Dynamo effect in a driven helical flow

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(Received 19 May 2003; published 8 October 2003)

The Roberts flow, a helical flow in the form of convectionlike rolls, is known to be capable of both kinematic and nonlinear dynamo action. We study the Roberts dynamo with particular attention being paid to the spatial structure of the generated magnetic field and its back-reaction on the flow. The dynamo bifurcation is decisively determined by the symmetry group of the problem, which is given by a subgroup of discrete transformations and a continuous translational invariance of the flow. In the bifurcation the continuous symmetry is broken while the discrete subgroup symmetry completely survives. Its actions help in understanding the spatial structures of the magnetic field and of the modified flow. In accordance with experimental observations, the magnetic field component perpendicular to the originally invariant direction is much stronger than the component in this direction. Furthermore, the magnetic field is largely concentrated in layers separating the convectionlike rolls of the flow and containing, in particular, its stagnation points, which are isolated for the modified flow while they are line filling for the original Roberts flow. The magnetic field is strongest near β-type stagnation points, with a two-dimensional unstable and a one-dimensional stable manifold, and is weak near α-type stagnation points, with a two-dimensional stable and a one-dimensional unstable manifold. This contrasts with the usual picture that dynamo action is promoted at the α points and impeded at the β points. Both the creation of isolated stagnation points and the concentration of strong fields at the β points may be understood as a result of the way in which the Roberts dynamo saturates. It is also found that, while the original Roberts flow is regular, the modified flow is chaotic in the layers between the convectionlike rolls where the magnetic field is concentrated. This chaoticity, which results from the back-reaction of the magnetic field on the flow, appears to merely enhance magnetic diffusion rather than to strengthen the dynamo effect.

DOI: 10.1103/PhysRevE.68.046302 PACS number(s): 47.20.Ky, 47.65.+a, 91.25.Cw, 95.30.Qd

I. INTRODUCTION

Long-lasting magnetic fields are a phenomenon that can be observed in many objects and on widely varying scales in our Universe. In most cases the only tenable explanation of their origin is induction by the motion of electrically conducting fluids. The geomagnetic field with its irregular reversals, the much stronger magnetic fields in the active regions on the Sun and the weak interstellar fields on the scale of galaxies are well-known examples. Despite their different appearance, all these fields are presumably generated and maintained by similar magnetohydrodynamic (MHD) dynamo processes. The theoretical study of MHD dynamos has a long research history; for comprehensive reviews we refer to Refs. [1–6].

Recently a number of successful attempts have been made to realize MHD dynamos in laboratory experiments under terrestrial conditions. Notably groups in Riga and Karlsruhe have reported the observation of self-excited dynamos in their experiments [7–9]; for reviews on dynamo experiments in the laboratory see Refs. [10–12]. These successes are presently stimulating further experimental efforts. Also the possibility of a dynamo effect in fast breeder reactors has found renewed interest [13,14]. Like the cosmic dynamos, also the dynamo experiments pose problems to the theory and motivate the study of specific aspects of the governing equations, aimed at better understanding the nature of the dynamo processes.

A real dynamo is characterized by a complex interaction between several physical processes. It can be modeled theoretically by a set of nonlinear partial differential equations including the Navier-Stokes equation (NSE), the magnetic induction equation, the heat equation, and the thermodynamic equation of state. In order to analyze the dynamo process in isolation from other processes, we shall study the problem in the framework of incompressible MHD with the fluid motion driven by an external mechanical body force. We wish to note, however, that there exist numerous studies of more complex and more realistic models, in particular, where the dynamo is driven by convection, as is presumably the case for the Earth and the Sun. For reviews discussing numerical simulations of convection-driven dynamos we refer to [10,6,15].

The governing equations for our study are the coupled NSE for the flow and induction equation for the magnetic field in the form

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=\nabla^2\boldsymbol{v}-\nabla p-\tfrac{1}{2}\nabla\boldsymbol{B}^2+(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}+\boldsymbol{f}, \quad (1)$$

$$\frac{\partial \boldsymbol{B}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{B}=\mathrm{Pm}^{-1}\,\nabla^2\boldsymbol{B}+(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}, \quad (2)$$

$$\nabla\cdot\boldsymbol{v}=0, \quad \nabla\cdot\boldsymbol{B}=0, \quad (3)$$

where $\boldsymbol{v}$, p, and $\boldsymbol{B}$ denote fluid velocity, pressure, and magnetic field. Pm is the magnetic Prandtl number and $\boldsymbol{f}$ the yet unspecified external body force. The third and fourth terms on the right-hand side of Eq. (1) constitute the Lorentz force. Equations (3) impose the incompressibility condition on the fluid and ensure the source-free property of the magnetic field. The equations have got this nondimensional form by means of a rescaling transformation based on a length scale L and the associated viscous time scale $T=L^2/\nu$, where ν is the kinematic viscosity (see Ref. [16]). A drawback of this rescaling is that the viscous Reynolds number is hidden in the forcing term. On the other hand, the viscous Reynolds number is essentially determined by the strength of the externally applied force and its control over the forcing strength corresponds to a more experimental approach. The other free parameter in Eqs. (1)–(3), besides the strength of the forcing, is the magnetic Prandtl number $\mathrm{Pm}=\nu/\eta$, given by the ratio between kinematic viscosity and magnetic diffusivity.

The body force $\boldsymbol{f}$ on the right-hand side of the NSE, Eq. (1), is the sum of all forces that drive the fluid. It pumps energy into the system. The question whether, given some initial seed field, a long-lasting magnetic field can be generated, not decaying in the limit of infinite time, represents the dynamo problem.

