Two-soliton collisions in a near-integrable lattice system

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We examine collisions between identical solitons in a weakly perturbed Ablowitz-Ladik (AL) model, augmented by either onsite cubic nonlinearity (which corresponds to the Salerno model, and may be realized as an array of strongly overlapping nonlinear optical waveguides) or a quintic perturbation, or both. Complex dependences of the outcomes of the collisions on the initial phase difference between the solitons and location of the collision point are observed. Large changes of amplitudes and velocities of the colliding solitons are generated by weak perturbations, showing that the elasticity of soliton collisions in the AL model is fragile (for instance, the Salerno's perturbation with the relative strength of 0.08 can give rise to a change of the solitons' amplitudes by a factor exceeding 2). Exact and approximate conservation laws in the perturbed system are examined, with a conclusion that the small perturbations very weakly affect the norm and energy conservation, but completely destroy the conservation of the lattice momentum, which is explained by the absence of the translational symmetry in generic nonintegrable lattice models. Data collected for a very large number of collisions correlate with this conclusion. Asymmetry of the collisions (which is explained by the dependence on the location of the central point of the collision relative to the lattice, and on the phase difference between the solitons) is investigated too, showing that the nonintegrability-induced effects grow almost linearly with the perturbation strength. Different perturbations (cubic and quintic ones) produce virtually identical collision-induced effects, which makes it possible to compensate them, thus finding a special perturbed system with almost elastic soliton collisions.

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I. INTRODUCTION

Soliton collisions constitute one of the central topics of nonlinear-wave dynamics. In integrable systems, solitons are well known to emerge unscathed from collisions [1]. However, even small nonintegrable perturbations may render the phenomenology much richer, causing various inelastic effects, such as trapping and formation of bound states, multiple-bounce interactions (where solitons separate after multiple collisions) [2], fractality in the outcome of the collision [3,4], and others. Such complex features are usually attributed to the excitation of soliton internal modes [2,4,5], but more recently it was realized that they also occur due to the possibility for radiationless energy exchange (even in the absence of internal modes) between the colliding solitons, should the conservation laws allow it [6,7]. The latter mechanism was both confirmed by direct simulations of the corresponding nonintegrable models, and might be expected to follow from the principle stating that any outcome compatible with the conservation laws may take place under appropriate initial conditions.

Such effects suggest that the integrability is essentially tantamount to the strictly elastic character of the collisions [1], and warrant the importance of further studies of strongly inelastic collision effects produced by small conservative perturbations added to basic integrable models. This general issue is of interest not only in its own right, but also for applications to nonlinear optical waveguides, as strong changes in the character of the interaction induced by a small perturbation may be naturally used in the context of switching, see, e.g., Ref. [8] and references therein.

The objective of the present work is to consider such effects in collisions of nontopological solitons in a *discrete* near-integrable system. As a matter of fact, the only integrable system which can be used in this case as the zeroth-order approximation is the Ablowitz-Ladik (AL) lattice [9]. Its well-known nonintegrable extension is the Salerno model (SM) [10], which is produced by adding the integrability-breaking perturbation in the form of the onsite nonlinearity to the integrable AL system with the intersite cubic nonlinearity. In order to test if the results that will be obtained below are generic, we will also consider an essentially different type of an integrability-breaking conservative perturbation, viz., the quintic onsite nonlinearity [its principal difference from the cubic counterpart is that it breaks the integrability of both the AL lattice and of its continuum limit, i.e., the nonlinear Schrödinger (NLS) equation].

