Quantum iterated function systems

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An iterated function system (IFS) is defined by specifying a set of functions in a classical phase space, which act randomly on an initial point. In an analogous way, we define a quantum IFS (QIFS), where functions act randomly with prescribed probabilities in the Hilbert space. In a more general setting, a QIFS consists of completely positive maps acting in the space of density operators. This formalism is designed to describe certain problems of nonunitary quantum dynamics. We present exemplary classical IFSs, the invariant measure of which exhibits fractal structure, and study properties of the corresponding QIFSs and their invariant states.

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I. INTRODUCTION

An *iterated function system* (IFS) may be considered as a generalization of a classical dynamical system, which permits a certain degree of stochasticity. It is defined by a set of k functions $f_i:\Omega\to\Omega$, $i=1,\dots,k$, which represent discrete dynamical systems in the classical phase space Ω. The functions f_i act randomly with given place-dependent probabilities $p_i:\Omega\to[0,1]$, $i=1,\dots,k$, $\Sigma_{i=1}^k p_i=1$ [1]. They characterize the likelihood of choosing a particular map at each step of the time evolution of the system.

There exist different ways of investigating such random systems. Having defined an IFS, one may ask, how is an initial point $x_0\in\Omega$ transformed by the random system. In a more general approach, one may pose a question that how does a probability measure μ on Ω change under the action of the Markov operator P associated with the IFS. If the phase space Ω is compact, the functions f_i are strongly contracting, and the probabilities p_i are Hölder continuous and positive (i.e., $p_i>0$), then there exists a unique invariant measure μ_* of P—see, for instance, [1–3], and references therein.

For a large class of IFSs, the invariant measure μ_* has a fractal structure. Such IFSs may be used to generate fractal sets in space Ω. In particular, iterated function systems leading to well-known fractal sets, such as the Cantor set or the Sierpiński gasket, can be found in Ref. [1]. These intriguing properties of IFSs allowed one to apply them for image compression, processing, and encoding [1,4,5].

Iterated function systems can also be used to describe several physical problems, where deterministic dynamics is combined with the random choice of interaction. In particular, IFSs belong to a larger class of random systems studied in Refs. [6,7]. Such a composition of deterministic and stochastic behavior is important in numerous fields of science, since very often an investigated dynamical system is subjected to an external noise.

Nondeterministic dynamics may also be relevant from the point of view of quantum mechanics. Although unitary time evolution of a closed quantum system is purely deterministic, the problem changes if one tries to take into account processes of quantum measurement or a possible coupling with a classical system. In the approach of event enhanced quantum theory (EEQT) developed by Blanchard and Jadczyk [8], the quantum time evolution is piecewise deterministic and in certain cases may be put into the framework of iterated function systems [9,10]. While some recent investigations in this area concentrate mostly on IFSs acting in the space of pure states [11], we advocate a more general setup, in which IFSs act in the space of mixed quantum states.

The main objective of this paper is to propose a general definition of quantum IFS (QIFS). Formally, it suffices to consider the standard definition of IFS and to take for Ω an N-dimensional Hilbert space $\mathcal{H}_N$. Instead of functions f_i, $i=1,\dots,k$, representing classical maps, one should use linear functions $V_i:\mathcal{H}_N\to\mathcal{H}_N$, which represent the corresponding quantum maps. Alternatively, one may consider the space $\mathcal{M}_N$ of density matrices of size N and construct an iterated function system out of k positive maps $G_i:\mathcal{M}_N\to\mathcal{M}_N$. The QIFSs defined in this way can be used to describe processes of quantum measurements, decoherence, dissipation, or coupling with an external environment. Moreover, QIFSs offer an interesting field of research on the semiclassical limit of quantum random systems. In particular, it is interesting to explore quantum analogs of classical IFSs, which lead to fractal invariant measures, and to investigate that how do quantum effects smear fractal structures out.

This paper is organized as follows. In the following section we recall the definition and basic properties of the clas-

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sical IFSs, and discuss several examples. In Sec. III we propose the definition of QIFSs, investigate their properties, and relate them to the notion of quantum channels and complete positive maps used in the theory of quantum dynamical semigroups. The quantum-classical correspondence is the subject of Sec. IV, in which we compare dynamics of exemplary IFSs and the related QIFSs. Concluding remarks are presented in Sec. V.

