8.7) for $\tilde{g}_{12}$ with Υ_{+12} of the form of Eq. (9.3) and transforming the result to g_{12} in accordance with Eq. (8.5), we get

$$g_{12}=-\frac{\pi}{\mu z}\frac{e^{-\mu}}{\cosh\mu}\frac{\exp(i\alpha-2i\theta x)}{\cosh[\pi(k-2\theta)/2\mu]}\frac{1-e^{-i\Lambda(\theta)t}}{\Lambda(\theta)}.$$

Here $\Lambda(\theta)=(k-2\theta)v_{gr}+2(\cosh\mu\cos k-\cos 2\theta)$ and, within the first-order approximation, we can take as v_{ph} and v_{gr} their initial values. Therefore, we arrive at the integrand I_{12} (8.8) which determines the integral (8.4). This integral can be calculated by residues. The dominant contribution is provided by the third-order residue in the point $z=\overline{z_1}$ (note that both $\cosh[\pi(k-2\theta)/2\mu]$ and $\Lambda(\theta)$ have simple zero in this point). The resulting expression is rather lengthy and does not reproduce here. Instead we plot in Fig. 3 the evolution of the perturbed AL soliton u_n (8.2) with account for this expression. The shape of the soliton changes periodically due to the discreteness of the system, with simultaneous decreasing of mass in virtue of damping. More detailed conclusion about the first-order contribution can be inferred from Fig. 4 where a difference $|u_n|-|u_s|$ is pictured. Here u_s is the AL soliton in the adiabatic approximation. The shape distortion is mainly localized within the soliton "envelope"

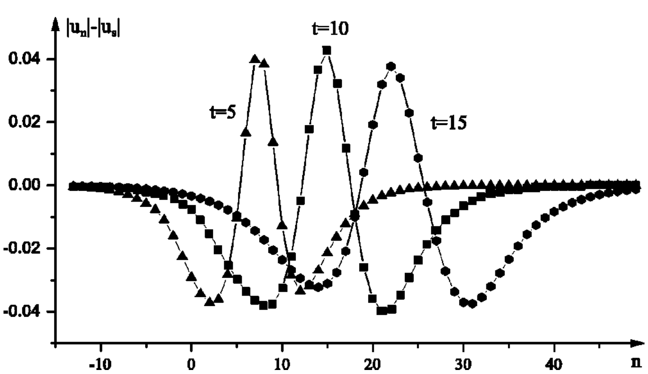

FIG. 4. The difference $|u_n|-|u_s|$ for different time intervals indicative of the shape distortion effect. Here u_s is the soliton in the adiabatic approximation and the soliton parameters are the same as in Fig. 2. There are no long-lived nonvanishing dispersive waves.

and in general is asymmetric. Such a behavior agrees with the asymptotic representation of the first-order correction for $n\rightarrow+\infty$:

$$u_{n(\mathrm{rad})}=-\epsilon\phi_{12}^{(1)}\Gamma_{22}(n,0)\approx-\frac{i\epsilon}{2H}\tanh\mu\, e^{-\mu+i\alpha+ik}(n-x)\times\left[n-x-\frac{\sin k}{\sinh\mu}\left(1-\frac{\sinh 2\mu}{2\mu}\right)H^{-1}\right]e^{(-\mu+ik)n}. \tag{9.4}$$

Here $H=\sinh\mu\cos k-i\cosh\mu\sin k+(i/\mu)\sinh\mu\sin k$. It follows from this expression that there are no nonvanishing linear waves at $n\rightarrow\infty$. The similar result takes place for $n\rightarrow-\infty$.

