Analytical results for a three-phase traffic model

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(Received 30 April 2003; published 14 October 2003)

We study analytically a cellular automaton model, which is able to present three different traffic phases on a homogeneous highway. The characteristics displayed in the fundamental diagram can be well discerned by analyzing the evolution of density configurations. Analytical expressions for the traffic flow and shock speed are obtained. The synchronized flow in the intermediate-density region is the result of aggressive driving scheme and determined mainly by the stochastic noise.

DOI: 10.1103/PhysRevE.68.046112 PACS number(s): 89.40.−a, 45.70.Vn, 05.40.−a

I. INTRODUCTION

Recently traffic dynamics has attracted considerable attention of physicists [1–3]. Many interesting effects on highways have been revealed by the advances of statistical physics. It is now widely accepted that vehicular traffic provides a prototype of the dynamic system driven far from equilibrium. Conventionally, the traffic on a highway is categorized by either free flow or congestion. A recent empirical study shows that a new phase could be identified and named synchronized flow [4,5]. Therefore, there are three different traffic phases: free flow, synchronized flow, and congestion. These three phases display distinct characteristics in the fundamental diagram, i.e., traffic flow versus vehicular density. In the free flow, the flow increases with the increase of density. In contrast, the flow decreases with the increase of density in the congestion, while in the synchronized flow, the high flow can be maintained as the density increases. The above characteristics are revealed by time averaged measurement of the fundamental diagram. When the fluctuations are considered, the synchronized flow displays another characteristic that the flow and density scatter into a two-dimensional area. Microscopically, the synchronized flow is also characterized by vehicles not being able to pass each other. And the motion becomes highly correlated and coherent [6]. As for the phase transitions among these three phases, it is conjectured that there is a sequence of two first order phase transitions: the transition from free flow to synchronized flow occurs first and only later the congestion emerges in the synchronized flow. Although the synchronized flow is mostly observed near on ramps or bottlenecks of a highway, it is still debated if these external disturbances are really indispensable to understand this newly identified traffic phase. It is desirable to be able to understand these three phases solely from the vehicular interactions. In this paper, we will address the stationary patterns of a single-lane traffic displayed in the fundamental diagram. The synchronized flow is understood specifically as the coherent motion of a vehicular queue.

In a recent publication [7], a variation of the well known Nagel-Schreckenberg traffic model has been proposed. A kind of aggressive driving scheme is considered and named takeover effect, which enhances the traffic flow significantly in the intermediate-density region. Three different traffic phases are observed on a homogeneous highway without on ramps or bottlenecks. Numerically, the fundamental diagram displays three distinct phases well separated according to the vehicular density. However, a simple mean-field theory fails to reproduce the characteristics.

In this paper, we present the analytical results of this interesting model. The model will be briefly reviewed in the following section. The takeover effects are discussed in Sec. III. The discussions and conclusion are summarized in the final section.

II. MODEL

The model adopts a cellular automaton approach to describe vehicles moving on a single-lane highway. With v and x denote, respectively, the velocity and position of a vehicle, these parameters are updated with the following four rules.

(*Rule 1*) Noise: If $v>0$, then $v\rightarrow v-1$ with a probability p.

(*Rule 2*) Braking: If $v>d$, then $v\rightarrow d$, where d denotes the headway.

(*Rule 3*) Acceleration: If $v<v_m$, then $v\rightarrow v+1$. (If the above acceleration advances a vehicle to the position of its preceding one, the acceleration will not succeed.)

(*Rule 4*) Motion: $x\rightarrow x+v$.

