# Soliton collisions in the discrete nonlinear Schrödinger equation

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We report analytical and numerical results for on-site and intersite collisions between solitons in the discrete nonlinear Schrödinger model. A semianalytical variational approximation correctly predicts gross features of the collision, viz., merger or bounce. We systematically examine the dependence of the collision outcome on initial velocity and amplitude of the solitons, as well as on the phase shift between them, and location of the collision point relative to the lattice; in some cases, the dependences are very intricate. In particular, merger of the solitons into a single one, and bounce after multiple collisions are found. Situations with a complicated system of alternating transmission and merger windows are identified too. The merger is often followed by symmetry breaking (SB), when the single soliton moves to the left or to the right, which implies momentum nonconservation. Two different types of the SB are identified, deterministic and spontaneous. The former one is accounted for by the location of the collision point relative to the lattice, and/or the phase shift between the solitons; the momentum generated during the collision due to the phase shift is calculated in an analytical approximation, its dependence on the solitons' velocities comparing well with numerical results. The spontaneous SB is explained by the modulational instability of a quasiflat plateau temporarily formed in the course of the collision.

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## I. INTRODUCTION

The discrete nonlinear Schrödinger (DNLS) equation is a ubiquitous dynamical-lattice model with many applications [1]. Its physical realizations include arrays of coupled optical waveguides [2,3] and Bose-Einstein condensates (BECs) trapped in strong optical lattices [4]. It is also relevant to many other problems, such as denaturation of the DNA double strand [5] and the envelope-wave expansion for nonlinear Klein-Gordon models on lattices [1].

Optical waveguide arrays (including virtual arrays induced by a system of laser beams in a photorefractive crystal [6]) offer the most straightforward experimental implementation of the DNLS system, with the number of lattice sites (guiding cores) up to 40 and the propagation length up to 15 mm. With the interchannel coupling constant $C \simeq 0.5\ \mathrm{mm}^{-1}$ [see Eq. (1) below] and nonlinearity coefficient $\simeq 5\ (\mathrm{m\,W})^{-1}$ in semiconductor waveguides, the beam power $\sim$500 W provides for formation of discrete solitons with the intrinsic dynamical length $\lesssim$1 mm [3]. Therefore, the available size of the samples is sufficient to test not only formation and stability of solitons, but also collisions between them. The study of collisions is a problem of fundamental significance by itself, and it may find applications in photonics, such as all-optical switching, steering, etc.

An array of BEC droplets trapped in a strong optical lattice, with $\sim 10^3$ atoms in each droplet, is another physical system which is directly described by the DNLS equation in the tight-binding limit [4]. In this case, a discrete soliton can be easily set in motion by means of a laser beam pushing the condensate.

The objective of the present work is the analytical and numerical study of collisions between DNLS solitons. Thus far, few publications have addressed the issue. In Ref. [7], collisions were studied in the *Salerno model* [8], i.e., a mixed Ablowitz-Ladik (AL)–DNLS system, but this was done close to the integrable [9] AL limit, while consequences of the strong nonintegrability of the ordinary DNLS system were not investigated. Actually, Ref. [7] was dealing with a collision of a soliton with a reflecting wall, which is tantamount to a collision between strictly in-phase solitons of the *on-site* (OS) type, while *intersite* (IS; see exact definitions below) collisions were not considered, nor the case of the phase difference $\Delta\phi$ between the solitons other than zero. In this work, we systematically consider various types of collisions in the DNLS model, as concerns the location of the collision point relative to the underlying lattice, and the value of $\Delta\phi$.

Numerical results reported in Ref. [7] for the Salerno model included quasielastic collisions in the case of a large relative velocity, and merger, interspersed with intermediate intervals of escape, for smaller velocities (seeds of such findings can be found in an earlier work [10]). However, no windows of multibounce escape (with solitons separating after several collisions), of the type known for kinks (topological solitons) in nonintegrable continuum models [11], were reported. Very recently, collisions of solitons in a weakly discrete NLS model were addressed in Ref. [12], but the system was actually approximated by a perturbed continuum NLS equation, while we aim to consider an essentially discrete case. The analysis developed in Ref. [12] was based on the use of the exact two-soliton solution of the unperturbed NLS equation; an effective small perturbation representing the weak discreteness and acting on the exact solution was shown to make the collision chaotic.

Collisions between solitons in a discrete model with a quadratic [$\chi^{(2)}$] nonlinearity were studied numerically in Ref. [13]. A strong distinction was found there between the above-mentioned OS and IS collisions. Generally, the OS and IS cases are well known to be energetically different in static configurations, due to the effective Peierls-Nabarro (PN) potential induced by the lattice [1]. It will be shown below that OS collisions in the DNLS model give rise to transmission or merger, while in the IS case alternation of different outcomes of the collision is more intricate, and an additional possibility is bounce after multiple collisions. In both cases, symmetry breaking (SB) is possible, its strongest manifestation being a merger of a symmetric soliton pair into a *single* soliton, which then moves in a certain direction. In fact, the latter outcome of the collision manifests not only the SB proper (which is also possible in soliton-soliton collisions in nonintegrable continuum models [14]), but also nonconservation of momentum: the momentum of the eventually emerging moving soliton is created "from nothing," as the initial momentum of the colliding soliton pair was zero. This effect is quite generic, as it was also observed in the case of the dynamical-lattice model with the $\chi^{(2)}$ nonlinearity in Ref. [13]. In this work, we give an explanation to this effect, which includes two different mechanisms: spontaneous and deterministic SBs. In the former case, the SB is initiated by small random perturbations, while in the latter case the SB is hidden in the initial conditions (initial positions and the above-mentioned phase difference $\Delta\phi$ of the two solitons).