Thus, we introduce a general dynamical model based on the following equation:

$$i\dot{\psi}_n+(2h^2)^{-1}(\psi_{n-1}-2\psi_n+\psi_{n+1})+\delta|\psi_n|^2\psi_n +(1/2)(1-\delta)|\psi_n|^2(\psi_{n+1}+\psi_{n-1})=\varepsilon|\psi_n|^4\psi_n\,. \tag{1}$$

Here ψ_n is the complex dynamical variable at the nth site of the lattice, the overdot stands for the time derivative, h is the lattice spacing, ε is a real constant controlling the quintic perturbation, and δ is a real parameter that accounts for the crossover between the AL ($\delta=0,\varepsilon=0$) and discrete-NLS

($\delta=1,\varepsilon=0$) [11] limits. Equation (1) conserves two dynamical invariants, namely, the norm

$$\mathcal{N}=\frac{1}{1-\delta}\sum_n \ln[1+h^2(1-\delta)|\psi_n|^2] \tag{2}$$

and the energy (Hamiltonian)

$$\mathcal{H}=-\sum_n \left\{\frac{h^2(1-\delta)+\varepsilon}{h^2(1-\delta)^3}\ln[1+h^2(1-\delta)|\psi_n|^2] - \frac{h^2(1-\delta)+\varepsilon}{(1-\delta)^2}|\psi_n|^2+\frac{h^2}{2}|\psi_n-\psi_{n-1}|^2 + \frac{\varepsilon h^2}{2(1-\delta)}|\psi_n|^4\right\}. \tag{3}$$

While the discrete NLS equation, corresponding to $\delta=1$ and $\varepsilon=0$ in Eq. (1), has numerous physical realizations, the most important one being arrays of nonlinear optical waveguides [12], the AL model does not directly apply to many physical systems because of the specific character of the nonlinear terms in it. However, a realization of the SM may be an array of *strongly overlapping* nonlinear optical waveguides, especially the one following a zigzag pattern (similar to an array introduced in Ref. [13]). Indeed, the overlapping between adjacent cores will give rise, through the Kerr effect, to a nonlinear correction to the linear coupling between the cores, in the form of the terms $\sim(1-\delta)$ in Eq. (1). It should be noted that, in this case, extra perturbation terms are expected too, such as $(|\psi_{n-1}|^2+|\psi_{n+1}|^2)\psi_n$ (cross-phase modulation). However, the results presented below clearly demonstrate that strong effects generated by small conservative perturbations are essentially the same for different perturbations, therefore we expect that taking into account all the possible perturbation terms corresponding to the optical waveguides with strong overlap between the cores will not alter the results significantly.

In some specific cases, soliton collisions in the SM have already been examined. In particular, a collision between a soliton and a reflecting wall, which is equivalent to a strictly symmetric collision between the soliton and its mirror image, was studied numerically in Ref. [14]. One of our aims is to explore the sensitivity of collisions to asymmetries in initial phases and positions of the solitons in the actual two-soliton collision. Very recently, collisions in the (strongly nonintegrable) discrete NLS equation were examined [15], and symmetry-breaking effects were found, along with sensitivity of the outcome to the location of the collision point. Here, we present results of collisions and their dependence on parameters in model (1) with small δ and ε, i.e., close to the integrable AL limit. Together with the already available findings for the strongly nonintegrable case [15], they provide a sufficiently comprehensive description of the collisional dynamics of nontopological solitons in fundamental lattice systems.

The AL model [Eq. (1) with $\delta=0$ and $\varepsilon=0$] has exact soliton solutions of the form

$$\psi_n(t)=\frac{1}{h}\frac{\sinh\mu}{\cosh\{\mu[n-x(t)]\}}\exp\{ik[n-x(t)]+i\alpha(t)\}, \tag{4}$$

where the instantaneous coordinate and phase of the soliton are

$$x(t)=x_0+\frac{t}{h^2}(\sin k)\frac{\sinh\mu}{\mu},$$

$$\alpha(t)=\alpha_0+\frac{t}{h^2}\left[(\cos k)\cosh\mu+k(\sin k)\frac{\sinh\mu}{\mu}-1\right]. \tag{5}$$

x_0 and α_0 are their initial values, while μ and k define the soliton's amplitude A and velocity V,