The ABC flow $\boldsymbol{v}_{\mathrm{ABC}}$ [16–20] (named after Arnold, Beltrami, and Childress) and the Roberts flow $\boldsymbol{v}_{\mathrm{R}}$ [21–25] are intensively studied examples of dynamo-effective velocity fields. In the following we concentrate on the latter and tie up to a preceding study of the bifurcations of the externally driven MHD equations under a body force of the Roberts type [26]. We extend the preceding investigations and describe the features of the generated magnetic field and its feedback to the velocity field. For this purpose we study in great detail the symmetry breaking effects connected with the onset of the dynamo and the role of different types of stagnation points of the flow. Additionally, we vary the magnetic Prandtl number, which was fixed to the value $\mathrm{Pm}=1$ in Ref. [26], and investigate its influence on the first instability of the basic flow.

Our starting point is the driving velocity field in the form of the Roberts flow which can be generated as a stationary solution of the incompressible NSE, Eq. (1) and first of Eqs. (3), if an external body force in the form $\boldsymbol{f}=-\nabla^2\boldsymbol{v}_{\mathrm{R}}$ is applied. This force compensates viscous losses and generates the required flow $\boldsymbol{v}_{\mathrm{R}}$. Furthermore, together with a vanishing magnetic field, the Roberts flow is a stable solution to the full MHD equations for small Reynolds numbers (small strengths of the forcing).

In the model considered here we impose periodic boundary conditions for the three-dimensional domain $\Omega=[0,2\pi]^3$, which implies the possibility to use the Fourier expansions

$$\boldsymbol{v}(\boldsymbol{x},t)=\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq\boldsymbol{0}}}\hat{\boldsymbol{v}}_{\boldsymbol{k}}(t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}),$$

$$\boldsymbol{B}(\boldsymbol{x},t)=\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq\boldsymbol{0}}}\hat{\boldsymbol{B}}_{\boldsymbol{k}}(t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}), \tag{4}$$

where $\hat{\boldsymbol{v}}_{\boldsymbol{k}}$ and $\hat{\boldsymbol{B}}_{\boldsymbol{k}}$ are the complex Fourier coefficients of the velocity field and of the magnetic field, respectively. Due to the periodic boundary conditions the pressure term in Eq. (1) can be easily eliminated and its Fourier decomposition is dropped here. The mean flow and the mean magnetic field, i.e., the modes with wave number $\boldsymbol{k}=(0,0,0)$, are conserved in time and are set equal to zero; these modes are excluded in the sums of Eq. (4). The fields are real and, hence, the complex mode coefficients have to fulfill the reality conditions $\hat{\boldsymbol{v}}_{-\boldsymbol{k}}=\hat{\boldsymbol{v}}_{\boldsymbol{k}}^*$ and $\hat{\boldsymbol{B}}_{-\boldsymbol{k}}=\hat{\boldsymbol{B}}_{\boldsymbol{k}}^*$ (an asterisk denotes the complex conjugate). When we plug the Fourier ansatz in the MHD equations, we obtain an infinite system of ordinary differential equations (ODEs) for the time evolution of the coefficients. However, to make the problem numerically feasible, only a finite number of modes can be taken into account. A truncation gives a finite-dimensional system of ODEs, which is eventually the model under consideration. In our numerics we have used a pseudospectral method with 16 grid points in each spatial direction. In Sec. II we examine, combining numerical calculations with a symmetry analysis, the dynamo bifurcation of the driven Roberts flow. Then, in Sec. III, we study in more detail the spatial structure of the generated magnetic field and the modification of the flow due to the back-reaction of the magnetic field. We end with some conclusions in Sec. IV.

II. DYNAMO EFFECT IN AN EXTERNALLY DRIVEN FLOW OF THE ROBERTS TYPE

In the following a dynamo model is studied that is based on a flow introduced by Roberts [21,22]. This flow is on the one hand kinematically very dynamo effective. On the other hand it resembles the roll solutions of thermal convection. In the convective zones of rotating celestial bodies, for instance, convection rolls parallel to the axis of rotation tend to be formed [27]. These facts have motivated an approximate realization of the Roberts flow in a laboratory experiment aimed at demonstrating the dynamo effect under terrestrial conditions [9,28,29]. The experimental setup, with helical conduits and liquid sodium in a cylindrical vessel, was proposed by Busse [30] who also gave a first kinematic analysis of the Roberts dynamo in a finite domain. Further kinematic studies related to this experiment are due to Rädler *et al.* [31–33] and Tilgner [34]. Rädler *et al.* applied mean-field dynamo theory [3], whose central mechanism is the α effect. Tilgner used direct numerical simulation of the induction equation. In the kinematic studies system parameters most suitable for dynamo excitation were determined. The modification of the Roberts flow (or of the approximate Roberts flow as realized in the experiment) due to the back-reaction of the magnetic field was studied by Tilgner and Busse [35,36] (see also discussion in Sec. III) and Rädler *et al.* [37]. In the following we study the dynamo effect in the context of the full MHD equations, taking into account the nonlinear feedback of the magnetic field to the velocity field. The forcing term in Eq. (1) is chosen such as to generate the Roberts flow $\boldsymbol{v}_{\mathrm{R}}$ as an exact solution of the incompressible NSE. Increasing the forcing strength above the dynamo threshold, we investigate how the flow is changed due to the back-reaction of the generated magnetic field. Specifically, certain correlations between the flow structure and the pat-