B. Quintic perturbation

It follows from the results of Sec. VII that for

$$\mathrm{Re}(R_n)=\sinh^5\mu\,\mathrm{sech}^5\mu(n-x),\qquad \mathrm{Im}(R_n)=0$$

we have in the adiabatic approximation

$$\mu=\mathrm{const},\qquad x_t=2\frac{\sinh\mu}{\mu}\sin k,$$

$$k_t=\frac{2\epsilon\pi}{3\mu}\frac{\sinh^4\mu}{\sinh\frac{\pi^2}{\mu}}\left[\frac{2\pi^4}{\mu^4}+\frac{2\pi^2}{\mu^2}\left(1-\frac{3}{\sinh^2\mu}\right)+\frac{1}{3}\right]\sin 2\pi x. \tag{9.5}$$

We do not write here the evolution equation for the phase α because of its lengthy form, though it is found quite immediately. Hence, soliton mass is preserved in the presence of the quintic perturbation, while the parameter $k(t)$ (and hence the group velocity) oscillates with a very small amplitude near the initial value. Linear stability analysis demonstrates that the fixed points of the evolutions (9.5), $k_s=\pi m$, $m=0,\pm1,\pm2,\ldots,$ and $x_s=l/2$, $l=0,\pm1,\pm2,\ldots,$ are stable for even (odd) m and odd (even) l.

Radiative corrections for the quintic perturbation are much the same as for damping. Indeed, the function Υ_{+12} takes the form

$$\Upsilon_{+12}=-\frac{\pi}{2\mu}e^{-\mu+i\alpha-2i\theta x}\frac{\sinh\mu}{\cosh[\pi(k-2\theta)/2\mu]}\times[P_3(\theta)-2e^{i(k-2\theta)}+\beta(\theta)P_4(\theta)],$$

where

$$P_3(\theta)=\frac{1}{3}\left(\cosh\mu+i\frac{k-2\theta}{\mu}\sinh\mu\right)^3+\left(1-\frac{4}{3}\sinh^2\mu\right)\times\left(\cosh\mu+i\frac{k-2\theta}{\mu}\sinh\mu\right)+\frac{2}{3}\cosh\mu,$$

$$P_4(\theta)=\frac{1}{24}\left(\frac{k-2\theta}{\mu}\sinh\mu\right)^4-\frac{1}{4}\left(2+\frac{1}{3}\sinh^2\mu\right)\times\left(\frac{k-2\theta}{\mu}\sinh\mu\right)^2+\frac{1}{2}(1+\cosh^2\mu)-\cosh\mu\cos(k-2\theta),$$

$$\beta(\theta)=\{\sinh\mu[\cosh\mu-\cos(k-2\theta)]\}^{-1}.$$

Therefore, the integrand I_{12} is written as

$$I_{12}=-\frac{\pi}{2\mu}e^{-\mu+i\alpha}\frac{\sinh\mu}{\cosh[\pi(k-2\theta)/2\mu]}\frac{z^2-z_1^2}{z^2-\overline{z}_1^2}z^{2(n-x)-1}\times[P_3(\theta)-2e^{i(k-2\theta)}+\beta(\theta)P_4(\theta)]\frac{1-e^{-i\Lambda(\theta)t}}{\Lambda(\theta)},$$

with the same $\Lambda(\theta)$, as before. The result of calculation of the integral (8.4) with the above integrand has the same structure as in Eq. (9.4). What is more, the function $P_3(\theta)-2e^{i(k-2\theta)}$ being zero for $z=\overline{z_1}$ does not contribute to the n^2 order of the radiative correction.

X. CONCLUSION

We have proposed a formalism suitable for analytical investigation of dynamics of the AL soliton subjected to a perturbation. This formalism provides a possibility of calculating both evolution of the soliton parameters and perturbation-induced radiation effects. Remarkably, it is the RH problem-based approach that has been proved to be efficient for treating continuous nonlinear equations, both integrable and nearly integrable, which turns out to be the natural basis to study discrete nonlinear systems. We have demonstrated within this approach how to consistently advance from an integrable to perturbed system, in so doing the only ingredient that should be added to the formalism to account for a perturbation is the evolution functional Π_+ (or Π_-) introduced by Shchesnovich [36]. A natural step to further extend the applicability of analytical methods in the theory of discrete nonlinear systems is to consider vector AL-type solitons [42]. Work in this direction is now in progress.