$$A=h^{-2}\sinh^2\mu, \quad V=(\mu h)^{-1}(\sinh\mu)\sin k. \tag{6}$$

The infinitely long AL system has an infinite series of dynamical invariants, the lowest ones being the norm, lattice momentum, and energy:

$$N=\sum_n \ln(1+h^2|\psi_n|^2), \tag{7}$$

$$P=ih^2\sum_n (\psi_n\psi_{n+1}^*-\psi_n^*\psi_{n+1}), \tag{8}$$

$$Q=\frac{1}{2}h^2\sum_n (\psi_n\psi_{n+1}^*+\psi_n^*\psi_{n+1}). \tag{9}$$

Note that the norm of the general nonintegrable model (1), given by expression (2), goes over into norm (7) in the limit $\delta=\varepsilon=0$, and Hamiltonian (3) of the nonintegrable model becomes, in the same limit, a linear combination of norm (7) and energy (9) of the AL integrable system: $H(\delta=\varepsilon=0)\equiv-(N+Q)$. It will be seen below that, as a matter of fact, the difference between the exact norm and Hamiltonian of the full perturbed model with small δ and ε and those of the AL model is negligible. However, all the other dynamical invariants of the AL model, *including* the lattice momentum (8), have no counterparts in the nonintegrable case. This is explained by the fact that each elementary dynamical invariant is generated by a certain continuum invariance of the underlying equation. In particular, the norm and energy conservation are accounted for by the invariance against phase and time shifts, respectively, that remain valid in the nonintegrable system, while a dynamical symmetry of the AL model which is responsible for the conservation of the lattice momentum is destroyed by the small perturbations. The momentum remains a dynamical invariant in continuum

nonintegrable models (e.g., the NLS equation with the quintic term), but in the discrete setting it is conserved solely in the integrable case—obviously, a generic lattice system is not invariant against arbitrary spatial translations. Further consideration of this issue can be found in a recent work [16].

An issue related to the lack of conserved momentum is the (non)existence of exact traveling soliton solutions in nonintegrable lattice models. While the problem still waits for its full solution, several important theoretical results have been obtained. The substitution of an appropriate traveling ansatz in the discrete equation gives rise to a differential-delay equation whose steady states are the traveling wave solutions of the original differential-difference equation (see, e.g., Refs. [17,18], as well as an earlier work by Feddersen, Ref. [19]). It is also pertinent to mention that moving solitons clearly persist in simulations of perturbed systems, without any conspicuous loss, for long times, which is sufficient to study their collisions in numerical experiments without ambiguity (see, e.g., Refs. [14] and [20]).

If the given system differs from the AL model by small terms, a natural question is how strong actual destruction of the former dynamical invariants, and especially of the momentum, which has a straightforward physical interpretation (and remains a virtually conserved quantity for free moving solitons, as explained above) will be in collisions between solitons. One of main objectives of the present work is to address this issue. We will conclude that the momentum conservation is *strongly* violated by the collisions, even if the perturbation parameters are quite small.

For two broadly separated AL solitons with parameters μ_j and k_j, expressions (7)–(9) take values

$$N_{\rm sol}=2\sum_{j=1}^{2}\mu_j, \quad Q_{\rm sol}=2\sum_{j=1}^{2}(\sinh\mu_j)\cos k_j, \tag{10}$$

$$P_{\rm sol}=2\sum_{j=1}^{2}(\sinh\mu_j)\sin k_j. \tag{11}$$

The fact that only two exact and, plausibly, one approximate conserved quantities (the latter one is the momentum) constrain possible outcomes of the soliton-soliton collisions, which are characterized by two amplitudes, two velocities, and, in addition, may depend on the initial relative phase $\Delta\alpha_0=\alpha_{02}-\alpha_{01}$ and positions $(x_0)_2$ and $(x_0)_1$ of the solitons, suggests that the above-mentioned radiationless energy exchange between the two solitons is quite feasible. Furthermore, for slow solitons (small k_j) the conservation of Q becomes an amplitude constraint [see Eq. (6)], and if amplitudes are small too ($\mu_j\to 0$), Q reduces to N, see Eq. (10), so that there actually remains the single constraint in this limit case.