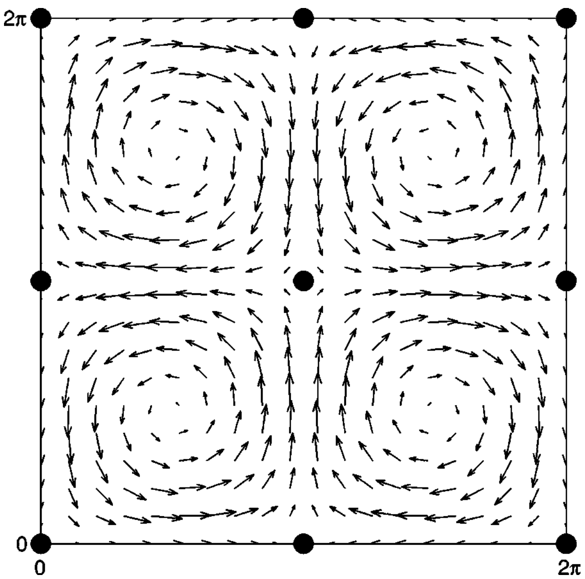

FIG. 1. Projection of the Roberts flow on the x-y plane. Black dots indicate stagnation points of the flow.

tern of the generated magnetic field are analyzed.

A. Roberts flow

The Roberts flow is given as a family of three-dimensional velocity fields which are independent of the (Cartesian) z coordinate, namely,

$$\boldsymbol{v}_{\mathrm{R}}=(g\sin x\cos y,-g\cos x\sin y,2f\sin x\sin y),\qquad(5)$$

where g and f are free parameters characterizing the strength of the horizontal and vertical velocity components. To give an impression of the flow structure, a projection of the velocity vectors on the x-y plane is plotted in Fig. 1. The flow consists of an array of rolls where the fluid spirals up and down in neighboring rolls. It has a helical structure with a nonzero kinetic helicity $H=\int_{\Omega}\boldsymbol{v}\cdot\nabla\times\boldsymbol{v}\,d^3\boldsymbol{x}$. A nonvanishing kinetic helicity is well known to be favorable for the large-scale dynamo action of small-scale velocity fields [3,38–40]. With respect to small-scale dynamos, where the magnetic field and the velocity field vary on comparable scales, a correlation between helicity and dynamo action has been demonstrated at least for certain flow families, e.g., generalized ABC flows [41].

B. Symmetry analysis

The appearance of the dynamo is accompanied by symmetry breaking. In a first step, thus, the symmetry group of the problem has to be determined, i.e., the equivariance group of the MHD equations, Eqs. (1)–(3) (for a review of the role of symmetries in bifurcations we refer to Ref. [42]). This group is decisively determined by the symmetry of the external forcing or equivalently by its defining flow, the Roberts flow $\boldsymbol{v}_{\mathrm{R}}$. The Roberts flow is translationally invariant in the z direction. Due to our periodic boundary conditions this continuous symmetry is isomorphic to the circle group S^1. The stagnation points indicated by black dots in Fig. 1 fill straight lines parallel to the z axis. These lines and their following mutation to chains of isolated stagnation points will need special attention in the study of the dynamo mechanism.

Additionally, the Roberts flow is symmetric with respect to a discrete group G consisting of 16 elements. From the two-dimensional projection of the flow shown in Fig. 1 one could get the impression there were a horizontal D_2 symmetry, namely, symmetry to reflections in the lines $x=\pi$, $y=\pi$, and to the product of these two reflections (the dihedral group D_2 is the commutative group with three nontrivial elements where each element is inverse to itself and the product of two elements gives the third element). However, all reflectional symmetries are excluded because there is a nonvanishing kinetic helicity, which as a pseudoscalar would change its sign under reflections. Actually the third component of the flow has to be taken into account as well. Though independent of z, the flow has a nonvanishing component in the z direction; neighboring vortices spiral in opposite directions (but all vortices or rolls possess the same helicity). In fact, the discrete group can be characterized as a semidirect product $D_2\times_S Z_4$. The group D_2 corresponds to rotations about the x, y, and z axes by the angle π and is a normal subgroup. The cyclic group Z_4 is the group of rotations which leave the square invariant, generated by a rotation about the z axis by the angle $\pi/2$. But to leave the Roberts flow invariant this rotation has to be combined with a shift by π in the x direction, in order to take into account the opposite, vertical flow directions in neighboring vortices. Nevertheless, the group is isomorphic, i.e., formally equivalent to Z_4. A more detailed description of the group structure of G can be found in the Appendix.

Finally, the MHD equations are invariant with respect to the special transformation $\boldsymbol{B}\rightarrow-\boldsymbol{B}$, $\boldsymbol{v}\rightarrow\boldsymbol{v}$. This Z_2 symmetry plays an essential role in the symmetry breaking bifurcation. Thus the whole equivariance group of the problem can be summarized now as

$$(D_2\times_S Z_4)\times S^1\times Z_2.\qquad(6)$$

C. Linear stability analysis for varying Prandtl number

In the first step we perform a linear stability analysis of the MHD equations, Eqs. (1)–(3), with an external force of the Roberts type,

$$\boldsymbol{f}=-\nabla^2\boldsymbol{v}_{\mathrm{R}}=2\boldsymbol{v}_{\mathrm{R}}.\qquad(7)$$

The parameters g and f of the Roberts flow in Eq. (5) are set equal to each other, that is, $g=f=\mathrm{Re}$. The Reynolds-like number Re measures the strength of the external force or, equivalently, the amplitude of the generated Roberts flow. The magnetic Prandtl number is varied between $0.5\leqslant\mathrm{Pm}\leqslant1.0$. As already mentioned, the Roberts flow $\boldsymbol{v}_{\mathrm{R}}$ with zero magnetic field is the only stable solution for small values of Re. By computing the eigenvalues of the Jacobian matrix of the equations linearized about the Roberts flow, the first instability of the Roberts flow solution is determined. The calculations are done in Fourier space using a pseudospectral code with 16 grid points in each spatial direction. Figure 2