The rest of the paper is organized as follows. In Sec. II, we start the analysis of the soliton-soliton collision with an analytical approach based on the variational approximation, which makes it possible to predict most basic features of the collision. In Sec. III, we summarize the collision phenomenology found from systematic numerical simulations. As the above-mentioned symmetry breaking and momentum nonconservation are quite remarkable features of the collisions, in Sec. IV we specially focus on them, combining analytical and numerical considerations. Section V concludes the paper.

## II. VARIATIONAL APPROXIMATION FOR THE COLLISION PROBLEM

We take the DNLS model in its usual form,

$$i\dot{u}_n = -C\Delta_2 u_n - |u_n|^2 u_n, \tag{1}$$

where $u_n$ is the complex amplitude of the electromagnetic field in the $n$th channel, in the case of the waveguide array, or the mean-field wave function at the $n$th site, in the BEC system. The overdot stands for the time derivative ("time" is actually the propagation distance in the case of the waveguide arrays), $\Delta_2 u_n = u_{n+1} + u_{n-1} - 2u_n$ is the discrete Laplacian, and $C$ is a positive coupling constant.

To gain analytical insight into the soliton-collision problem, we make use of the variational approximation (VA; a recent review of the technique can be found in Ref. [15]). For immobile DNLS solitons, VA can be implemented analytically [17] and/or numerically [18]. However, in the case of the collision between discrete solitons, direct VA generates equations which are difficult even to write down in an explicit form. For this reason, we take a simpler path that yields a tractable set of equations, and, eventually, leads to very approximate but meaningful results.

To this end, the DNLS equation is replaced by its continuum counterpart, with the commonly known Lagrangian, $\int_{-\infty}^{+\infty}[i(u^*\dot{u} - u\dot{u}^*) - |u_x|^2 + |u|^4]dx$, while the *Ansatz* (trial wave form) is taken as a combination of two *spikes*, which are mirror images to each other:

$$u = A\exp[-W^{-1}(|x|-\xi) + ib(|x|-\xi)],\ |x|>\xi, \tag{2}$$

$$u = A\exp[-W^{-1}(\xi-|x|) + ic(\xi-|x|)],\ |x|<\xi \tag{3}$$

(spikes in variational equations were considered in some works in different contexts [16]). It should be mentioned that the continuum limit of the DNLS equation is, by itself, an integrable NLS equation, in which collisions between *smooth* solitons are elastic. However, we employ the spikes to emulate *discrete* solitary waves with centers located at the points $x = \pm\xi(t)$ in the version of the system which is far from the continuum limit.

It should also be highlighted that, in a certain sense, the substitution of a spike-shaped *Ansatz* (i.e., a discreteness-motivated one) in the continuum Lagrangian of the model is the exact inverse of the time-honored approach of inserting a *continuum Ansatz* in the *discrete* Lagrangian of the model (see, e.g., Ref. [1], and references therein). Hence, just as the latter *continuum-in-discrete* approach has proved very successful in the study of discreteness effects (chiefly, static ones), we expect (and confirm *a posteriori* by the results obtained below) that the *discrete-in-continuum* approach proposed here can capture key aspects of the discrete-soliton dynamics.

Besides $\xi$, other variational parameters in *Ansätze* (2) and (3) are the complex amplitude $A(t)$, real width $W(t)$, and outer and inner wave numbers $b(t)$ and $c(t)$. Using the *Ansätze*, we derive a system of the Euler-Lagrange equations for the variational parameters:

$$b = \frac{1}{2}\left[(2-e^{-\eta})\frac{d}{dt}\left(\frac{W}{2-e^{-\eta}}\right) + \frac{d}{dt}(W\eta)\right], \tag{4}$$

$$c = \frac{2-e^{-\eta}}{2(1-e^{-\eta})}\frac{d}{dt}\left[\frac{W[1-(1+\eta)e^{-\eta}]}{2-e^{-\eta}}\right] - \frac{1}{2}\frac{d}{dt}(W\eta), \tag{5}$$

$$\frac{db}{dt} + [1-(1+\eta)e^{-\eta}]\frac{dc}{dt} + \eta(2-e^{-\eta})\frac{d}{dt}\left[\frac{b-c(1-e^{-\eta})}{2-e^{-\eta}}\right] = \frac{2}{W^3}(2-e^{-\eta}) - \frac{E}{2W^2}\frac{2-e^{-2\eta}}{2-e^{-\eta}}, \tag{6}$$

$$\frac{d}{dt}\left[-\frac{1}{W^2} - \frac{b^2+(1-e^{-\eta})c^2}{2-e^{-\eta}} + \frac{E}{2W}\frac{2-e^{-2\eta}}{(2-e^{-\eta})^2}\right] = 0. \tag{7}$$

Here $\eta \equiv 2\xi/W$, and $E \equiv \Sigma_{-\infty}^{+\infty}|u_n|^2 = |A|^2 W[2-\exp(-2\xi/W)]$ is the conserved power in the waveguide array, or number of atoms in BEC.

For two far separated solitons, which correspond to $\eta \to \infty$, the *Ansatz* splits into solitary spikes. In this case, Eqs. (4)–(7) show that $b,c$, and $W$ are constant, so that $W=8/E$ (hence, $|A|=E/4$) and

$$b=-c=\dot{\xi}, \tag{8}$$

i.e., according to Eqs. (2) and (3), each individual spike is symmetric, $\pm b = \mp c$ being its velocity.

Equations (4)–(7) were solved numerically for various initial values $2\xi_0$ of the separation between the solitons and their widths $W_0$ and velocities $\pm \dot{\xi}_0$, the latter being defined by the initial values $b_0=-c_0$ as per Eq. (8). Typical findings are shown in Fig. 1. A drastic difference between the cases of large and small initial velocities is seen in the evolution of the inner wave number $c(t)$, and in the difference between temporal scales: in the former case, a *bounce* of the spikes is predicted, which is seen in the explosion of $c(t)$ (the explosion does not make it possible to explicitly continue the integration for larger values of $t$); divergence of $c$ implies a very large velocity of the solitons, according to Eq. (8), i.e., a bounce indeed. On the contrary, for small initial velocities the solution gets stuck with $b,c,\xi \to 0$, and $W \approx$ const. In this limit, Eqs. (4)–(7) describe a *single* immobile symmetric spike with the power $E$. Thus, VA predicts that the collision of two solitons with large velocities leads to bounce, while the collision with small velocities gives rise to *merger* of the solitons. Despite the simplistic nature of the approximation, it correctly predicts basic features of the collision; see below.