II. NUMERICAL RESULTS

A. Setting up the problem

To perform simulations of the collisions, we notice that h can actually be scaled out from Eq. (1), leaving δ and ε/h^2 as independent control parameters, therefore in what follows, we fix $h=0.8$. The parameters are varied in ranges corresponding to the weak quintic perturbation, $\varepsilon\in[-0.01,0.01]$, and moderately weak Salerno's perturbation, $\delta\in[-0.08,0.08]$. Equation (1) was integrated by means of an implicit Crank-Nicholson scheme with the accuracy of $O((\Delta t)^2)$ and with reflecting boundary conditions. The initial condition was taken as a superposition of two far separated solitons (4) that would be exact solutions of the AL model, and the numerical integration was run until outgoing solitons were separated well enough. Their amplitudes and velocities after the collision, $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{V}_1$, and $\tilde{V}_2$, were measured, and then the corresponding parameters $\tilde{\mu}_1$, $\tilde{\mu}_2$, $\tilde{k}_1$, and $\tilde{k}_2$ were found inverting Eq. (6). We also checked to what extent the dynamical invariants given by the unperturbed expressions (10) and (11), as well as by the exact ones (2) and (3), were conserved.

To present the results, we will focus on the symmetric collisions: $\mu_1=\mu_2=\mu$, $k_1=-k_2=k$. While simulations were run for various values of the amplitudes and velocities, we display results for a case that turned out to be a typical one, adequately representing many others, with $\mu=0.75$, $k=0.1$. This implies $A_1=A_2=A=1.057$, $V_1=-V_2=V=0.137$. The initial phase difference $\Delta\alpha_0$ was controlled by setting $\alpha_{01}=0$ and choosing α_{02} from the interval $(-\pi,\pi)$. The initial positions of the solitons were taken as $(x_0)_1=-x_0+x_c$ and $(x_0)_2=x_0+x_c$, with $x_0=12$; this provides for the large initial separation $2x_0=24$ between them, while x_c was chosen from the interval $[0,1)$ to control the location of the collision point.

The presentation of results is structured as follows: we first examine the effect of variation of the initial phase difference $\Delta\alpha_0$ and collision point x_c on the outcome of the collision. Then, we analyze how the approximate conservation of expressions (10) and (11) correlates with the results. Finally, we examine the effect of combining the perturbation parameters ε and δ, in order to demonstrate that the two perturbations may almost exactly cancel each other, thus making the collisions virtually elastic. We stress that results obtained at other values of parameters are completely tantamount to those displayed below, provided that ε and δ remain small.

B. Sensitivity to the phase and position of the collision

In Fig. 1, values of the soliton parameters after the collision are presented as functions of the initial phase difference $\Delta\alpha_0$ for the case of the SM perturbation. In the part of the interval $(-\pi,\pi)$ which is not included, the collision is almost completely elastic. It is obvious that, in the interval $|\Delta\alpha_0|\lesssim 0.5$, the small perturbation is, in fact, a singular one, resulting in very strong effects (in other words, the elasticity of soliton collisions in the AL lattice is a very fragile feature). Note that, in the case of a weakly perturbed *continuum* NLS equation considered in Ref. [6], noteworthy inelastic effects in the collision of two solitons also took place at relatively small values of $|\Delta\alpha_0|$.