II. CLASSICAL ITERATED FUNCTION SYSTEMS

Consider a compact metric space Ω and k functions $f_i:\Omega\to\Omega$, where $i=1,\dots,k$. Let us specify k probability functions $p_i:\Omega\to[0,1]$ such that for each point $x\in\Omega$ the condition $\Sigma_{i=1}^{k}p_i(x)=1$ is fulfilled. Then the functions f_i may be regarded as classical maps, which act randomly with probabilities p_i. The set $\mathcal{F}_{\mathrm{Cl}}:=\{\Omega,f_i,p_i:i=1,\dots,k\}$ is called an *iterated function system*.

Let $\mathcal{M}(\Omega)$ denote the space of all probability measures on Ω. The IFS $\mathcal{F}_{\mathrm{Cl}}$ generates the following *Markov operator* P acting on $\mathcal{M}(\Omega)$:

$$(P\mu)(B)=\sum_{i=1}^{k}\int_{f_i^{-1}(B)}p_i(x)d\mu(x), \tag{1}$$

where B is a measurable subset of Ω and μ belongs to $\mathcal{M}(\Omega)$. This operator represents the corresponding Markov stochastic process defined on the code space consisting of infinite sequences built out of k letters which label maps f_i. On the other hand, P describes the *evolution of probability measures* under the action of $\mathcal{F}_{\mathrm{Cl}}$.

Consider an IFS defined on an interval in $\mathbb{R}$ and consisting of invertible C^1 maps $\{f_i:i=1,\dots,k\}$. This IFS generates the associated Markov operator P on the space of densities [12], which describes one-step evolution of a classical density γ

$$P[\gamma](x)=\sum_i p_i(f_i^{-1}(x))\gamma(f_i^{-1}(x))\left|\frac{df_i^{-1}(x)}{dx}\right|, \tag{2}$$

where for $x\in\Omega$ the sum goes over $i=1,\dots,k$, such that $x\in f_i(\Omega)$.

Let $d(x,y)$ denote the distance between two points x and y in the metric space Ω. An IFS $\mathcal{F}_{\mathrm{Cl}}$ is called *hyperbolic*, if it fulfills the following conditions for all $i=1,\dots,k$.

(i) f_i are Lipschitz functions with the Lipschitz constants $L_i<1$, i.e., they satisfy the contraction condition $d(f_i(x),f_i(y))\le L_i d(x,y)$ for all $x,y\in\Omega$.

(ii) The probabilities p_i are Hölder continuous, i.e., they fulfill the condition $|p_i(x)-p_i(y)|\le K_i d(x,y)^{\alpha}$ for some $\alpha\in(0,1]$, $K_i\in\mathbb{R}^+$ for all $x,y\in\Omega$.

(iii) All probabilities are positive, i.e., $p_i(x)>0$ for any $x\in\Omega$.

The Markov operator P associated with a hyperbolic IFS has a unique *invariant probability measure* μ_* satisfying the equation $P\mu_*=\mu_*$. This measure is *attractive*, i.e., $P^n\mu$ converges weakly to μ_* for every $\mu\in\mathcal{M}(\Omega)$ as $n\to\infty$. In other words, $\int_\Omega u\, dP^n\mu$ tends to $\int_\Omega u\, d\mu_*$ for every continuous function $u:\Omega\to\mathbb{R}$. Let us mention that the hyperbolicity conditions (i)–(iii) are not necessary to assure the existence of a unique invariant probability measure—some other, less restrictive, sufficient assumptions were analyzed in Refs. [2,3,13–17].

Observe that in the above case, in order to obtain the exact value of an integral $\int_\Omega u\, d\mu_*$, it is sufficient to find the limit of the sequence $\int_\Omega u\, d(P^n\mu)$ for an arbitrary initial measure μ. This method of computing integrals over the invariant measure μ_* is purely *deterministic* [1]. Sometimes it is possible to perform the integration over the invariant measure analytically, even though μ_* displays fractal properties [18]. Alternatively, a *random iterated algorithm* may be employed by generating a random sequence $x_j\in\Omega$, $j=0,1,\dots,$ by the IFS, which originates from an arbitrary initial point x_0. Due to the ergodic theorem for IFSs [2,19,20], the mean value $(1/n)\Sigma_{j=0}^{n-1}u(x_j)$ converges with probability one in the limit $n\to\infty$ to the desired integral $\int_\Omega u\, d\mu_*$ for a large class of u.