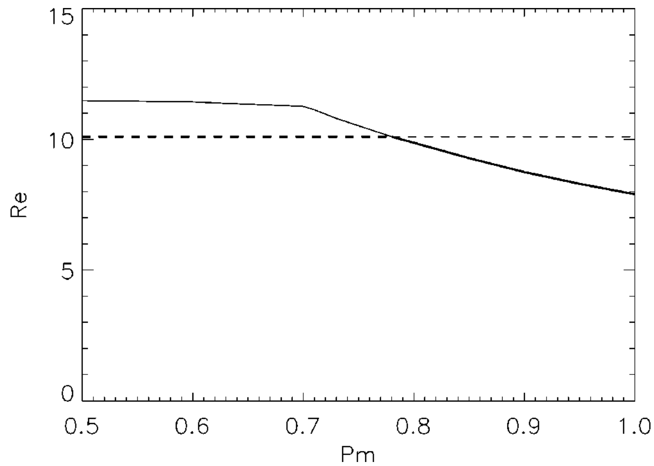

FIG. 2. Linear stability analysis of the primary Roberts flow. The solid line marks a magnetic instability whereas the dashed line gives an instability of the flow in which no magnetic components are excited.

shows the result of the linear stability analysis. The solid line marks a steady-state bifurcation in which two real eigenvalues go through zero. These eigenvalues belong to a magnetic mode which generates a dynamo for Pm$>$0.78. For smaller Prandtl numbers the Roberts flow becomes unstable by a symmetry breaking Hopf bifurcation where the real parts of two pairs of complex conjugate eigenvalues become zero. This instability does not produce a dynamo but generates a nonmagnetic time-periodic modified flow. It is remarkable that the value of the Reynolds number at which the Hopf bifurcation occurs, namely, Re$=$10.1, is independent of the magnetic Prandtl number. However, this is explained by the fact that the magnetic Prandtl number cannot influence the flow as long as the magnetic field is absent.

Investigations of the nonlinear dynamics give a steady-state dynamo above the thick solid line in Fig. 2 and a time-periodic flow with a zero magnetic field above the thick dashed line. The dynamics in the vicinity of the point of intersection of the two curves is rather complex and not the topic of the present investigations. The dynamo effect for Pm$>$0.78 is qualitatively the same for all those values of Pm. The dynamo bifurcation for Pm$=$1.0 is described in more detail in the following sections.

A remaining question then is whether there exists a dynamo for Pm$<$0.78 and larger values of Re as a result of a secondary bifurcation of the modified flow. For smaller values of Pm the increased magnetic diffusivity works against the dynamo. Thus, for Pm$<$0.4 we did not see any dynamo effect. However, for the interval 0.5$\leq$Pm$\leq$0.78 dynamo activity in the form of intermittent bursts is observed. The driving flow is at first nearly time periodic and feeds slowly all scales of the magnetic field which finally results in strong magnetic bursts. During the bursts the magnetic field and the velocity field are interacting in a chaotic manner. This dynamo activity is rather short and a relaminarization process extinguishes the dynamo. The dynamics of the velocity field becomes regular and the cycle starts again. We could not classify this dynamics as a result of a local bifurcation. However, a similar behavior was found for the ABC dynamo by Sweet *et al.* [43], also for the case of small magnetic Prandtl numbers. These authors, who studied the scaling behavior of the bursts with respect to the magnetic Reynolds number, explained the phenomenon as a result of blowout bifurcations.

D. Symmetry breaking bifurcations

Next we study in more detail the dynamo generation for the special Prandtl number Pm$=$1 which is representative for the qualitative solution behavior for Pm$>$0.78, cf. Fig. 2. As already mentioned in the preceding section, the primary Roberts flow loses its stability by a symmetry breaking pitchfork bifurcation where two real eigenvalues become equal to zero. New stationary solution branches with nonvanishing magnetic fields bifurcate. The dynamo operating here will be called Roberts dynamo. Like for the ABC dynamo, a sequence of bifurcations finally leads to chaos. A detailed discussion of these transitions has already been given in Ref. [26]. In the following we focus on the first, symmetry-breaking bifurcation. We investigate the structure of the magnetic field and its influence on the velocity field after the onset of the dynamo. Two real eigenvalues of the Jacobian matrix become equal to zero and the associated two-dimensional linear eigenspace consists of purely magnetic modes where one can be transformed into the other by a shift in the z direction. The magnetic eigenmodes have wave number $k_z=1$, that is, they are z dependent with the maximum wavelength 2π. The bifurcation generates a family of new, dynamo-active steady states. The original continuous (S^1) symmetry is broken and now both the magnetic field and the flow depend on the z coordinate. Any translation in the z direction leads to an equivalent solution of the steady-state family.

The discrete symmetry group $(D_2\times_S Z_4)\times Z_2$ survives the bifurcation and determines the spatial structure of the modified flow and of the generated magnetic field. But the actions of this discrete group must now be given in a modified form compared to the original symmetry transformations described in Sec. II B. Specifically, z translations have to be added to most of the original transformations. This is a remnant of the S^1 symmetry, that is to say, the new actions of the group $(D_2\times_S Z_4)\times Z_2$ correspond to a discrete subgroup symmetry of the original $(D_2\times_S Z_4)\times Z_2\times S^1$ symmetry. They are really nontrivial and their knowledge gives information on the spatial structure of the fields.