## III. NUMERICAL SIMULATIONS

Proceeding to numerical analysis of the collisions, it is necessary to mention a long-standing debate on the existence of exact traveling-soliton solutions in the DNLS model [19]. Due to the presence of the PN potential, one may expect resonances produced by motion of the soliton, similar to ones that are well known for kinks in discrete models [20], where they give rise to permanent leakage of the kinetic energy from the moving soliton. However, this issue is a rather formal one: even if traveling solitons do not exist in the rigorous sense, numerical works clearly show that the distance at which such structures cease propagating is long, hence they may be readily observed in the experiment, which makes it relevant to consider collisions between them.

The motion of a soliton at a velocity $v$ is supported by a phase gradient $k$ across it [see Eq. (8)]. In Eq. (1), a rough relation between them is $v \sim C\Delta\psi/a$, where $\Delta\psi$ is the phase shift between the fields at adjacent lattice sites and $a$ is the lattice spacing. In the case of discrete solitons, the most interesting situation for collisions is expected when the characteristic soliton's diffraction time/distance, $t_{\mathrm{diffr}} \sim a^2N^2/C$ ($N$ is the number of sites in the soliton), is comparable to the collision time/distance, $t_{\mathrm{coll}} \sim aN/v$. It follows from these estimates that nontrivial collisions are expected if the solitons are "pushed" by lending them the intersite phase shift $\Delta\psi \sim 1/N$ (note that $a$ and $C$ drop out from the estimate).

For systematic simulations, we used initial conditions suggested by the AL model, where analytical expressions for moving discrete solitons are available [the spike *Ansatz* based on Eqs. (2) and (3) was also used as the initial configuration, yielding similar results]. Thus, a superposition of two far separated pulses was taken at $t=0$, with common amplitude $B$ and width $W$,

$$\begin{aligned} u_0 = {} & B\,\mathrm{sech}[W^{-1}(n-x_1)]\exp[ia(n-x_1)+(i/2)\Delta\phi] \\ & + B\,\mathrm{sech}[W^{-1}(n-x_2)]\exp[-ia(n-x_2)-(i/2)\Delta\phi], \end{aligned} \tag{9}$$

where $x_{1,2}$ are the positions of their centers, $\Delta\phi$ is the initial phase shift between them, and the wave number $a$ determines the initial speed, cf. Eq. (8). We fix $W=1$ (other values of the width produce very similar results) and use $a$ as a main control parameter, as changing $a$ is tantamount to varying the initial velocity.

Analysis of numerical results demonstrates that three different values of the amplitude, viz., $B=\sinh(1/W)\approx 1.175$, which corresponds to the AL soliton, $B=1$, corresponding to the continuum NLS limit, and a smaller value, $B=1/\sinh(1/W)\approx 0.851$, provide for adequate description of the relevant phenomenology. Although these values are not very different, the results obtained for them may differ considerably. We also varied the initial coordinates $x_1$ and $x_2$, so as to place the collision center at different positions relative to the lattice. We will thus consider OS and IS collisions, with the central point located, respectively, on site or at a midpoint between sites, as well as a "quarter-site" collision. Finally, the comparison of results obtained for different values of the phase shift $\Delta\phi$ also reveals important peculiarities, which will be considered in detail below; in this section, we present basic results for the collisions between in-phase solitons, with $\Delta\phi=0$. Outcomes of the collisions are readily identified, plotting trajectories of the center of mass of each soliton in the $(x,t)$ plane, see Figs. 2 and 3 below.

We start with OS collisions between in-phase solitons ($x_{1,2}=\pm 30, \Delta\phi=0$) in the case $B=1$. The first result is that, if the velocity parameter takes values from $0<a<0.7755$, the colliding solitons merge into a single pulse (subsequently remaining at the collision point). Further, two distinct intervals are identified inside this region: $0<a<0.711$, where the two solitons fuse into one after a single collision, and $0.711<a<0.7755$, where the fusion takes place after multiple collisions (the latter case may be employed for control purposes in optical applications: a solitary pulse which temporarily reappears between two collisions can be affected by an external signal). For $a>0.7756$, the collisions is quasi-elastic, i.e., the solitons separate. Note that basic features of this phenomenology (barring sophisticated peculiarities, such as the fusion after multiple collisions) are correctly predicted by VA.

In the same OS configuration, but with a larger amplitude, $B=1.175$, the solitons cannot collide at all if $a$ belongs to a "stop band," $a<0.550$, as in this case free solitons are