If probabilities p_i are constant we say that an IFS is *of the first kind*. Such IFSs are often studied in the mathematical literature (see Ref. [1], and references therein). Moreover they also have some applications in physics. For example, they were used to construct multifractal energy spectra of certain quantum systems [21], and to investigate second-order phase transitions [22]. On the other hand, IFSs with place-dependent probabilities can be associated with some classical and quantum dynamical systems [3,20,23–27]. In analogy with the position-dependent gauge transformations, such IFSs may be called *iterated function systems of the second kind* [18].

If Ω is a compact subset of $\mathbb{R}^n$ and d_E represents the Euclidean distance, or Ω is a compact manifold (e.g., sphere S^2 or torus T^n) equipped with the natural (Riemannian) distance d_R, then an IFS will be called *classical*. For concreteness we provide below some examples of classical IFSs. The first example demonstrates that even simple linear maps f_i may lead to a nontrivial structure of the invariant measure.

Example 1. $\Omega=[0,1]$, $k=2$, $p_1=p_2=1/2$, and two affine transformations are given by $f_1(x)=x/3$ and $f_2(x)=x/3+2/3$ for $x\in\Omega$. Since both functions are contractions with Lipschitz constants $L_1=L_2=1/3<1$, this IFS is hyperbolic. Thus, there exists a unique attractive invariant measure μ_*. It is easy to show [1] that μ_* is concentrated uniformly on the Cantor set of the fractal dimension $d=\ln 2/\ln 3$.

The following example presents an IFS of the second kind.

Example 2. As before, $\Omega=[0,1]$, $k=2$, $f_1(x)=x/3$, and $f_2(x)=x/3+2/3$ for $x\in\Omega$. The probabilities are now place dependent, $p_1(x)=x$ and $p_2(x)=1-x$. Although this IFS is not hyperbolic [condition (iii) is not fulfilled], a unique invariant measure μ_* still exists. It is also concentrated on the Cantor set, but now in a nonuniform way [18]. The measure μ_* displays in this case multifractal properties, since its generalized dimension depends on the Rényi parameter.

Example 3. $\Omega=[0,1]\times[0,1]\subset\mathbb{R}^2$, $k=4$, $p_1=p_2=p_3=p_4=1/4$. Four affine transformations are given by

$$f_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/3&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix},\quad f_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/3&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}2/3\\0\end{pmatrix},$$

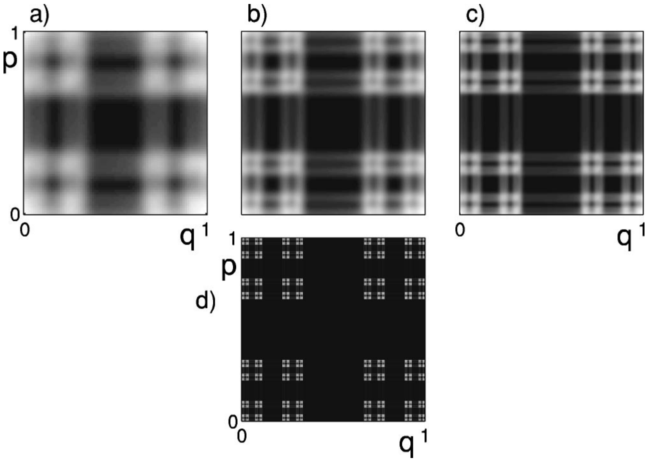

FIG. 1. "Tartanlike" invariant density of the QIFS defined in Example 14 for (a) $N=3^4$-, (b) $N=3^5$-, and (c) $N=3^6$-dimensional Hilbert space, shown in the generalized Husimi representation. Invariant measure of the corresponding classical IFS on the torus [Eq. (3)] occupies a fractal set (d).

$$f_3\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1&0\\0&1/3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix},\quad f_4\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1&0\\0&1/3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\2/3\end{pmatrix}. \tag{3}$$

Also, this IFS is not hyperbolic, since the transformations f_i are not globally contracting; the former two contract along x axis, while the latter two contract along the y axis only. An invariant measure μ_* for this IFS is presented in Fig. 1(d). The support of μ_* is the Cartesian product of two Cantor sets. Thus, its fractal dimension is $d=2\ln 2/\ln 3$.