The transformation $\boldsymbol{B}\rightarrow-\boldsymbol{B}$, $\boldsymbol{v}\rightarrow\boldsymbol{v}$ (corresponding to the Z_2 symmetry) has to be combined with a translation in the z direction by π. Under this translation the x-y projection of the magnetic field vector rotates by an angle of 180° about the z axis (and describes one complete rotation about the vertical z axis over the full z period 2π). The simulations show a three-dimensional magnetic field with nonvanishing components in all three spatial directions. But the z component is weak in comparison to the horizontal field (the projection on the x-y plane), which will be denoted here as the main field. Thus, the action of the Z_2 symmetry on the main field gives a first impression of its spatial structure: it rotates under translations in the z direction as described above.

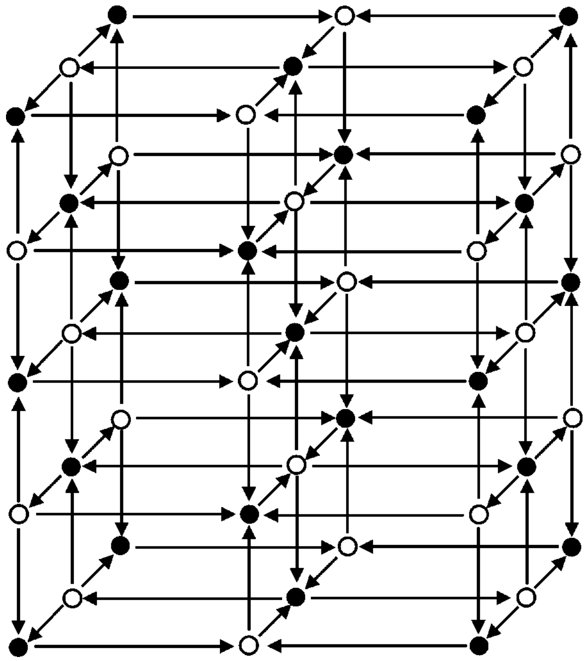

FIG. 3. Stagnation points and their connecting heteroclinic orbits after the bifurcation. Full black dots: α-type stagnation points. Empty circles: β-type stagnation points.

The rotation of the horizontal average of magnetic field, which because of $\nabla \cdot \boldsymbol{B} = 0$ is purely horizontal, is a well-known characteristic of the kinematic Roberts dynamo [22–25]. In the kinematic case just one vertical wave number $\pm k_z$ is excited and the horizontally averaged or mean field rotates without changing its modulus and proportionally to z in a spiral-staircase-like fashion about the z axis. This rotation of the mean field is an essential ingredient in heuristic models of the Roberts dynamo and in the search for kinematically growing modes it was generally assumed at the outset.

Also the generating transformation of the Z_4 symmetry is modified compared to the corresponding original transformation for the undisturbed Roberts flow with vanishing magnetic field. A translation in the z direction by $\pi/2$ has to be added to the original transformation (rotation about the z axis by $\pi/2$ and shift in the x direction by π).

The action of the surviving D_2 symmetry, finally, is more subtle. Before giving this action we characterize the deformed flow after the onset of the dynamo. Of particular interest are the stagnation points of the flow, around which strong magnetic fields are concentrated. Due to the translational invariance in the z direction, the original Roberts flow possesses lines of stagnation points, which are connected by a family of heteroclinic orbits (see Fig. 1). The bifurcation splits these lines up into a discrete set of 16 stagnation points within the periodicity box $\Omega = [0,2\pi]^3$. A skeleton of the stagnation points together with the connecting heteroclinic orbits after the bifurcation is sketched in Fig. 3 (for counting the stagnation points the periodicity has to be taken into account).

The stagnation points can be classified by the eigenvalues of the linearized flow fields in their vicinity, i.e., by the eigenvalues of the matrices $A_{ij} = \partial v_i / \partial x_j |_{\boldsymbol{x}=\boldsymbol{x}_0}$, where $\boldsymbol{x}_0$ is the position of a stagnation point. There are eight stagnation points with two negative eigenvalues and one positive eigenvalue and eight stagnation points with opposite signs of the eigenvalues. Following the terminology introduced by Dombre *et al.* [44], stagnation points with two negative eigenvalues $(-,-,+)$ are denoted as of α type and such with one negative eigenvalue $(-,+,+)$ as of β type.

The symmetry breaking bifurcation produces a family of equivalent solutions which can be obtained from one of them by translations in the z direction. We have selected one solution here by choosing the coordinates $x = y = z = 0$ for one of the α-type stagnation points.

Now the actions of the surviving D_2 symmetry for the selected solution are given: These actions are rotations by π about the x, y, and z axes, combined with translation by π in the z direction for the rotations about the x and z axes (but without such a translation for the rotation about the y axis). Obviously, these transformations form really a D_2 group. The translations in the z direction in the cases of the rotations about the x and z axes are necessary to transform the magnetic field into itself; the rotation of the horizontal field about the z axis is described above. For the selected solution the horizontal magnetic field in the origin is parallel to the y axis. For other solutions of the family, obtained by z translations, the actions of the group are different, leading to conjugate subgroups of type D_2. A complete description of the actions of the symmetry group in terms of coordinates is given explicitly in the Appendix.

III. SPATIAL STRUCTURES OF THE MAGNETIC AND VELOCITY FIELDS

In this section we shall try to give an idea of the structure of the magnetic field after the onset of the dynamo and of the back-reaction of the magnetic field on the flow; some properties of the magnetic field and of the flow were already given in connection with the symmetries in Sec. II D. The flow is modified by the influence of the Lorentz force. It has got 16 saddle-type stagnation points, classified according to the signs of their eigenvalues as of α or β type. The skeleton of the modified flow is sketched in Fig. 3.