These update rules are applied to all vehicles in parallel. Basically the first three rules prescribe a vehicle to adjust its velocity, which is then carried out in the fourth rule. There are only two parameters: the stochastic noise p and the speed limit v_m. By changing the order of rule (1) (noise) and rule (3) (acceleration), the model is reduced to the Nagel-Schreckenberg traffic model [8,9]. In this model, the braking is applied before the acceleration. Thus there is no overreaction and the spontaneous jam formation is suppressed. As the stochastic noise is now applied in the first rule, the effects will be shadowed by the subsequent rules. With naive expectation, the randomness will be suppressed. A linear relation between the traffic flow f and vehicular density ρ is then expected. In this work, extensive numerical simulations are carried out on a cyclic road of 5000 sites. The simulations are started with random configurations. The traffic flow is averaged over 10^4 time steps with the first 10^4 discarded; another average over 100 random initial configurations is taken at each density. As shown in Fig. 1, three distinct

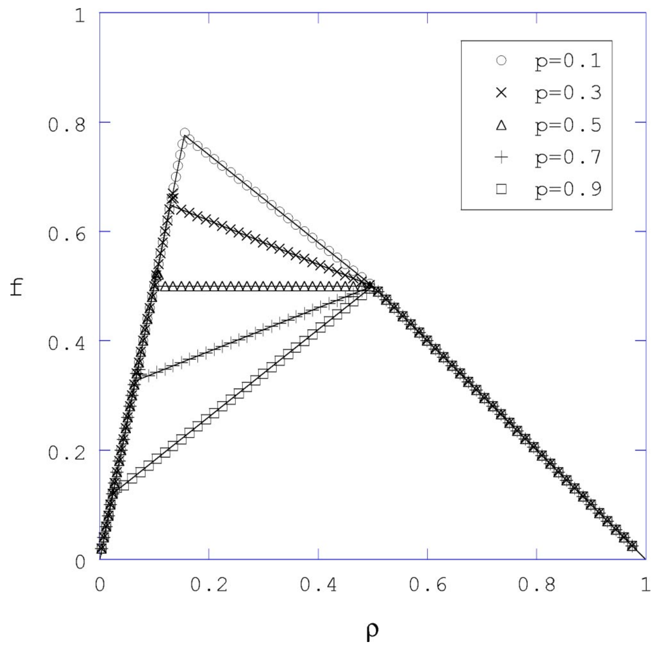

FIG. 1. Fundamental diagram in the cases without takeover. The speed limit $v_m = 5$. The analytic results are shown by the solid lines.

phases are displayed in the fundamental diagram. The resultant traffic flow is linear in vehicular density for all three phases. When the density is low, vehicles will not block each other; the second rule is ineffective. The deceleration in the first rule will be compensated by the acceleration in the third rule. Thus all the vehicles move with the highest speed v_m and the traffic flow becomes $f=\rho v_m$. In contrast, the behavior in the high-density region is dictated by the second rule. The traffic flow is restricted by the available headway and becomes $f=1-\rho$. In the intermediate-density region, the traffic flow is determined by the stochastic noise p. The prescribed rules dictate a stable structure of density profile as the following (vehicles are moving to the right),

$$\cdots 1\times 1\times 1\times 1\times 1\times\times\times\times\times\cdots, \tag{1}$$

where the position of a vehicle is marked by an integer showing its velocity and each empty cell is marked by a symbol $(\times)$. The vehicles are moving with the same velocity and keeping the same headway. The stochastic noise p will only affect the first vehicle within such a queue. All the following vehicles will maintain at the velocity 1 and the headway 1. With a probability p, the first vehicle will keep the velocity in the next time step, and the queue will propagate forward with a speed $+1$ as the following,

$$\cdots\times 1\times 1\times 1\times 1\times 1\times\times\times\times\cdots. \tag{2}$$

However, the first vehicle may also accelerate with a probability $(1-p)$ and moves away from the queue as the following,

$$\cdots\times 1\times 1\times 1\times 1\times\times 2\times\times\times\cdots. \tag{3}$$