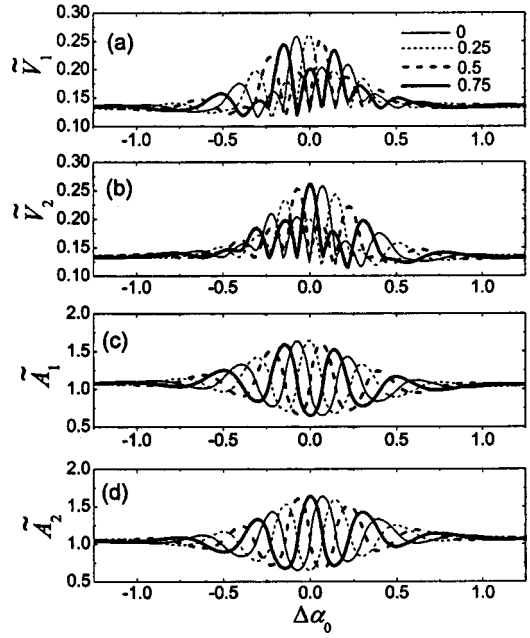

FIG. 1. Velocities and amplitudes of the solitons after the collision vs the phase difference $\Delta\alpha_0$, in the case $\delta=0.04$, $\varepsilon=0$ (the Salerno model without quintic terms). This and all the other cases are shown for $\mu=0.75$ and $k=0.1$, see Eq. (6). Four different curves correspond to different positions of the collision point, $x_c=0$, 0.25, 0.5, and 0.75. The collisions are strongly inelastic in the vicinity of $\Delta\alpha_0=0$.

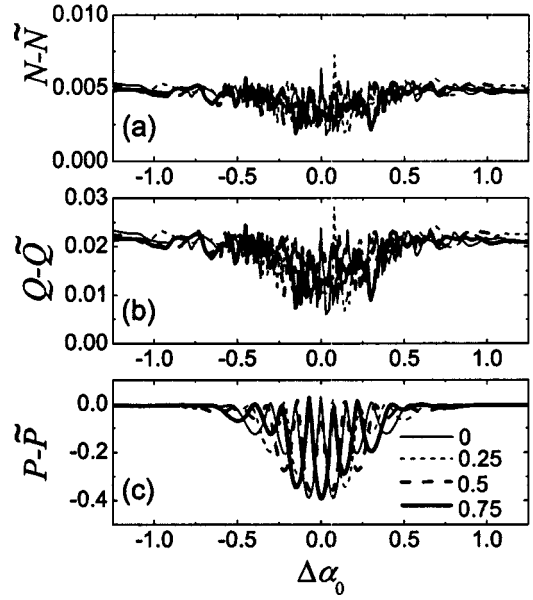

FIG. 2. The collision-induced changes of the net norm, energy, and momentum for the two solitons, defined as per Eqs. (10) and (11), vs $\Delta\alpha_0$ for different values of x_c in the Salerno model (recall that the tilde refers to the postcollision values of the corresponding quantities). The quantities displayed in this figure are obtained by adding up their values for the two solitons (rather than by direct calculation for the whole system). If the norm and energy are defined by the exact expressions (2) and (3), rather than the approximate ones (10), they are completely conserved.

C. Dynamical invariants

In the present case, the initial values of expressions (10) and (11) are $N=3$, $Q=3.273$, and $P=0$ (the absolute values of the initial momenta are $|P_1|=|P_2|=0.1642$). Using the exact expressions (2) and (3), we have checked that the net values of the norm and energy for the initial solitons and those observed after the collision are equal, in the case of the ordinary numerical accuracy employed, up to 10^{-4} (in relative units); running simulations with higher accuracy (smaller Δt), it was possible to check the norm and energy conservation with the accuracy of up to 10^{-6}. Norm and energy loss due to radiation loss remained completely negligible in all the cases considered. As concerns the difference between the unperturbed expressions used in Eq. (10) and the exact ones (2) and (3), Fig. 2 demonstrates that the largest relative difference between them, which reflects a direct effect of the small perturbations, is $\Delta N/N\sim 10^{-3}$ for the norm and $\Delta Q/Q\sim 3\times 10^{-3}$ for the energy. However, the bottom panel in Fig. 2 shows that the momentum is *not* conserved in any approximation, in accordance with the fact that the perturbed system has no translational symmetry.