Example 4. Let $\Omega=S^2$. Take $k=2$, $p_1=p_2=1/2$, and choose f_1 to be the rotation along z axis by angle χ_1 [later referred to as $R_z(\chi_1)$]. In the standard spherical coordinates, $f_1(\theta,\phi)=(\theta,\phi+\chi_1)$. The second function f_2 is a rotation by angle χ_2 along an axis inclined by angle β with respect to z axis. Since both classical maps are isometries, this IFS is by no means hyperbolic. The properties of the Markov operator depend on the angle β and the commensurability of the angles χ_i. However, the Lebesgue measure on the sphere is always an invariant measure for this IFS.

Example 5. $\Omega=[0,1]$, $k=2$, $p_1=p_2=1/2$, $f_1(x)=2x$ for $x<1/2$, and $f_1(x)=2(1-x)$ for $x\geq 1/2$ (*tent map*); $f_2(x)=2x$ for $x<1/2$ and $f_2(x)=2x-1$ for $x\geq 1/2$ (*Bernoulli map*). Both classical maps are expanding (and chaotic), thus the IFS is not hyperbolic. The Lebesgue measure on $[0,1]$ is an invariant measure for this IFS.

III. QUANTUM ITERATED FUNCTION SYSTEMS

A. Pure states QIFSs

To describe a quantum dynamical system we consider a complex Hilbert space $\mathcal{H}$. When the corresponding classical phase space Ω is compact, the Hilbert space $\mathcal{H}_N$ is finite dimensional and its dimension N is inversely proportional to the Planck constant $\hbar$ measured in the units of the volume of Ω. Analyzing quantum systems, N is usually treated as a free parameter, and the semiclassical limit is studied by letting $N\rightarrow\infty$.

A quantum state can be described by an element $|\psi\rangle$ of $\mathcal{H}_N$ normalized according to $\langle\psi|\psi\rangle=1$. Since for any phase α the element $|\psi'\rangle=e^{i\alpha}|\psi\rangle$ describes the same physical state as $|\psi\rangle$, we identify them, and so the space of all pure states $\mathcal{P}_N$ has $2N-2$ real dimensions. From the topological point of view, it can be represented as the complex projective space $\mathbb{C}P^{N-1}$ equipped with the Fubini-Study (FS) metric given by

$$D_{FS}(|\phi\rangle,|\psi\rangle)=\arccos|\langle\phi|\psi\rangle|. \tag{4}$$

It varies from zero for $|\phi\rangle=|\psi\rangle$ to $\pi/2$ for any two orthogonal states. In the simplest case of a two-dimensional Hilbert space $\mathcal{H}_2$, the space of pure states $\mathcal{P}_2$ reduces to the Bloch sphere, $\mathbb{C}P^1\simeq S^2$, and the FS distance between two quantum states equals the natural (Riemannian) distance between the corresponding points on the sphere of radius 1/2.

Definition 1. To define a *(pure states)* QIFS it is sufficient to use the general definition of IFS given in Sec. II, taking for Ω the space $\mathcal{P}_N$. We specify the following two sets of k linear invertible operators.

(1) $V_i:\mathcal{H}_N\rightarrow\mathcal{H}_N$ $(i=1,\ldots,k)$, which generate maps $F_i:\mathcal{P}_N\rightarrow\mathcal{P}_N$ $(i=1,\ldots,k)$ by

$$F_i(|\phi\rangle):=\frac{V_i(|\phi\rangle)}{\|V_i(|\phi\rangle)\|}. \tag{5}$$

(2) $W_i:\mathcal{H}_N\rightarrow\mathcal{H}_N$ $(i=1,\ldots,k)$, forming an operational resolution of identity, $\sum_{i=1}^k W_i^\dagger W_i=\mathbb{I}$, which generate probabilities $p_i:\mathcal{P}_N\rightarrow[0,1]$ $(i=1,\ldots,k)$ by

$$p_i(|\phi\rangle):=\|W_i(|\phi\rangle)\|^2 \tag{6}$$

for any $|\phi\rangle\in\mathcal{P}_N$.