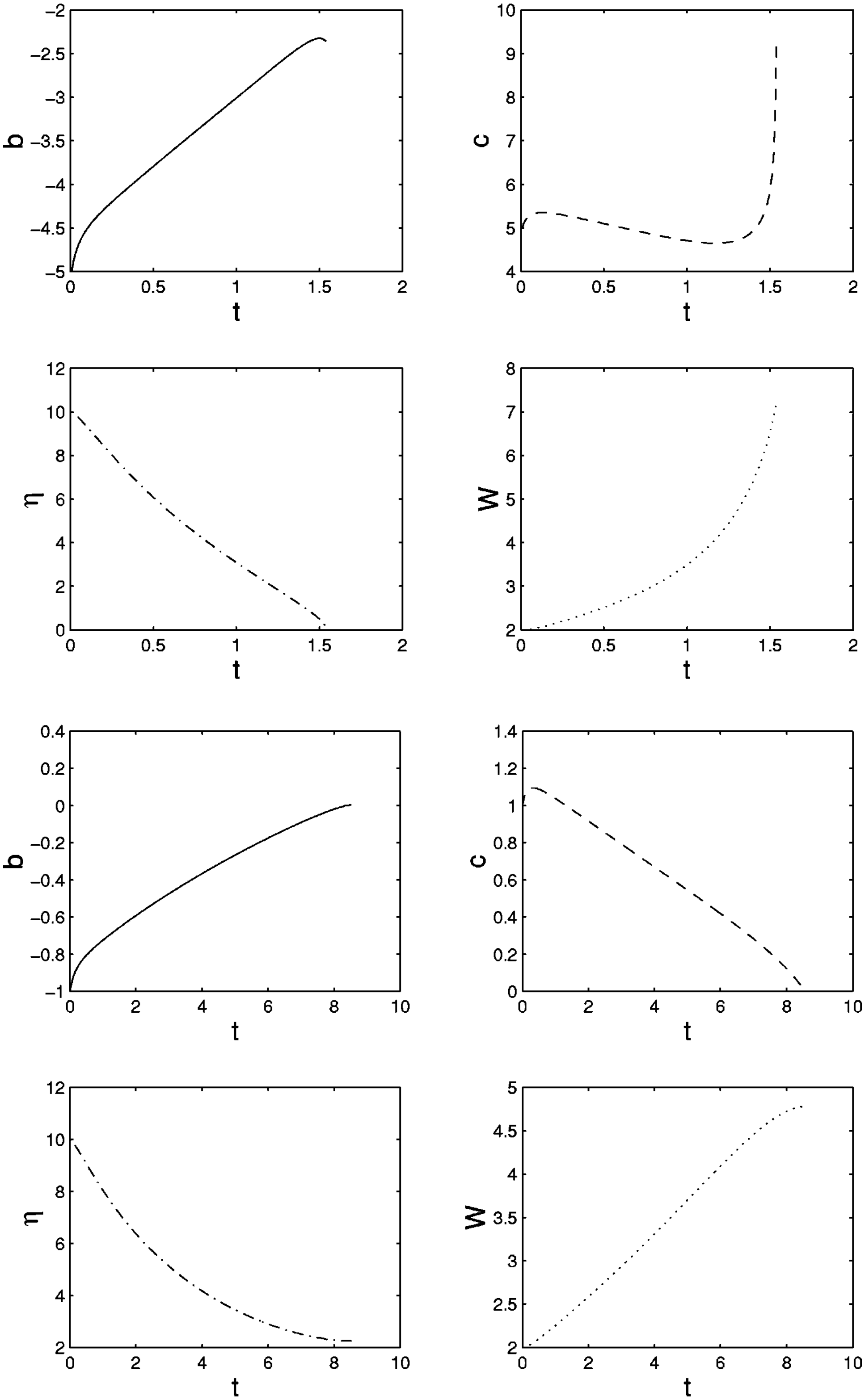


FIG. 1. Two different cases of the collision as predicted by VA for the initial separation $\eta_0 = 10$. Four top and bottom panels correspond, respectively, to the collisions with large $(c = -b = 5)$ and small $(c = -b = 1)$ initial velocities. In the former case, the speed $c$ explodes as the solitons approach each other, which implies bounce, while in the latter case $c$ drops to zero, implying merger of the solitons.

quickly trapped by the lattice. This is explained by the fact that taller pulses encounter a higher PN barrier, hence they need larger kinetic energy to overcome it. Above the stop band, viz., in the interval $0.550 < a < 2.175$, the solitons move freely and collide, which results in merger (after multiple collisions, if $a$ is close to the upper border of this interval). Quasielastic collisions take place if $a > 2.175$.

For the OS configuration, but with a smaller initial amplitude, $B = 0.851$, a different feature is found in intervals $a < 0.203$ and $0.281 < a < 0.3$. There, the solitons merge after multiple collisions, which is accompanied by notable *symmetry breaking* (SB): the resultant pulse moves to the left or to the right, at a well-defined value of the velocity, as is seen in Fig. 2. This feature resembles strong SB observed in Ref.

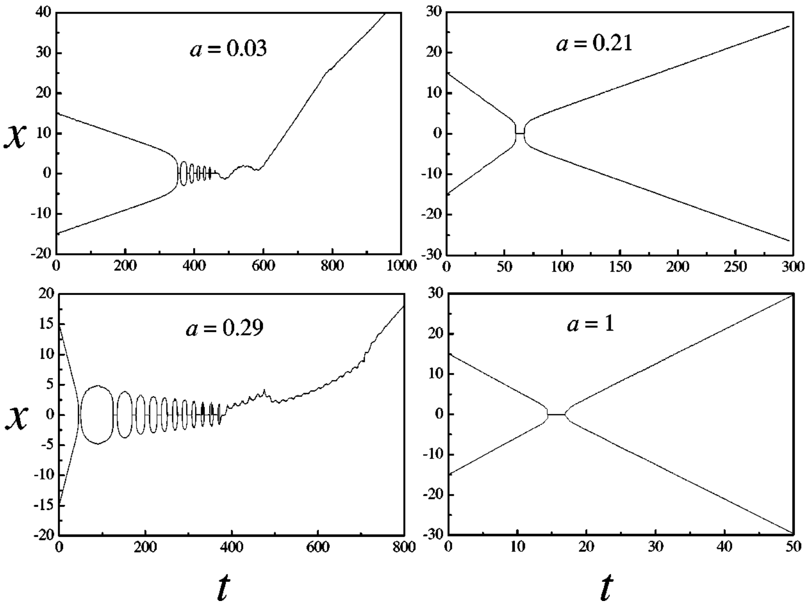


FIG. 2. The on-site collision ($x_{1,2}=\pm 30$) in the most interesting case, with the amplitude $B=0.851$. Intervals of merger with spontaneous symmetry breaking are separated by regions of quasielastic collisions. In all the cases displayed here, $C=0.5$.