The conservation of N suggests that a simple relation between the soliton amplitudes after the collision may be expected: according to Eqs. (10), $\mu_1+\mu_2$ must keep the original value with the accuracy $\sim 10^{-3}$. On the contrary, the momentum nonconservation promises a much worse accuracy in the prediction of a relation between the velocities. The conservation of Q does not provide for an essential additional information for small values of k (see above), while large k implies the collision between fast solitons, when nontrivial effects will be very weak.

The comparison between the actual results of the collision (dots) and predictions based on the approximate conservation laws for the values N and P of the two solitons in the form of Eqs. (10) (dashed lines) is displayed in Fig. 3. As is seen, the amplitude relation indeed follows from the norm conservation in a very accurate form, while the conservation of the momentum may be traced in a very crude form only.

Another salient feature of Fig. 3 is strong deviation of the dots from the diagonal point (1.054,1.054) corresponding to the values of the parameters before the collision. Such a feature was impossible in the case of the collision of a soliton with its mirror image in the SM, examined in Ref. [14]. A typical example of an inelastic collision (inducing this effect) is shown in Fig. 4. The major cause of the effect is the location of the collision central point x_c relative to the underlying lattice.

Besides that, the phase difference between the colliding solitons may produce a similar symmetry-breaking effect. Indeed, if the AL solitons, described by Eqs. (4) and (5), moving to the right and to the left (with $k>0$ and $k<0$, accordingly) are given phase shifts $+\Delta\phi$ and $-\Delta\phi$, this is equivalent to the shift of the coordinate x, but *solely* in the expressions for the solitons' phases, by $\Delta x=\phi_0/k$, which has *equal* signs for both solitons. This means that the phase pattern of the two-soliton configuration gets shifted by Δx relative to the shapes of the colliding solitons, which is an obvious cause for the symmetry breaking. The fact that the

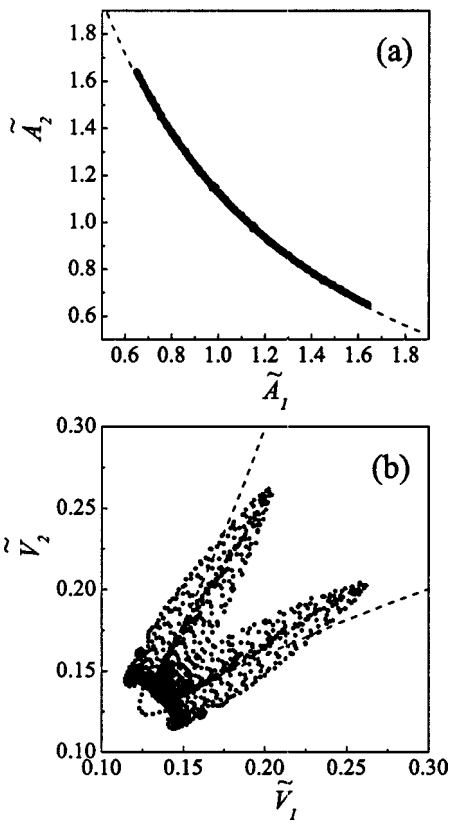

FIG. 3. Relations between the solitons' amplitudes (a) and velocities (b) after the collision in the Salerno model. The dashed lines show the relations predicted by the norm, energy, and momentum conservation for integrable AL chain, Eqs. (10) and (11). Dots are numerical results for 2500 collisions, with values of the initial phase difference $\Delta\alpha_0$ taken from the interval $[-1.25,1.25]$ with a step of 0.01, and the collision-point's coordinate x_c taken from $[0,1)$ with a step of 0.1. Note that the norm was taken in the approximate form of Eq. (10), which pertains to the unperturbed Ablowitz-Ladik lattice. The relative nonconservation of this norm after the collision is $\sim 10^{-3}$ (see the text), while the exact norm of the perturbed model, as given by Eq. (2), is conserved exactly, within the numerical accuracy.

above-mentioned position and phase factors do not affect the collision symmetry in the integrable AL model is another specific manifestation of its integrability.