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APPENDIX A: PERTURBED EVOLUTION OF EIGENVECTOR

In this appendix we derive Eq. (6.17). Taking the total time derivative of Eq. (6.15) gives the following with account of Eqs. (6.8) and (6.16):

$$\Big\{V(n)\Psi_+(n)-\Psi_+(n)\Omega-i\epsilon\Psi_+(n)[E^n\Pi_+^{(\mathrm{reg})}E^{-n}+(z-z_1)^{-1}\mathrm{Res}_{z=z_1}(E^n\Pi_+E^{-n})]+z_t\frac{\partial}{\partial z}\Psi_+(n)\Big\}_{z=z_1}|n\rangle+\Psi_+(n,z_1)|n\rangle_t=0. \quad (\mathrm{A1})$$

Let us introduce a holomorphic function $\widetilde{\Pi}=-i\epsilon(z-z_1)E^n\Pi_+E^{-n}$ which evidently gives

$$\widetilde{\Pi}(z_1)|n\rangle=-i\epsilon\mathrm{Res}_{z=z_1}[E^n\Pi_+(z)E^{-n}]|n\rangle. \quad (\mathrm{A2})$$

On the other hand, representing $\Pi_+(z)$ from Eq. (6.8) as

$$-i\epsilon E^n\Pi_+(z)E^{-n}=\Psi_+^{-1}\Psi_{+t}-\Psi_+^{-1}V\Psi_++\Omega,$$

we obtain

$$\widetilde{\Pi}(z_1)|n\rangle=[(z-z_1)\Psi_+^{-1}\Psi_{+t}]_{z_1}|n\rangle=-z_{1t}|n\rangle. \quad (\mathrm{A3})$$

Comparing Eqs. (A2) and (A3), we arrive at

$$i\epsilon\mathrm{Res}_{z=z_1}[E^n\Pi_+(z)E^{-n}]|n\rangle=z_{1t}|n\rangle. \quad (\mathrm{A4})$$

Applying now $\Psi_+(z_1)$ to both sides of Eq. (A4), we obtain the important identity

$$\Psi_+(n,z_1)\mathrm{Res}_{z=z_1}(E^n\Pi_+E^{-n})=0. \quad (\mathrm{A5})$$

Equations (A4) and (A5) permit us to considerably simplify Eq. (A1). Indeed, the last term in square brackets in Eq. (A1) is rearranged by means of Eq. (A5) as

$$-i\epsilon\left[\frac{\Psi_+(z)-\Psi_+(z_1)}{z-z_1}\mathrm{Res}_{z=z_1}[E^n\Pi_+(z)E^{-n}]\right]_{z_1}|n\rangle=-i\epsilon\left[\frac{\partial}{\partial z}\Psi_+(z)\right]_{z_1}\mathrm{Res}_{z=z_1}(E^n\Pi_+(z)E^{-n})|n\rangle=-z_{1t}\left[\frac{\partial}{\partial z}\Psi_+(z)\right]_{z_1}|n\rangle$$

and cancels the same term in Eq. (A1). As a result, the evolution equation for the vector $|n\rangle$ takes the form

$$|n\rangle_t=\Omega(z_1)|n\rangle+i\epsilon E^n(z_1)\Pi_+^{(\mathrm{reg})}(z_1)E^{-n}(z_1)|n\rangle.$$

APPENDIX B: ADIABATIC APPROXIMATION

Here we obtain within the RH problem approach Eqs. (7.1)–(7.4) which govern the adiabatic dynamics of the AL soliton.