[13] in simulations of solitons collisions in lattices with the quadratic nonlinearity. We have checked that SB in the present model (as well as the other above-mentioned outcomes) is not a numerical artifact: rerunning the simulations with higher accuracy produces no change in the results. Between these intervals, i.e., at $0.203<a<0.281$ and at $a>0.3$, only quasielastic collisions occur.

Symmetry breaking was also observed in collisions of solitons in nonintegrable continuum models of the NLS type [14]. However, in continuum systems SB is constrained by the momentum conservation. The lack of the momentum conservation in the (nonintegrable) lattice makes the SB more dramatic in the DNLS model. In fact, there are two different mechanisms that explain this effect: *spontaneous* SB in the case of in-phase collisions, with $\Delta\phi=0$, and an additional *deterministic* SB in the case $\Delta\phi\neq 0$. Both mechanisms will be considered in detail in the following section.

For IS collisions we anticipate a significant change in the phenomenology, as in this case the collision point is at a local maximum of the PN potential, whereas in the OS case it was at a local minimum, hence trapping of the resulting pulse was more energetically favorable, while the outcome of IS collisions may be more sensitive to the initial kinetic energy (differences between OS and IS collisions in $\chi^{(2)}$ dynamical lattices were reported in Ref. [13]). In fact, changes in the initial velocity, which give rise to different outcomes of the IS collision, decrease by an order of magnitude in comparison to the OS case (see below).

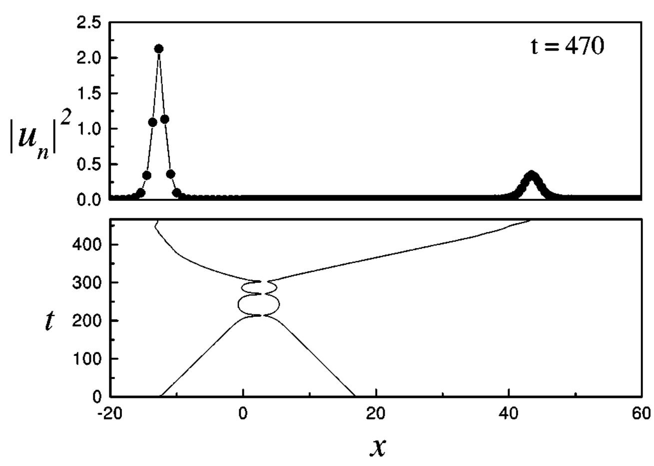


FIG. 3. The intersite collision with $B=1$. Trajectories of the solitons and their eventual profiles (in terms of $|u_n|^2$) are displayed.

IS collisions at the intermediate value of the amplitude, $B=1$, lead to straightforward merger for $a<0.061$. In the interval $0.062<a<0.075$, spontaneous SB occurs, with mutual bounce of the solitons after multiple collisions, see Fig. 3. To the best of our knowledge, this is the first example of a *multiple-bounce* window in a dynamical-lattice model, which may be compared to what was found for kinks in continua [11]; however, in kink-bearing continuum models, spontaneous SB (in kink-antikink collisions) is impossible due to the momentum conservation. The IS collision leads to ordinary merger for $0.075<a<0.089$, and quasielastic collisions occur at $a>0.089$.

As in the OS case, IS collisions of pulses with the larger amplitude ($B=1.175$) are simpler: the pulses may propagate and collide only if $a>0.53$; they merge in the interval $0.53<a<0.795$, and quasielastic collisions take place if $a>0.795$. On the contrary, in the case of the smaller amplitude, $B=0.851$, an intricate system of intervals of multibounce merger with spontaneous SB was found ($0<a<0.04$; $0.042<a<0.044$; $0.046<a<0.049$; $0.053<a<0.055$), interspersed with windows of quasielastic collisions; only quasielastic collisions occur if $a>0.056$. It is noteworthy that, for quasielastic collisions, the time elapsed between the initial collision and eventual separation is almost independent of $a$.

Last, in quarter-site collisions (not shown here in detail), at all the values of $a$ examined (with $B=0.851$), separation of the solitons upon a single bounce was observed, but *always* with SB, resulting in asymmetric soliton pairs with amplitudes and speeds different from original ones. In this case, however, the SB is not necessarily spontaneous, as it may be induced in a straightforward way by asymmetry of the initial configuration relative to the lattice. In fact, a mixture of spontaneous and deterministic SB takes place in such a situation; see below.

## IV. DETERMINISTIC AND SPONTANEOUS SYMMETRY BREAKING, AND MOMENTUM NONCONSERVATION IN SOLITON-SOLITON COLLISIONS

### A. Deterministic symmetry breaking

As mentioned above, a salient feature of the observed phenomenology is SB. In fact, SB in collisions between solitons was observed in various nonintegrable models, both continuum [14] and discrete [12,13] ones. As was mentioned above, in the case where the collision center does not coincide with an OS or IS point, the symmetry breaking has a straightforward (deterministic) explanation. In view of the lack of momentum conservation in nonintegrable dynamical-lattice models, SB also explains the generation of momentum from nothing (before the collision, the net momentum was equal to zero due to the symmetry of the two-soliton configuration).

A more subtle but similar situation takes place in the case where the solitons collide with a nonzero phase difference, i.e., $\Delta\phi\neq 0$ in Eq. (9). Indeed, Eq. (9) implies that, while the collision center is located at the point $n=(x_1+x_2)/2\equiv x_0$, the phase-center point is at $\tilde{n}=x_0-\Delta\phi/(2a)$. The difference between the two points (in the case $\Delta\phi\neq 0$) is a natural source of the deterministic SB.