D. The role of the perturbation strength

In Fig. 3 one can observe that, for $\delta=0.04$ and $\varepsilon=0$, the maximum possible soliton amplitude after the collision is $\tilde{A}_{\max}=1.64$, and the maximum possible postcollision velocity is $\tilde{V}_{\max}=0.263$. These values [and, in particular, their deviation from initial ones, $(A,V)=(1.054,0.137)$] may be regarded as a measure of the departure of the perturbed model from the integrability. In Fig. 5, we use $\tilde{A}_{\max}$ and $\tilde{V}_{\max}$ to gauge the deviation from the integrable case with the increase of the perturbation strength (the cases of both the SM and quintic perturbations are shown). It is concluded that the weak perturbations generate quite large inelastic effects, the inelasticity increasing almost linearly with the perturbation

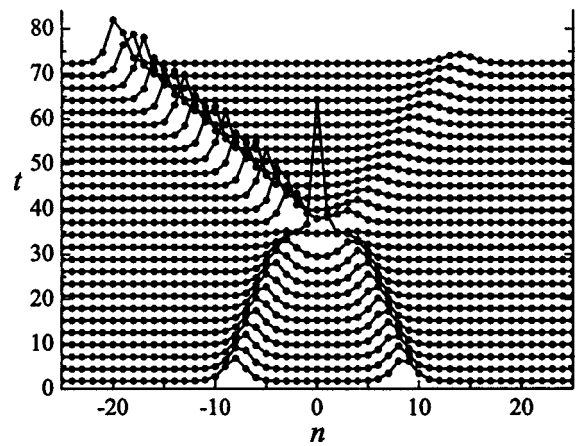

FIG. 4. An example of a strongly inelastic collision between solitons in the AL system with a small onsite quintic perturbation, $\delta=0$ and $\varepsilon=-0.01$. In this case, $\Delta\alpha_0=0$ and $x_c=0.2$.

parameter. For instance, the value $\delta=+0.08$ of the relative perturbation parameter in the SM model gives rise to collision-induced changes of the soliton's amplitude by a factor of $\simeq 2$, and of the absolute value of the velocity by a factor of $\simeq 3$.

Another noteworthy feature is the asymmetry of the plots in Fig. 5. The asymmetry is due to the fact that internal modes in the colliding solitons can be excited only when $\delta>0$ (for $\varepsilon=0$) or when $\varepsilon<0$ (for $\delta=0$) [21]. Hence, in these cases, we observe a combined effect of the radiationless energy exchange and the internal-mode excitation, while for $\delta<0$, $\varepsilon=0$ and $\delta=0$, $\varepsilon>0$, only the former occurs. Naturally, the net nonintegrability-induced effects are stronger in the cases where the internal mode can be excited.

E. Compensation of perturbation effects

In Ref. [6], it was found that, in continuum models, inelastic effects in soliton collisions can be strongly suppressed if contributions from different perturbations cancel each other. We have observed a similar feature in the present

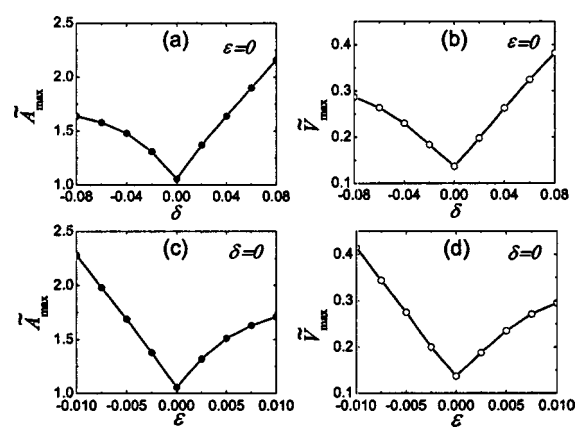

FIG. 5. The maximum possible amplitude $\tilde{A}_{\max}$ (filled circles) and velocity $\tilde{V}_{\max}$ (empty circles) of the soliton after the collision vs the perturbation strengths δ and ε: (a,b) $\varepsilon=0$ (the Salerno model); (c,d) $\delta=0$ (the quintic model).