The queue is then broken and the front propagates backward with a speed -1. Thus the shock front will propagate with an average speed $(2p-1)$. When p assumes a small value $(p\sim 0)$, the shock front propagates backward and the traffic flow can be maintained at a large value; when p assumes a large value $(p\sim 1)$, the shock front propagates forward and the traffic flow is suppressed. In contrast, the congestion in the high-density region provides a much more compact profile and can be represented as the following:

$$\cdots 0\ 0\ 0\ 0\ 0\ 0\ 0\times\times\times\times\times\times\times\cdots. \tag{4}$$

In the next time step, the first vehicle will move away deterministically as the following:

$$\cdots 000000\times 1\times\times\times\times\times\times\cdots. \tag{5}$$

Thus the shock front in the high-density region propagates backward with a speed -1.

In summary, the analytical expressions for the traffic flow can be obtained

$$f=\begin{cases} \rho v_m, & \rho<\frac{1}{2}\left(\frac{1-v_s}{v_m-v_s}\right)\\ \frac{1}{2}-v_s\left(\frac{1}{2}-\rho\right), & \frac{1}{2}\left(\frac{1-v_s}{v_m-v_s}\right)<\rho<\frac{1}{2}\\ 1-\rho, & \rho>\frac{1}{2}, \end{cases} \tag{6}$$

where $v_s=(2p-1)$ is the shock speed. The numerical results can be correctly reproduced. It is interesting to note that the low-density and the intermediate-density regions are characterized by speed limit v_m and stochastic noise p, respectively; while in the high-density region, both v_m and p becomes irrelevant.

III. TAKEOVER

Next, we consider the effects of takeover, which was proposed to model a kind of aggressive driving behavior. Basically it is an anticipation effect to take into account the motion of the preceding vehicle [10–13]. We note that the takeover does not imply passing. The ordering of vehicles is kept strictly in this model. Rule (3) is modified as the following.

Acceleration: If $v<v_m$, then $v\to v+1$. (If the above acceleration advances a vehicle to the position of its preceding one, the acceleration will succeed only when the preceding vehicle has a positive velocity and moves away from that position in the same time step, i.e., the position is taken over.)

The results are shown in Fig. 2. Basically, there are still three phases displayed in the fundamental diagram. For both the low-density and the high-density regions, the traffic flow remains the same as in the cases without takeover; for the intermediate-density region, the traffic flow enhances significantly yet still preserves the linear dependence to the vehicular density. The increase of the traffic flow can also be observed by the decrease of the shock front speed v_s as shown in Fig. 3. When p is small, the decrease of traffic flow attributed to the random noise can be fully restored by the take-

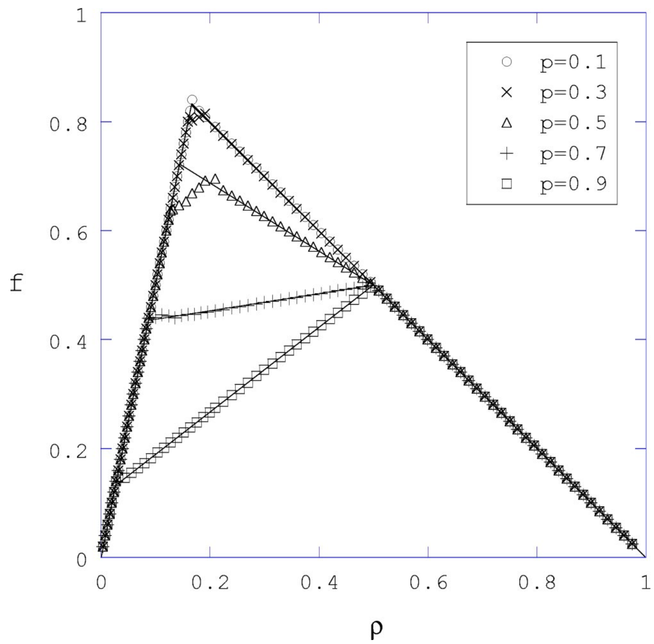

FIG. 2. Fundamental diagram in the cases with takeover. The speed limit $v_m = 5$. The analytic results are shown by the solid lines.