Clearly, for every $|\phi\rangle\in\mathcal{P}_N$ the normalization condition $\sum_{i=1}^k p_i(|\phi\rangle)=1$ is fulfilled. In this situation a QIFS may be defined as the set

$$\mathcal{F}_N=\{\mathcal{P}_N;\ F_i:\mathcal{P}_N\rightarrow\mathcal{P}_N;\ p_i:\mathcal{P}_N\rightarrow[0,1]:i=1,\ldots,k\}. \tag{7}$$

Such a QIFS may be realized by choosing an initial state $|\phi_0\rangle\in\mathcal{P}_N$ and generating randomly a sequence of pure states $(|\phi_j\rangle)_{j\in\mathbb{N}}$. The state $|\phi_0\rangle$ is transformed into $|\phi_1\rangle=F_i(|\phi_0\rangle)$ with probability $p_i(|\phi_0\rangle)$, later $|\phi_1\rangle$ is mapped into $|\phi_2\rangle=F_j(|\phi_1\rangle)$ with probability $p_j(|\phi_1\rangle)$, and so on. If we choose $W_i=\sqrt{p_i}\,\mathbb{I}$, then the probabilities are constant: $p_i(|\phi\rangle)=p_i$ for $i=1,\ldots,k$. An arbitrary QIFS $\mathcal{F}_N$ determines by formula (1) the operator P acting on probability measures on $\mathcal{P}_N$.

Such defined QIFS $\mathcal{F}_N$ cannot be hyperbolic, since the quantum maps F_i are not contractions with respect to the Fubini-Study distance in $\mathcal{P}_N$.

Example 6. $\Omega=\mathcal{P}_N\simeq\mathbb{C}P^{N-1}$, $k=2$, $p_1=p_2=1/2$, $F_1(|\psi\rangle)=U_1(|\psi\rangle)$, and $F_2(|\psi\rangle)=U_2(|\psi\rangle)$, where the operators U_i $(i=1,2)$ are unitary. In this case, both quantum maps are isometries. Thus, the natural Riemannian (Fubini-Study) measure in $\mathcal{P}_N$ is invariant, but as we shall see in the following section, its uniqueness depends on the choice of U_1 and U_2.

B. Mixed states QIFSs

Mixed states are described by N-dimensional density operators ρ, i.e., positive Hermitian operators acting in $\mathcal{H}_N$ with trace normalized to unity, $\rho=\rho^\dagger$, $\rho\geqslant 0$ and $\mathrm{tr}\rho=1$. They may be represented (in a nonunique way) as convex combinations of projectors. We shall denote the space of density operators by $\mathcal{M}_N$.

Definition 2. Now we can formulate the general definition of a *mixed states QIFS* as a set

$$\mathcal{F}_N:=\{\mathcal{M}_N,\ G_i:\mathcal{M}_N\to\mathcal{M}_N,\ p_i:\mathcal{M}_N\to[0,1]:\ i=1,\ldots,k\},\tag{8}$$

where the maps G_i, $i=1,\ldots,k$ transform density operators into density operators, and for every density operator $\rho\in\mathcal{M}_N$ the probabilities are normalized, i.e., $\Sigma_{i=1}^k p_i(\rho)=1$.

The above definition of QIFS is more general than the previous one, since in particular G_i and p_i may be defined by

$$G_i(\rho):=\frac{V_i\rho V_i^\dagger}{\mathrm{tr}(V_i\rho V_i^\dagger)}\tag{9}$$

and

$$p_i(\rho):=\mathrm{tr}(W_i\rho W_i^\dagger)\tag{10}$$

for $i=1,\ldots,k$ and $\rho\in\mathcal{M}_N$, where the linear maps V_i and W_i are as in Definition 1. Thus, each QIFS on $\mathcal{P}_N$ can be extended to a QIFS on $\mathcal{M}_N$. Note that in this case $p_i(\rho)=\mathrm{tr}(W_i^\dagger W_i\rho)$. Hence, we can alternatively define the probabilities by $p_i(\rho)=\mathrm{tr}(\mathcal{L}_i\rho)$ $(i=1,\ldots,k,\ \rho\in\mathcal{M}_N)$, where the linear operators $\mathcal{L}_i$ are Hermitian, positive, and fulfill the identity $\Sigma_{i=1}^k\mathcal{L}_i=1$.