It should be noted that a similar deterministic mechanism, based on the phase difference, i.e., mismatch between the collision center and phase-center point, can also explain SB in soliton-soliton collisions in nonintegrable continuum models. However, SB-induced effects in continuum models are strongly restricted by the (total) momentum conservation, while the momentum is no longer a conserved quantity in the nonintegrable dynamical-lattice setting. Indeed, a natural definition of the lattice momentum is

$$P=i\sum_{n=-\infty}^{+\infty}(\psi_{n+1}\psi_n^*-\psi_{n+1}^*\psi_n) \tag{10}$$

(this expression goes over into the correct conserved momentum in the continuum limit, and coincides with the conserved momentum in the integrable AL model [9]). As follows directly from the underlying equation (1), an exact evolution equation for $P$ is

$$\frac{dP}{dt}=\sum_{n=-\infty}^{+\infty}|\psi_n|^2\psi_n^*(\psi_{n+1}-\psi_{n-1})+\text{c.c.}, \tag{11}$$

where c.c. stands for the complex-conjugate expression. The fact that the linear part of Eq. (1) yields no contribution to the evolution of the momentum is natural, as the momentum is conserved in the linear dynamical lattice. The derivation of Eq. (11) assumes, as usual, the boundary conditions $\psi(n=\pm\infty)=0$.

To proceed with the analysis of the momentum nonconservation, it is necessary to calculate the right-hand side of Eq. (11), and then perform the time integration. In fact, the only possibility of obtaining an analytical result is to employ a quasicontinuum approximation, setting

$$\psi_{n+1}-\psi_{n-1}\approx 2\partial\psi/\partial n+(2/3)\partial^3\psi/\partial n^3, \tag{12}$$

where $n$ is treated as a continuous variable. Then, the lowest-order continuum limit of Eq. (1) is

$$i\psi_t+C\psi_{nn}+|\psi|^2\psi=0. \tag{13}$$

Substituting approximation (12) in Eq. (11) and assuming the boundary conditions $\psi(n=\pm\infty)=0$, we arrive at a result

$$\frac{dP}{dt}=2\int_{-\infty}^{+\infty}dn\left|\frac{\partial\psi}{\partial n}\right|^2\frac{\partial}{\partial n}(|\psi|^2). \tag{14}$$

Note that the first term (the first derivative) on the right-hand side of Eq. (12) gives no contribution to expression (14), which precisely corresponds to the momentum conservation in the continuum approximation. As it follows from Eq. (14), the net momentum change generated by the collision, which is a measure of the momentum nonconservation, is

$$\Delta P\equiv\int_{-\infty}^{+\infty}dt\frac{dP}{dt}=2\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}dn\left|\frac{\partial\psi}{\partial n}\right|^2\frac{\partial}{\partial n}(|\psi|^2). \tag{15}$$

One can try to use an exact solution of Eq. (13) for the soliton-soliton collision, which is provided by the inverse scattering transform [21], to calculate the integral expression in Eq. (15). This solution describes the collision between two symmetric moving solitons; when they are far separated, it reduces to the linear superposition,

$$\begin{aligned}\psi(n,t)=&B\,\mathrm{sech}\left(\frac{B}{\sqrt{2C}}[(n-x_0)+2Cat]\right)\\&\times\exp\left(ia(n-n_0)+\frac{i}{2}(B^2-2Ca^2)t+\frac{i}{2}\Delta\phi\right)\\&+B\,\mathrm{sech}\left(\frac{B}{\sqrt{2C}}[(n+x_0)-2Cat]\right)\\&\times\exp\left(-ia(n+n_0)+\frac{i}{2}(B^2-2Ca^2)t-\frac{i}{2}\Delta\phi\right),\end{aligned} \tag{16}$$

where $B$ is the common amplitude of the solitons, $\pm 2Ca$ are their velocities, $\pm x_0$ are their initial positions, and $\Delta\phi$ is the phase shift between them. Note that $B$, $a$, and $\Delta\phi$ have the same meaning as in expression (9).

Substituting the exact two-soliton solution in Eq. (15), one can first of all see that $\Delta P$, calculated this way, exactly vanishes if $\Delta\phi$ is 0, $\pi/2$, $\pi$, or any multiple of $\pi/2$. Actually, the quasicontinuum approximation assumes that the solitons are broad, i.e., $B/\sqrt{2C}\ll 1$. This condition facilitates the calculation of integral (15) with the exact two-soliton

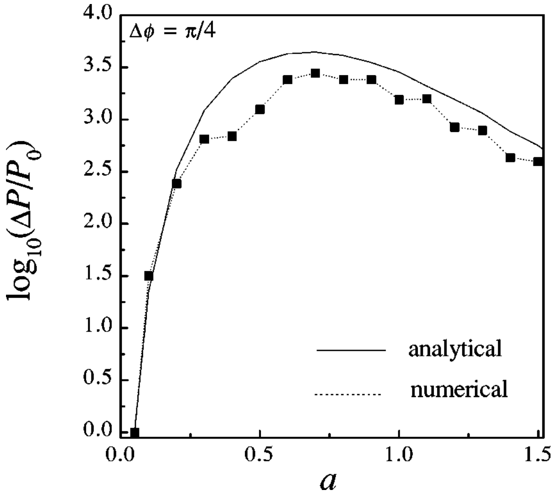


FIG. 4. The comparison (on the log scale) of the dependence of the momentum generated "from nothing" by the soliton-soliton collision for $\Delta\phi=\pi/4$, $B=0.851$ [see Eq. (16)], as predicted by Eq. (18) and found from direct simulations of Eq. (1) with $C=0.5$, for the on-site collisions.

solution. Nevertheless, the accurate result is very cumbersome. What can be obtained in a simple form is the dependence of the generated momentum on the velocity parameter $a$ at fixed values of $\Delta\phi$ and $B$:

$$\Delta P=\text{const}\times a^5\exp\left(-2\pi\frac{\sqrt{2C}a}{B}\right). \tag{17}$$

which assumes that $B/\sqrt{2C}\ll a$.