over. The traffic flow can be maintained at the same value as in the cases without noise. The density profile is characterized by the compact jam shown in Eq. (4), which leads to the shock speed -1. On the other hand, when p is large, the density profile is then characterized by the synchronized motion shown in Eq. (1). With takeover, however, any vehicle within the queue may accelerate to speed 2 in the next time step. Thus the breakaway of the queue is no longer limited to the first vehicle. For example, if the first two vehicles accelerate and the third one remains at speed 1, the configuration becomes

$$\cdots \times 1 \times 1 \times 1 \times \times 2 \times 2 \times \times \times \cdots, \tag{7}$$

which can be associated with a probability $p\,(1-p)^2$. As the first two vehicles move away from the queue, a shock speed -3 is assigned. In general, the average shock speed can be expressed as

$$v_s = \sum_{n=0}^{\infty} (2n-1) p (1-p)^n = 3 - \frac{2}{p}, \tag{8}$$

where the index n denotes the number of vehicles moving away from the queue in the next time step, which can be associated with a shock speed $(2n-1)$ and a probability $p(1-p)^n$. The analytical expressions in these two limits, $p \to 0$ and $p \to 1$, already give a satisfiable description to the highly nonlinear relation between v_s and p, see Fig. 3. In contrast, the cases without takeover can be fully described by a simple linear relation $v_s = (2p-1)$.

The above asymptotic results can be rephrased as $(v_s + 1) \to 0$ in the limit $p \to 0$ and $(v_s - 3 + 2/p) \to 0$ in the limit $p \to 1$. For a general p, the following relation can be expected phenomenologically:

$$(v_s + 1)\left(v_s - 3 + \frac{2}{p}\right) = A, \tag{9}$$

where A is a small parameter to smear the transformation from $p=0$ to $p=1$. Naively, the asymptotic results can be reproduced by taking $A=0$. With a simple Gaussian fluctuations around $p=0.5$ as the following:

$$A = 0.15\, e^{-50\ (p-0.5)^2}, \tag{10}$$

the shock speed v_s can be fairly described from $p=0$ to $p=1$; see Fig. 3. With this parametrization of v_s, the traffic flow can be expressed by the same analytical formula as in Eq. (6). The numerical results can be fairly reproduced; see Fig. 2.

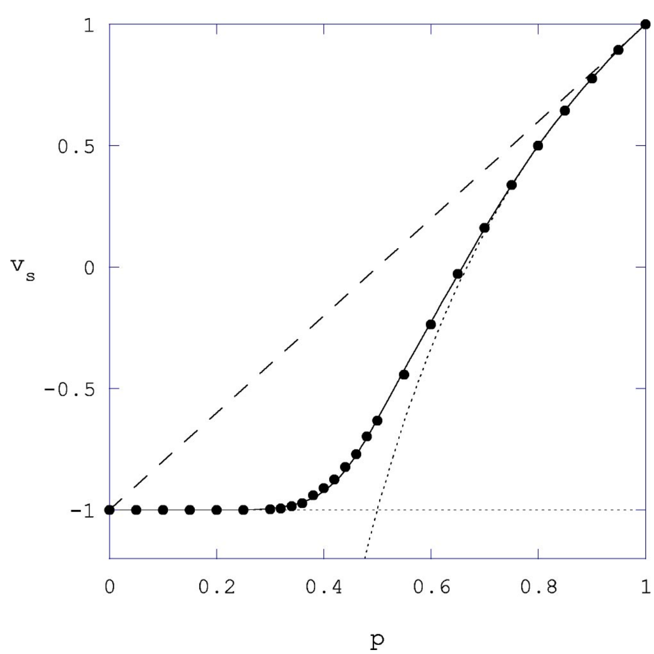

FIG. 3. The shock speed v_s as a function of the stochastic noise p in the intermediate-density region. The solid line shows the analytical results in Eqs. (9) and (10). The asymptotic results at $p \to 0$ and $p \to 1$ are shown by the dotted lines. The dash line shows the linear relation in the cases without takeover.