First of all we turn to Eq. (6.20). In accordance with Eqs. (6.5), (4.1), (5.1), and (5.2) we write

$$\begin{aligned}\mathrm{Res}_{z=z_1}\Upsilon_{+11}(z) &= \mathrm{Res}_{z=z_1}\left[\frac{1}{z}\sum_{n=-\infty}^{\infty}\Gamma^{-1}(n+1,z)\psi_+^{-1}(n+1)\right.\\ &\quad\left.\times\hat{R}_n\psi_+(n)\Gamma(n,z)\right]_{11}\\ &= \frac{\sinh\mu}{2z_1}\left[\sum_{n=-\infty}^{\infty}F_-(n+1)\right.\\ &\quad\left.\times\begin{pmatrix}0 & \Gamma_{22}^{-1}(n,0)r_n\\ \Gamma_{22}(n+1,0)r_n^* & 0\end{pmatrix}\Gamma(n,z_1)\right]_{11}.\end{aligned}$$

With account of explicit expressions (5.6) and (5.7) for F_- and Γ we obtain

$$\begin{aligned}z_{1t} &= -\frac{i\epsilon}{4}z_1\sinh\mu\\ &\quad\times\sum_{n=-\infty}^{\infty}\frac{R_n e^{\mu(n-x)}-R_n^* e^{-\mu(n-x)}}{\cosh\mu(n+1-x)\cosh\mu(n-1-x)},\end{aligned}$$

where $R_n=r_n\exp[-ik(n-x)-i\alpha]$. Then from the definition $z_1=\exp[(1/2)(\mu+ik)]$ we easily derive Eqs. (7.1) and (7.2).

In order to obtain Eqs. (7.3) and (7.4), we should at first calculate $\Upsilon^{(\mathrm{reg})}(z_1)$:

$$\begin{aligned}\Upsilon^{(\mathrm{reg})}(z_1) &= [\Upsilon(z)-(z-z_1)^{-1}\mathrm{Res}_{z=z_1}\Upsilon(z)]_{z_1}\\ &= \sum_{n=-\infty}^{\infty}E^{-(n+1)}\\ &\quad\times\left(\mathbb{1}+\frac{\sinh\mu}{4z_1}F_+(n+1)\right)\mathcal{R}_n\Gamma(n,z_1)E^n\\ &\quad+\lim_{z\to z_1}\frac{\sinh\mu}{2(z-z_1)}\sum_{n=-\infty}^{\infty}[E^{-(n+1)}(z)F_-(n+1)\\ &\quad\times\mathcal{R}_n\Gamma(n,z)E^n(z)-E^{-(n+1)}(z_1)F_-(n+1)\\ &\quad\times\mathcal{R}_n\Gamma(n,z_1)].\end{aligned}\tag{B1}$$

Here $\mathcal{R}_n=\psi_+^{-1}(n+1)\hat{R}_n\psi_+(n)$. The second term on the rhs of Eq. (B1) gives the ratio 0/0 in the limit $z\to z_1$. Using the l'Hôpital rule, we ultimately arrive at

$$\begin{aligned}\Upsilon^{(\mathrm{reg})}(z_1) &= \sum_{n=-\infty}^{\infty}E^{-(n+1)}(z_1)\left\{\left(\mathbb{1}-\frac{\sinh\mu}{2z_1}\sigma_3F_-(n+1)\right.\right.\\ &\quad\left.+\frac{\sinh\mu}{4z_1}F_+(n+1)\right)\mathcal{R}_n\Gamma(n,z_1)\\ &\quad+\frac{\sinh^2\mu}{4}F_-(n+1)\mathcal{R}_n\left(\frac{\tilde{F}_-(n)}{(z_1-\overline{z_1})^2}+\frac{\tilde{F}_+(n)}{(z_1+\overline{z_1})^2}\right)\\ &\quad\left.-n\frac{\sinh\mu}{2z_1}[\sigma_3,F_-(n+1)\mathcal{R}_n\Gamma(n,z_1)]\right\}E^n(z_1).\end{aligned}$$