Equation (17) is more convenient for the comparison with results of direct simulations in a logarithmic form,

$$\log\left(\frac{\Delta P}{\Delta P_0}\right)=5\,\log\left(\frac{a}{a_0}\right)-\frac{2\pi\sqrt{2C}}{B\log 10}(a-a_0), \tag{18}$$

where $a_0$ is an arbitrarily fixed value, $\Delta P_0$ being the corresponding value of $\Delta P$. A typical result of the comparison of the analytical prediction (18) with numerical findings is displayed in Fig. 4. As is seen, the agreement is reasonable, and (as it often happens with results obtained by means of asymptotic methods) it actually extends to a region where the assumed condition $B\ll\sqrt{2C}a$ does not hold.

### B. Spontaneous symmetry breaking

The dependence of the generated momentum on $\Delta\phi$ is quite complicated, as is seen in Fig. 5. Nevertheless, for not too small values of the velocity parameter $a$, the generated momentum indeed shows the trend to vanish at $\Delta\phi=0$, as is predicted by Eq. (15). For very small $a$, the situation is different (see the panel of Fig. 5 pertaining to $a=0.03$): conspicuous momentum generation is observed when $\Delta\phi=0$ (which is also implied in Fig. 2). Even in the case of $a=0.29$, the value of $\Delta P$ corresponding to $\Delta\phi=0$ is not very small, see Fig. 5. On the other hand, the initial configuration (9) with $\Delta\phi=0$ is completely symmetric (even), so

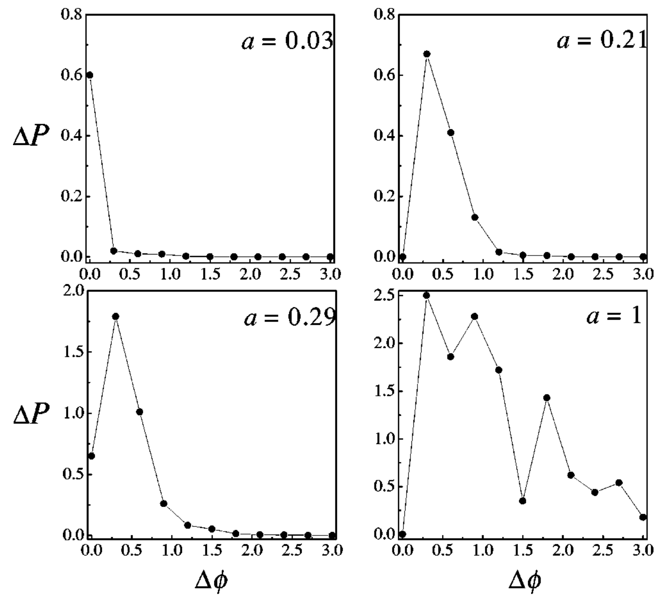


FIG. 5. The generated momentum vs the phase shift between the colliding solitons for different fixed values of the velocity parameter $a$. In all the cases, $C=0.5$ and the on-site collisions were simulated.

that $u_0(-n)=u_0(n)$. As the underlying equation (1) is compatible with the symmetry $n\to -n$, no deterministic SB is possible in this case.

An explanation for the SB and momentum generation in the case $\Delta\phi=0$ is possible in terms of *spontaneous* SB under the action of small random perturbations (numerical noise, which emulates noise in the real physical system). To demonstrate this possibility, we take a typical case, with $a=0.29$, when conspicuous momentum generation (although smaller than at finite $\Delta\phi$) is observed at $\Delta\phi=0$, see Figs. 2 and 5.

In Fig. 6 a set of snapshots of the lattice field is displayed, around the moment of $t=390$, at which the SB takes place, and the momentum generation commences, see the panel pertaining to $a=0.29$ in Fig. 2. As is seen (and it is a typical

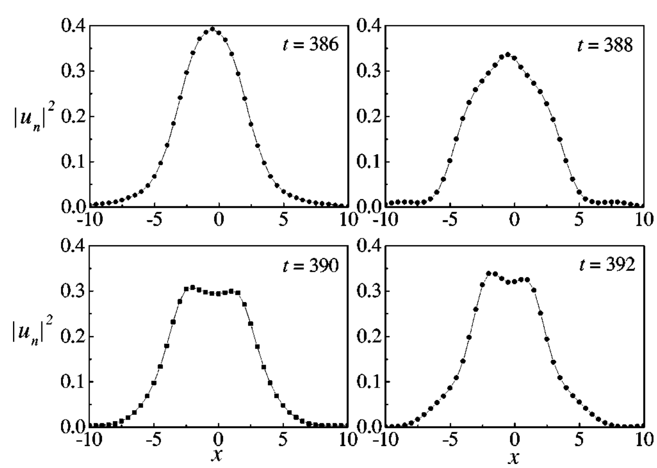


FIG. 6. A set of instantaneous profiles of the lattice fields, $|u_n|^2$, in the case of the on-site collision between solitons with $a=0.29$, $B=0.851$, and $\Delta\phi=0$.

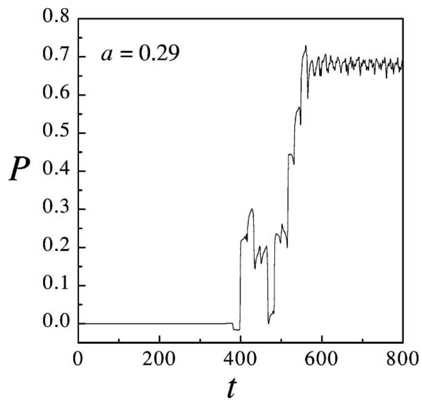


FIG. 7. The total momentum of the lattice field vs time, in the case of the soliton-soliton collision, details of which are displayed in Fig. 6.

feature observed in many other cases), the collision gives rise to temporary formation of a broad quasiflat configuration, which may be subject to the modulational instability (MI). In fact, the panels of Fig. 6 which pertain to $t=390$ and 392 do show the beginning of spontaneous SB that may be attributed to MI. Direct numerical computation of lattice momentum (10) as a function of time (see Fig. 7) demonstrates a clear correlation between the beginning of the spontaneous SB and the commencement of the momentum generation.