The discrepancy in the transition from low-density to intermediate-density regions should be attributed to yet another synchronized traffic phase, which is exclusively accounted by takeover. With takeover, the following configuration becomes possible,

$$\cdots 2 \times 2 \times 2 \times 2 \times 2 \times \times \times \times \times \cdots, \tag{11}$$

where each vehicle moves synchronously into the position occupied by its preceding one in the previous time step. With a probability p, the first vehicle will keep the velocity in the next time step, and the queue will propagate forward with a speed $+2$ as the following:

$$\cdots 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times \times \times \cdots. \tag{12}$$

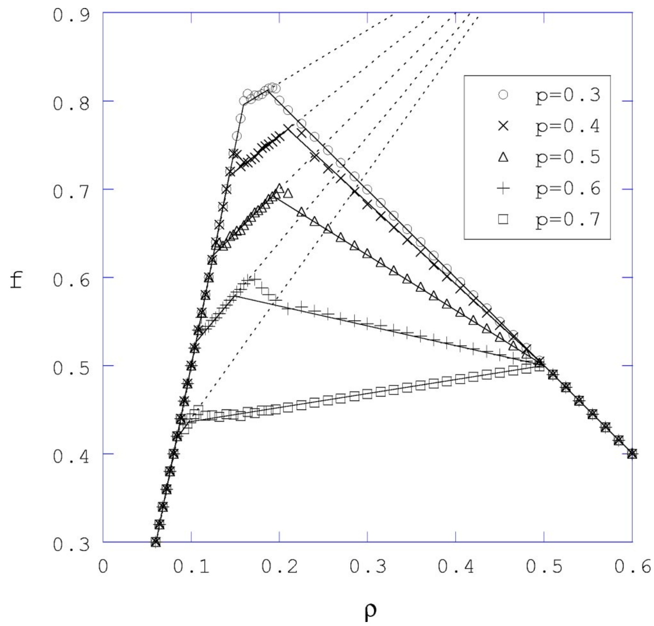

FIG. 4. The enlarged view of a portion of Fig. 2 to emphasize the transition from low-density to intermediate-density regions. More data are shown. Compared to Fig. 2, the solid lines show the analytical results including the synchronous motion of Eq. (11). The dotted lines show the extrapolation of the synchronized phase in Eq. (11).

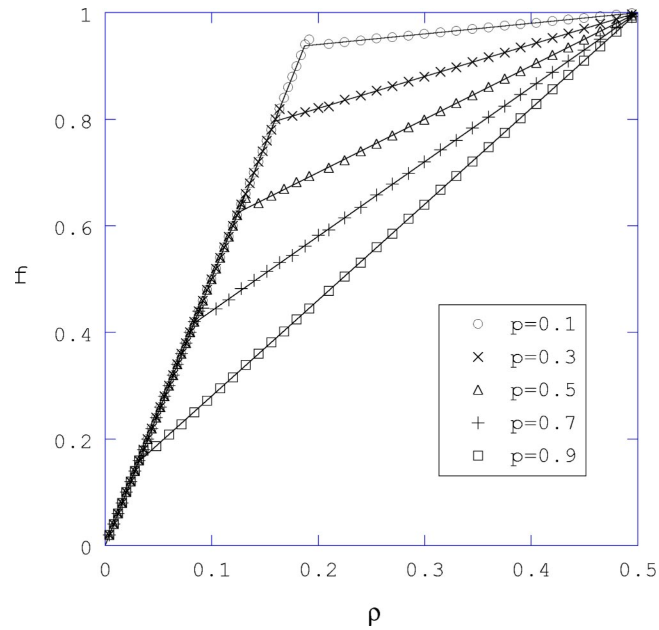

FIG. 5. Fundamental diagram in the cases with takeover. Except for a larger headway prescribed in the initial configurations, the simulations are the same as in Fig. 2. The synchronized phase shown in Fig. 4 can be sustained to higher densities. The analytic results are shown by the solid lines.