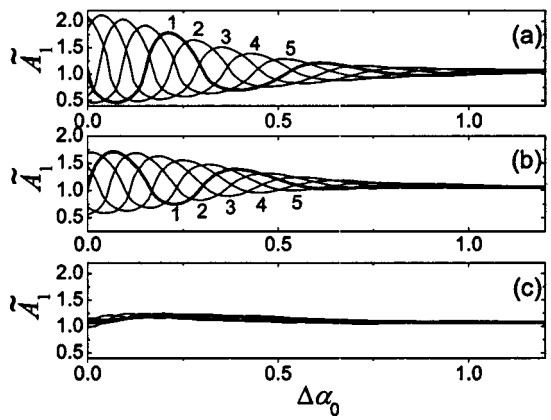

FIG. 6. The amplitude of the first soliton after the collision, $\tilde{A}_1$, vs $\Delta\alpha_0$ for $x_c=0$, 0.2, 0.4, 0.6, and 0.8, curves 1–5, respectively. The perturbations have (a) $\delta=0.08$, $\varepsilon=0$; (b) $\delta=0$, $\varepsilon=0.01$; (c) $\delta=0.08$, $\varepsilon=0.01$.

model. In particular, in Fig. 6 we show the postcollision amplitude $\tilde{A}_1$ versus $\Delta\alpha_0$ for three different perturbations. It is clear that the cancellation takes place in case (c), making the norm and momentum exchange an order of magnitude smaller than in the other cases. Similar compensation effects were observed for $\delta=0.08$ and $\varepsilon=0.01$ in a wide range of soliton parameters, including nonsymmetric collisions (between nonidentical solitons). In fact, the possibility of the mutual compensation between the Salerno and quintic perturbation is a strong proof to the assertion that different conservative perturbations produce virtually identical effects, hence essentially the same results are expected from other perturbations.

III. DISCUSSION AND CONCLUSIONS

In this work, we have quantified properties of collisions between solitons in the AL model with weak Hamiltonian perturbations. We have observed complex dependences of the outcomes of the collisions on the initial phase difference between the solitons and exact location of the collision point. Strong inelastic effects, in the form of radiationless energy and momentum exchange between colliding solitons, are generated by weak perturbations (for instance, a perturbation with the relative strength $\delta=0.08$ gives rise to a change of the solitons' amplitudes by a factor exceeding 2). The effects produced by different conservative perturbations are quite similar, suggesting that the results reported in this paper are generic. The exact and approximate conservation laws of the perturbed system were examined, with a conclusion that the small perturbations very weakly affect the norm and energy conservation, but strongly destroy the conservation of the lattice momentum, which is explained by the absence of the translational symmetry in nonintegrable lattice models. Statistical data collected for a very large number of collisions validate this conclusion. Symmetry-breaking effects in the collisions (which are simply explained by the dependence of the result on the location of the central point of the collision relative to the lattice, and by the phase difference between the colliding solitons) were highlighted, and their magnitude was used to gauge the deviation of the perturbed model from integrability. It was also shown that, properly combining two different perturbations, it is possible to almost exactly cancel their integrability-destroying effects, thus constructing a perturbed system in which collisions are practically elastic.

In this paper, we were dealing with collisions between solitons with relatively large initial velocities. It would naturally be of interest to see how the picture presented is modified for smaller collision velocities, and, in particular, to examine whether a fractal structure, similar to that observed in Ref. [7], can be found in the present model. This issue will be considered elsewhere.

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