Therefore,

$$\begin{aligned}\Upsilon_{11}^{(\mathrm{reg})}(z_1) &= \frac{1}{8}\sum_{n=-\infty}^{\infty}[(3R_ne^{\mu(n-x)}-R_n^*e^{-\mu(n-x)})\sinh\mu+2(R_n\cosh\mu+R_n^*e^{-\mu})e^{\mu(n-x)}-4R_n\mathrm{sech}\mu(n+1-x)]\\ &\quad\times\mathrm{sech}\mu(n+1-x)\mathrm{sech}\mu(n-1-x),\end{aligned}\tag{B2}$$

$$\begin{aligned}\Upsilon_{21}^{(\mathrm{reg})}(z_1) &= \frac{1}{4}\sum_{n=-\infty}^{\infty}\frac{z_1^{2n+1}e^{-ik(n-x)-i\alpha}}{\cosh\mu(n+1-x)\cosh\mu(n-1-x)}\left[\left(\cosh\mu-\frac{3}{2}\sinh\mu\right)R_n\right.\\ &\quad\left.+\left(e^{-\mu}+2e^{-\mu(n-1-x)}\cosh\mu(n-x)+\frac{1}{2}e^{-2\mu(n-x)}\sinh\mu\right)R_n^*-2n\sinh\mu(R_n+e^{-2\mu(n-x)}R_n^*)\right].\end{aligned}\tag{B3}$$

Then we obtain from Eq. (6.21) the following evolution equations for the components of the vector $|p\rangle$:

$$p_{1t}=i\epsilon\Upsilon_{11}^{(\mathrm{reg})}(z_1)p_1,$$

$$p_{2t}=i\epsilon\Upsilon_{21}^{(\mathrm{reg})}(z_1)\exp\left[i\int^t(z_1^2+z_1^{-2}-2)\mathrm{d}t\right]p_1.$$

Because $(p_1/p_2)=\exp(a+i\varphi)$, we have the following from Eq. (B2):

$$\frac{d}{dt}(a+i\varphi)=\frac{i\epsilon}{4}\sum_{n=-\infty}^{\infty}[(3R_n e^{\mu(n-x)}-R_n^* e^{-\mu(n-x)})\sinh\mu +2n(R_n e^{\mu(n-x)}-R_n^* e^{-\mu(n-x)}) \times\sinh\mu-2R_n^* e^{\mu}\cosh\mu(n-x)] \times\text{sech}\mu(n+1-x)\text{sech}\mu(n-1-x)$$

which results in

$$a_t=-\epsilon\sinh\mu\sum_{n=-\infty}^{\infty}\left(n+\frac{3}{2}\right)\times\frac{\text{Im}(R_n)\cosh\mu(n-x)}{\cosh\mu(n+1-x)\cosh\mu(n-1-x)}, \qquad \text{(B4)}$$

$$\varphi_t=\epsilon\sum_{n=-\infty}^{\infty}[n\,\text{sech}\mu\,\text{sech}\mu(n-x)-\cosh\mu\cosh\mu(n-x) +\tfrac{1}{2}\sinh\mu\sinh\mu(n-x)]\text{Re}(R_n)\text{sech}\mu(n+1-x) \times\text{sech}\mu(n-1-x).$$

It follows from Eqs. (5.8) and (5.10) that

$$x_t=\frac{2}{\mu}\sinh\mu\sin k-\frac{1}{\mu}\left[\left(x+\frac{3}{2}\right)\mu_t+a_t\right],$$

$$\alpha_t=2\left(\cosh\mu\cos k+\frac{k}{\mu}\sinh\mu\sin k-1\right) +\left(x+\frac{1}{2}\right)k_t-\left(x+\frac{3}{2}\right)\frac{k}{\mu}\mu_t-\frac{k}{\mu}a_t+\varphi_t.$$

Inserting here Eqs. (7.1), (7.2), (B3), and (B4), we finally obtain Eqs. (7.3) and (7.4).

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