With a probability $(1-p)$, the first vehicle may accelerate and move away from the queue as the following:

$$\cdots 2\times 2\times 2\times 2\times 2\times\times 3\times\times\cdots. \tag{13}$$

The queue can be taken as stationary and assigned a front speed 0. Thus the shock front will propagate with an average speed $v_s=2p$. The flow becomes $f=1-p+2p\rho$. The numerical data can be correctly described, see Fig. 4. It is interesting to note that such a synchronized traffic phase cannot be supported in the extreme limits of p. In the case of $v_m=5$, a restriction $0.251<p<0.744$ can be easily calculated by the parametrization in Eqs. (9) and (10). This traffic phase can be observed only when $v_m\geqslant 4$. With the analytical expressions in the asymptotic limits, the following restriction can be obtained:

$$\frac{1}{v_m-1}<p<\frac{v_m-2}{v_m-1}. \tag{14}$$

Thus, this synchronized phase is most prominent around the setting $p\sim 0.5$; see Fig. 4.

IV. DISCUSSIONS

In this paper, we present the analytical results for a three-phase traffic model. This simple model is very transparent, yet it can reproduce three different traffic phases on a homogeneous highway. In the low-density region, the free flow is determined by the speed limit v_m. In the intermediate-density region, the synchronized flow is controlled by the stochastic noise p. In the high-density region, the congestion is subjected to the available headway, i.e., the flow is determined by the density, and independent of both v_m and p. The characteristics displayed in the fundamental diagram can be well discerned by analyzing the evolution of density configurations. The numerical simulations can be well reproduced by the analytical formulas.

In the usual meaning, the synchronized flow implies both the synchronization among lanes and the high flow in the atypical congested traffic on a multilane highway. In the above study of the single-lane traffic, we are looking for the synchronized movement of a vehicular queue. The seemingly different characteristics reflect basically the same traffic dynamics. On a multilane highway, the synchronization among neighboring lanes implies the passing maneuvers are difficult to carry out. Also as the high flow is maintained, the lane changing behavior is expected to be suppressed. And each vehicle follows its preceding one much more strictly. Thus the synchronization among different lanes leads to the synchronization within the same lane. The coherent behavior within the vehicular queue is then enhanced. The vehicular movement becomes much more efficient which results in the high flow. In the simple model we studied, such coherent behavior is further enhanced and becomes dominant over a certain density range.

The effects of takeover are most significant in the intermediate-density region. Such aggressive driving scheme is unnecessary in the free flow of low-density region. On the other hand, it becomes useless in the congestion of high-density region. We note that although the traffic flow in the high-density region is unaffected by the takeover, the aggressive driving does affect the density profile. With takeover, the range of compact jams is enlarged. The fluctuations in the vehicular distribution are enhanced. On part of the highway vehicles may move with a higher speed; while on the other part of the highway vehicles would be trapped within the traffic jams for a longer time. Therefore, the overall traffic flow is the same as in the cases where such aggressive driving is absent.

In the intermediate-density region, the takeover provides a mechanism of synchronous motion to enhance the traffic flow. It is interesting to note that the same mechanism incapable of boosting traffic flow in the high-density region becomes quite effective to enhance traffic flow in the intermediate-density region. As the takeover also enhances the density fluctuations, the increase of traffic flow does not imply that vehicles will move more smoothly. When the noise is small ($p \sim 0$), the aggressive driving restores the traffic flow to the cases without noise. However, the density profile is very different from the cases without takeover. Without the aggressive driving, vehicles distribute uniformly in this density region. With takeover, the compact jams begin to emerge even when the density is not as large as in the congestion. When the noise is large ($p \sim 1$), the effects of takeover are diminished. Both the traffic flow and the density profile conform to the cases without takeover.