There is a close correlation between the location of these stagnation points and the regions of strong magnetic fields. Figure 4 shows a surface-level plot of the modulus of the magnetic field. Bright gray tones indicate regions of strong magnetic fields. Comparing this figure with Fig. 3, one recognizes that regions of strong fields enclose the stagnation points of β type. Similarly the field is weak in the neighborhood of the stagnation points of α type. A correlation between the stagnation points and the regions of strong magnetic fields has already been found for the ABC dynamo [16,20]. However, for the ABC dynamo the strong magnetic fields are concentrated around the α-type stagnation points. What is the reason for this contrasting behavior of the two dynamos?

To elucidate this point, it is helpful to compare the nonlinearly saturated, steady magnetic field with kinematically generated ones, obtained by solving the magnetic induction

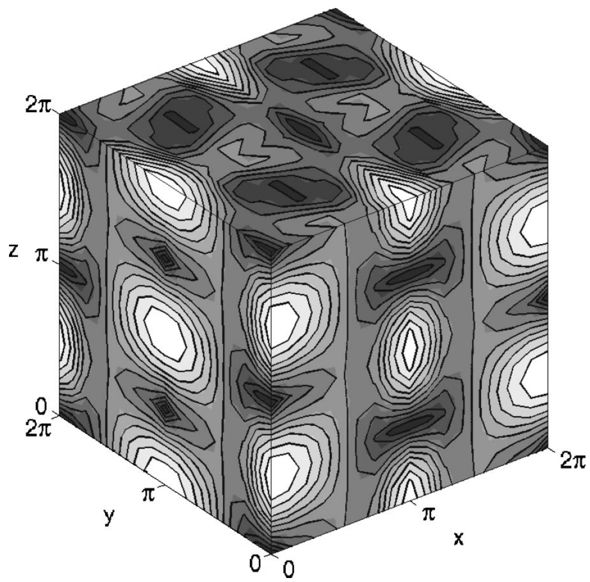

FIG. 4. Surface-level plot of the modulus of the magnetic field $|B|$. Bright gray tones indicate regions of strong magnetic field while dark regions correspond to weak fields.

equation, Eq. (2), with the flow $\boldsymbol{v}$ prescribed. The field generated kinematically by the undisturbed Roberts flow $\boldsymbol{v}_R$ looks already very much like the nonlinearly saturated one, with, in particular, alternating strong-field and weak-field regions as shown in Fig. 4, though the flow is independent of z. We have, then, done additional kinematic simulations by taking the flow that was obtained as final, steady state in the nonlinear simulations of the full MHD equations as prescribed velocity field. Again, the kinematically generated field showed the alternation of strong-field and weak-field regions and, interestingly, the regions of strong fields were located at the α type and the regions of weak fields at the β-type stagnation points of the flow. This is in accordance with the usual picture [19,45] that, through forming and stretching ropelike field structures, dynamo action is promoted at the α points, while at the β points local expansion in the flow leads to more diffuse structures.

Thus, the fact that in the nonlinear Roberts dynamo the field is strong at the β and weak at the α points must result from the way in which the dynamo saturates. The simplest, though not very detailed, explanation is this: In the kinematic phase, the undisturbed Roberts flow generates a field with alternating strong-field and weak-field regions. The back-reaction of the generated magnetic field on the flow by the Lorentz force sets in first where the field is strong. The Lorentz force in the strong-field regions acts such as to impede dynamo action by generating isolated, β-type stagnation points of the flow. The formation of the α-type stagnation points may then be understood as a consequence of this, since matter ejected from the β points along the positive and negative z directions meets in the weak-field regions to form stagnation points with inflow from the positive and negative z directions.

Figure 4 also shows that the magnetic field is very weak inside the driven convectionlike rolls where it reaches its minimal value. The magnetic field is pushed out of the rolls, an effect known as flux expulsion, and is accumulated in layers which separate counterrotating rolls and contain the stagnation points. For the kinematic Roberts dynamo and the limit of large magnetic Reynolds number (small magnetic diffusivity), the generation of magnetic fields confined to these layers, which are very thin for large magnetic Reynolds number, has been studied using boundary layer methods [23–25,45].

Tilgner and Busse [35] studied the modification of the Roberts flow due to the Lorentz force in the frame of a mean-field model. They decomposed the magnetic field into the mean field $\overline{\boldsymbol{B}}=B_0(\cos kz,\sin kz,0)$ and the residual $\boldsymbol{b}=\boldsymbol{B}-\overline{\boldsymbol{B}}$ and used linearized equations for $\boldsymbol{b}$ and the velocity perturbation to $\boldsymbol{v}_R$. The resulting modified velocity field is of the form $\boldsymbol{v}=(1-\gamma)\boldsymbol{v}_R+\boldsymbol{v}_1$, where $\gamma=0$ and $\boldsymbol{v}_1=\boldsymbol{0}$ for $\boldsymbol{B}=\boldsymbol{0}$. $\boldsymbol{v}_1$ varies sinusoidally in all three spatial directions. Its horizontal wave numbers are those of $\boldsymbol{v}_R$, there being a phase shift of $\pi/2$ with respect to $\boldsymbol{v}_R$ in both horizontal directions, and its wave number in the z direction is $2k$. If we translate this to our model, then $\overline{\boldsymbol{B}}$ corresponds to the Fourier component with wave vector $\boldsymbol{k}=(0,0,\mp 1)$ and $\boldsymbol{v}_1$ to that with wave vector $\boldsymbol{k}=(\mp 1,\mp 1,\mp 2)$. For the velocity field in the saturated steady state we find the mode $(\mp 1,\mp 1,\mp 2)$ to be clearly excited (among the modes with the same horizontal wave numbers as $\boldsymbol{v}_R$, i.e, ∓ 1, it is the second largest after that corresponding to $\boldsymbol{v}_R$). Its superposition with $\boldsymbol{v}_R$ already gives the basic structure of the flow with its alternation of type-α and type-β stagnation points. Thus the modified flows in the mean-field model of Tilgner and Busse and in our calculations have the same basic structure.