MI in the DNLS equation can be analyzed following Ref. [22]. To this end, a quasiflat field configuration is taken in the form

$$\psi_n(t)=[A+b_n(t)+ic_n(t)]\exp(iA^2t), \tag{19}$$

where $A$ is a constant, and $b_n$ and $c_n$ are small real perturbations. Spatially even and odd eigenmodes of the perturbation are

$$(b_n,c_n)=\left(1,\frac{\sigma(k)}{4C\sin^2(k/2)}\right)b^{(0)}\exp[\sigma(k)t]\times\begin{cases}\cos(kn)\\ \sin(kn),\end{cases} \tag{20}$$

with an arbitrary infinitesimal perturbation amplitude $c^{(0)}$, real wave number $k$, and the corresponding instability growth rate

$$\sigma(k)=\sqrt{8C\sin^2(k/2)[A^2-2C\sin^2(k/2)]}. \tag{21}$$

Note that characteristic values of the instability growth rate (21) which correspond to the quasiflat configuration observed in Fig. 6 are $\sim 1$ [recall that Eq. (1) was simulated with $C=0.5$], hence the time interval within which the quasiflat configuration exists, which is $\Delta t\simeq 5$, is sufficient for the development of the instability.

The quasiflat background field configuration shown in Fig. 6 may be roughly approximated by boundary conditions (BCs)

$$a_n(n\geqslant \pm N)=0, \tag{22}$$

where the size of the background is $2N-1$ (in the above-mentioned example, one may adopt $N=5$, assuming, in the crudest approximation, that only nine inner sites carry nonzero field). The BCs (22) imply, in the case of the *odd* eigenmode (20), $\sin(kN)=0$, or

$$k_{\rm odd}=\frac{\pi m}{N}, m=1,2,3,\ldots. \tag{23}$$

Now, one can substitute the perturbed wave field (19) into the right-hand side of Eq. (11), in order to find a contribution of the perturbation to the momentum nonconservation. After simple manipulations, we obtain

$$\left(\frac{dP}{dt}\right)_{\rm pert}=2A^3\sum_{n=-\infty}^{+\infty}(b_{n+1}-b_{n-1}). \tag{24}$$

Adopting the above-mentioned approximation, according to which the field, including the perturbation, is limited to the domain $|n|\leqslant N-1$, we obtain the following result from Eq. (24):

$$\left(\frac{dP}{dt}\right)_{\rm pert}=2A^3[b_{N-1}-b_{-(N-1)}]. \tag{25}$$

From expression (25) it follows that only odd perturbation modes may contribute to the momentum nonconservation, see Eq. (20). Indeed, the presence of a small symmetry-breaking perturbation on top of the quasiflat background, which may be accounted for by an odd mode, is evident in Fig. 6. If it is taken in the form (20), and expression (23) for $k$ is taken into regard, the eventual result is

$$\left(\frac{dP}{dt}\right)_{\rm pert}=2(-1)^{m-1}b^{(0)}A^3\sin\left(\frac{\pi m}{N}\right)\exp\left[\sigma\left(\frac{\pi m}{N}\right)t\right] \tag{26}$$

[recall that $\sigma(k)$ is defined in Eq. (21)]. An important feature of this result is its essentially discrete character: the continuum limit implies fixing $m$ and letting $N\to\infty$, then the expression (26) vanishes.

To the best of our knowledge, the analyses presented above for the cases of the deterministic and spontaneous SBs constitute the first explicit consideration of the collision-induced momentum generation in nonintegrable dynamical-lattice models. In the general case ($\Delta\phi\neq 0$), the SB and momentum generation are contributed to by both the deterministic and spontaneous mechanisms.

## V. CONCLUSION

In this work, we have studied in an analytical approximation and, in detail, numerically collisions between solitons in the discrete NLS equation. We have observed and classified different outcomes, whose most notable features are various manifestations of the SB, leading to the appearance of either a single moving pulse, or a pair of pulses with different amplitudes and speeds, after single or multiple bounces. The dependence of the outcome on the type of the collision (on-site or intersite), initial velocity, and amplitude of the pulses, as well as the phase shift between them, was quantified. The analysis of the SB and collision-induced momentum generation from nothing were developed for two qualitatively different cases, when they are accounted for by the phase shift between the solitons, or modulational instability of the broad plateau which is temporarily formed in the course of the collision. If the collision is asymmetric relative to the lattice, the SB is explained in a simpler way by the presence of the effective Peierls-Nabarro potential. The results suggest straightforward experimental realizations in optical waveguide arrays, and in Bose-Einstein condensates trapped in a strong optical lattice. The variety of different outcomes of the collision, and the possibilities to control them suggest potential applications to the design of multifunctional photonic devices based on waveguide arrays.

*Note added*. Recently it was brought to our attention that moving pulses in the DNLS equation, subject to periodic boundary conditions, were found to bifurcate from an exact constant-amplitude traveling-continuous-wave solution [23]. The pulses vanish at some critical value of the nonlinearity coefficient. It is plausible that, in the limit of the infinitely long system, this result implies the existence of genuine moving solitons in the DNLS equation. Additionally, we have been informed that experiments are currently in progress regarding the interaction of two solitary waves in arrays of coupled optical waveguides [24], for which the relevant mathematical model is the one discussed herein. The focus in these experiments is on the case of the interaction of solitons with zero initial velocities (in terms of the present paper), and a result coinciding with our findings is that, in the limit of the zero collision velocity, two solitons with the zero phase difference always merge into one.

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