The enhancement of traffic flow in the intermediate-density region can be attributed to the traffic queue broken by the synchronous motion of vehicles. In the transition from low-density to intermediate-density regions, another synchronous motion of vehicles is observed. As the takeover is basically a kind of velocity-dependent randomness [14], the results are expected to have a strong dependence on the initial configurations in the simulations. Instead of a random configuration, if the simulations start with a configuration where moving vehicles are well separated, such synchronous motion can be sustained to a higher density, see Fig. 5. Obviously, when $\rho > 0.5$, such synchronized phase ceases to exist.

In this simple model, the flow is linear to density in various traffic phases. The fundamental diagram consists of three segments, which represent different phases. However, such lines do not imply a unique relation between flow and density. In practice, the lines represent the data after averaging over space and time. When the local flow and the local density are considered, the data become scattered over the fundamental diagram; see Fig. 6. In the low-density free flow, the data collapse into the averaged line. In the high-density congestion, the data scatter into a narrow band. In the intermediate-density synchronized flow, much larger fluctuations are observed. The data scatter over a two-dimensional area, which has also been noted empirically as the characteristic feature of synchronized flow. We note that the fluctuations are different from the erratic behavior observed empirically on a real highway. Naively, this wide scattering of data invites an interpretation of phase separation and the gas-liquid analogy of traffic dynamics [15,16]. In Fig. 6, the data do not form a continuous band across the intermediate-density region, where the scattering is much more wild. With a closer look, as shown by a typical event at $\rho = 0.3$, the data accumulate into two separate regions. The first one forms a narrow band connecting the low-density free flow to the point $(\rho, f) = (0.5, 1)$, which can be easily identified with the synchronized traffic phase shown in Fig. 5; the second one is a fixed point at $(\rho, f) = (0.5, 0.5)$, which can then be taken as the breakdown of the synchronized phase and also the emergence of high-density congestion. Within the intermediate-density region, the system configurations self-organize into these two different phases. The phase separation can be discerned and a first-order phase transition is implied.

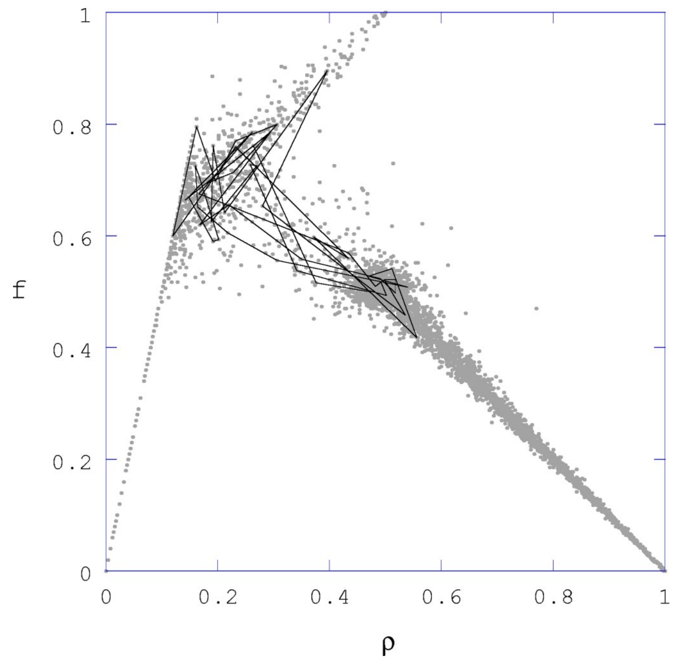

FIG. 6. Fundamental diagram in the cases with takeover. The stochastic noise $p = 0.5$ and speed limit $v_m = 5$. Data show the local flow and local density. A typical event at $\rho = 0.3$ is marked with solid lines to show the wide fluctuations.

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