The back-reaction of the magnetic field by the Lorentz force modifies the Roberts flow in a way fully analogous to what happens in a perturbed integrable Hamiltonian system. According to the Kolmogorov-Arnold-Moser (KAM) theorem [46], more and more KAM tori breakup when the strength of the perturbation is increased. In our case the perturbation is caused by the onset of the dynamo and its strength is controlled by the forcing parameter. As a result of the process of torus destruction, chaotic layers, that is, regions with chaotic streamlines, are generated, sandwiched between surviving KAM tori. This is the general case where a perturbation destroys the integrability of a system. In order to identify chaotic and regular regions, visualization by means of Poincaré sections is an appropriate tool. Starting from a set of initial points, the flow lines are traced and their points of intersection with a fixed plane sampled. This procedure is equivalent to the tracking of passive tracers moving with the flow. Figure 5 shows a Poincaré section of the modified velocity field after the onset of the dynamo. The x-y plane at $z=0$ is the cutting plane and the points shown correspond to trajectories started from points distributed randomly in the periodicity cube. One recognizes the regular structure in the interior of the driven rolls where the velocity reaches its maximum value. The outer parts of the original, unperturbed rolls have been replaced by a chaotic layer. This layer looks rather homogeneous in Fig. 5, but a closer examination reveals that it consists of sublayers which in turn are separated by preserved KAM regions. Detailed descrip-

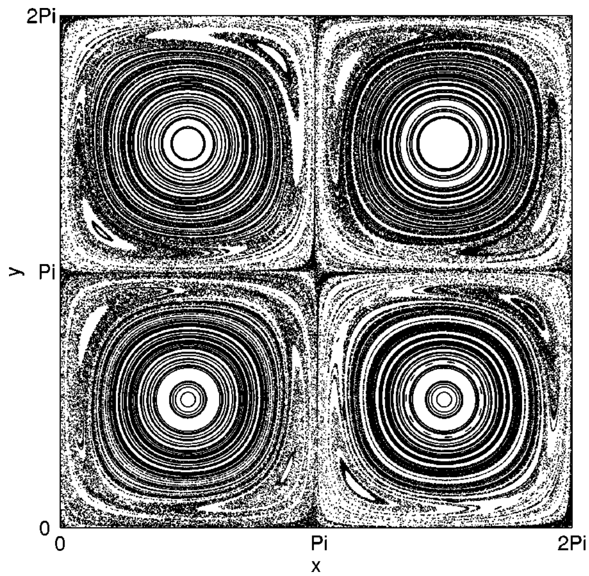

FIG. 5. Poincaré section of the modified flow after the onset of the dynamo with the x-y plane at $z=0$ as cutting plane.

tions of the fractal nature of this kind of pattern may be found in books on Hamiltonian dynamics [47]. As a noticeable effect of the chaoticity of the flow, passive tracers are no longer captured in one roll. Tracers moving in the outermost part of a roll are switching chaotically to neighboring rolls.

Another explanation for the origin of the chaotic part of the flow can be given in terms of the invariant manifolds of the fixed points. The heteroclinic orbits of the Roberts flow, namely, the straight line segments parallel to the x or y axis which connect neighboring stagnation points, survive the bifurcation, as shown in Fig. 3. Also shown in Fig. 3 are new heteroclinic orbits parallel to the z axis, coinciding with segments of the lines originally filled with stagnation points. The heteroclinic orbits form the one-dimensional unstable manifolds of the α points and the one-dimensional stable manifolds of the β points. The symmetry of the flow implies that the heteroclinic orbits are straight line segments. However, the two-dimensional stable manifolds of the α points and the two-dimensional unstable manifolds of the β points, which are spanned by the heteroclinic lines, are not pieces of planes but curved surfaces. We suppose that like for the ABC flow discussed by Dombre *et al.* [44], there exists an infinite number of intersections between stable and unstable manifolds for the modified Roberts flow. We suppose further that the closure of this set generates a horseshoe dynamics [47,48] which in turn is the germ of the chaotic region.

The presence of chaotic regions in the flow, where nearby particle trajectories separate at an exponential rate and the field lines of frozen-in magnetic fields are correspondingly stretched exponentially, is often considered as favorable for dynamo action, which just requires stretching of the magnetic field lines. Unlike, e.g., the ABC flow [44], the undisturbed Roberts flow is regular (in no finite region chaotic) and exponential field line stretching is confined to the stagnation points. Now the Roberts flow is modified and becomes chaotic in finite regions after the onset of the dynamo. It must be noted, however, that the presence of chaotic streamlines by oneself is not sufficient for a dynamo. On the contrary, the chaoticity of a flow in general only enhances magnetic diffusion, which counteracts the dynamo. The chaoticity of the modified Roberts flow is a secondary effect and appears to merely enhance magnetic diffusion, rather than to strengthen the dynamo effect.

IV. CONCLUSION

The dynamo bifurcation of the Roberts flow is decisively determined by the symmetries of the problem, i.e., by the symmetry group of the incompressible MHD equations with Roberts forcing. This group is given by a subgroup of discrete transformations and the continuous translational invariance of the flow. In the bifurcation the continuous symmetry is broken while the discrete subgroup symmetry completely survives the bifurcation. Its actions give information on the spatial structure of the dynamo-generated magnetic field and on the flow, which is modified compared to the original Roberts flow due to the back-reaction of the magnetic field.

In particular, the action of a surviving Z_2 symmetry, stemming from the invariance of the MHD equations with respect to the transformation $\boldsymbol{B}\rightarrow-\boldsymbol{B}$, $\boldsymbol{v}\rightarrow\boldsymbol{v}$, reveals a twisted structure of the magnetic field. The magnetic field has a strong component in the horizontal plane perpendicular to the originally invariant, vertical direction. On translation in the vertical direction, this horizontal field rotates about the vertical axis. The vertical field component is relatively weak compared to the horizontal one. A decomposition of the magnetic field into a strong horizontal and a small vertical component (fluctuating due to turbulence) was also measured in the Karlsruhe dynamo experiment [9]. Despite the idealizations of the model considered here, there is a good agreement between theory and experiment at least with respect to this special property of the dynamo.

Due to flux expulsion, the magnetic field is largely concentrated in layers separating the convectionlike rolls of the flow. These layers contain, in particular, the stagnation points. In contrast to the original Roberts flow, whose stagnation points fill vertical lines, the stagnation points of the modified flow are isolated and either of α type, with a two-dimensional stable and a one-dimensional unstable manifold, or of β type, with a two-dimensional unstable and a one-dimensional stable manifold. It is found that the magnetic field is strongest in regions around the β points and comparably weak at the α points. This contrasts with results for the ABC dynamo, where the magnetic field is strongest at the α points. For the Roberts dynamo considered here, both the creation of isolated stagnation points from the original continuous line of stagnation points and the fact that the magnetic field is strong at the β points and weak at the α points may be understood as a result of the way in which the dynamo saturates.

It is also found that, while the original Roberts flow is regular, the modified flow after the bifurcation is chaotic in the layers between the convectionlike rolls where the magnetic field is concentrated. This chaoticity, which results from the back-reaction of the magnetic field on the flow, appears to merely enhance magnetic diffusion rather than to strengthen the dynamo effect.

APPENDIX

The symmetries of the incompressible MHD equations with Roberts forcing are determined by the symmetries of the Roberts flow, given by Eq. (5).

The invariance of the Roberts flow in the z direction, combined with the z periodicity, gives the continuous circle symmetry S^1. Additionally, the flow is invariant with respect to a discrete group with the formal structure $D_2 \times_S Z_4$. The dihedral group D_2 is given by the three rotations by 180° about the x, y, and z axes. It is a normal subgroup in the semidirect product. The group Z_4 is the cyclic group of rotations which leave the square invariant. In its action on the Roberts flow it is generated by a rotation about the z axis by 90° combined with a shift in the x direction by π. In general, the discrete transformations leaving the Roberts flow invariant consist of a rotation A and a translation T_x:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + T_x. \tag{A1}$$

Equation (A1) describes a mapping of the position vector $\boldsymbol{x} = (x,y,z)$ on the position vector $\boldsymbol{x}' = (x',y',z')$. The tranformed fields $\boldsymbol{v}', \boldsymbol{B}'$ are given by

$$\boldsymbol{v}'(\boldsymbol{x}') = A\boldsymbol{v}(\boldsymbol{x}), \quad \boldsymbol{B}'(\boldsymbol{x}') = A\boldsymbol{B}(\boldsymbol{x}) \tag{A2}$$

and invariance of the fields means

$$\boldsymbol{v}(\boldsymbol{x}) = A\boldsymbol{v}[A^{-1}(\boldsymbol{x} - T_x)],$$
$$\boldsymbol{B}(\boldsymbol{x}) = A\boldsymbol{B}[A^{-1}(\boldsymbol{x} - T_x)]. \tag{A3}$$

The translation along the x axis appears only for the transformations of the group Z_4. The generators of both subgroups, D_2 and Z_4, are given in Table I. The last column of the table has no meaning at this point since z translations have only to be taken into account after the symmetry breaking.

The MHD equations also possess a Z_2 symmetry resulting from their invariance to the transformation $\boldsymbol{B} \to -\boldsymbol{B}$, $\boldsymbol{v} \to \boldsymbol{v}$. Thus, the full equivariance group of the problem is given by

TABLE I. The generating transformations of the symmetry subgroups D_2 and Z_4. For the original Roberts flow the translation T_z has to be left off.

Symmetry subgroup	Generator A	Rotation axis	Angle	T_x	T_z
D_2	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	x	180°		$\begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix}$
	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	z	180°		$\begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix}$
Z_4	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	z	90°	$\begin{pmatrix} \pi \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ \pi/2 \end{pmatrix}$

$$(D_2 \times_S Z_4) \times Z_2 \times S^1. \tag{A4}$$

After the onset of the dynamo the S^1 symmetry is broken and both the velocity field and the generated magnetic field depend on the z coordinate. The symmetry to the discrete subgroup $(D_2 \times_S Z_4) \times Z_2$ survives the bifurcation. However, its actions are more subtle now. Both the transformations of the $D_2 \times_S Z_4$ group and that of the Z_2 group have to be combined with translations along the z axis. That means, a translation T_z along the z axis has to be added to the group action in Eqs. (A1)–(A3). This is in a certain sense a remnant of the broken S^1 symmetry. The generators of the resulting symmetry transformations of the $D_2 \times_S Z_4$ group are given as before in Table I, but now with taking into account the action of z translations T_z.

The Z_2 symmetry ($\boldsymbol{B} \to -\boldsymbol{B}$, $\boldsymbol{v} \to \boldsymbol{v}$) also survives. However, the transformation has to be combined with a translation along the z axis by π. Thus, after the bifurcation the equivariant subgroup Z_2 is generated by the transformation

$$\boldsymbol{B}(x,y,z) \to -\boldsymbol{B}(x,y,z+\pi),$$
$$\boldsymbol{v}(x,y,z) \to \boldsymbol{v}(x,y,z+\pi). \tag{A5}$$

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