Now the dynamics takes place in the convex body of all density matrices $\mathcal{M}_N$. The space of mixed states $\mathcal{M}_N$ has N^2-1 real dimensions in contrast to the $(2N-2)$-dimensional space of pure states $\mathcal{P}_N$. For $N=2$ it is just the three-dimensional *Bloch ball*, i.e., the volume bounded by the Bloch sphere.

The special class of QIFSs is a class of *homogenous* QIFSs introduced in a more general setting by one of the authors [27]. A QIFS is called *homogenous* if both p_i and G_ip_i are affine maps for $i=1,\ldots,k$. The mixed states QIFS being a generalization of a pure state QIFS and defined by formulas (9) and (10) is homogenous if $W_i=V_i$ for $i=1,\ldots,k$. Interesting examples of such systems acting on the Bloch sphere were recently analyzed by Jadczyk and Öberg [11]. For a homogenous QIFS, p_i and G_i may be interpreted in terms of a discrete measurement process as the probability that the measurement outcome is i, and the state of the system after the measurement if the result was actually i, respectively.

A homogenous QIFS generates not only the Markov operator P acting in the space of probability measures on $\mathcal{M}_N$, but also the linear, trace-preserving, and positive operator $\Lambda:\mathcal{M}_N\to\mathcal{M}_N$ defined by

$$\Lambda(\rho):=\sum_{i=1}^{k}p_i(\rho)G_i(\rho)=\sum_{i=1}^{k}V_i\rho V_i^\dagger\tag{11}$$

for $\rho\in\mathcal{M}_N$.

A mixed state $\tilde{\rho}$ is Λ invariant if and only if it is the *barycenter* of some P invariant measure $\tilde{\mu}$, i.e.,

$$\tilde{\rho}=\int_{\mathcal{M}_N}\rho\,d\tilde{\mu}(\rho),\tag{12}$$

see Ref. [27].

Example 7. $\Omega=\mathcal{M}_N$, $k=2$, $p_1=p_2=1/2$, $G_1(\rho)=U_1\rho U_1^\dagger$, and $G_2(\rho)=U_2\rho U_2^\dagger$. This is just Example 6 in another form; the normalized identity matrix, $\rho_*=\mathbb{I}/N$ is Λ invariant irrespective of the form of unitary operators U_i, $i=1,2$. Note that $\tilde{\rho}=\rho_*$ may be represented as Eq. (12), where the measure $\tilde{\mu}$, uniformly spread over $\mathcal{P}_N$ (the *Fubini-Study measure*), is P invariant.

To define hyperbolic QIFSs one needs to specify a distance in the space of mixed quantum states. There exist several different metrics in $\mathcal{M}_N$, which may be applicable (see, e.g., Refs. [28,29], and references therein). The standard distances: the *Hilbert-Schmidt (HS) distance*

$$D_{HS}(\rho_1,\rho_2)=\sqrt{\mathrm{tr}[(\rho_1-\rho_2)^2]},\tag{13}$$

the *trace distance*

$$D_{\mathrm{tr}}(\rho_1,\rho_2)=\mathrm{tr}\sqrt{(\rho_1-\rho_2)^2}=||\rho_1-\rho_2||_{\mathrm{tr}},\tag{14}$$

and the *Bures distance* [30]

$$D_{Bures}(\rho_1,\rho_2)=\sqrt{2\{1-\mathrm{tr}[(\rho_1^{1/2}\rho_2\rho_1^{1/2})^{1/2}]\}},\tag{15}$$

the latter based on the idea of purification of mixed quantum states [31,32], are mutually bounded [33]. They generate the same natural topology in $\mathcal{M}_N$. Having endowed the space of mixed states with a metric, we may formulate immediate conclusion from the theorem on hyperbolic IFSs. We define a hyperbolic QIFS as in the preceding section, and the following proposition holds.

Proposition 1. If a QIFS (8) is homogenous and hyperbolic (that is, the quantum maps G_i are contractions with respect to one of the standard distances in $\mathcal{M}_N$, p_i are Hölder continuous and positive), then the associated Markov operator P possesses a unique invariant measure $\tilde{\mu}$. This invariant measure determines a unique Λ-invariant mixed state $\tilde{\rho}\in\mathcal{M}_N$ given by Eq. (12).

Note that for a homogenous hyperbolic QIFS, the sequence $\Lambda^n(\rho_0)$ tends in the limit $n\to\infty$ to a unique invariant state $\tilde{\rho}$ irrespective of the choice of an initial state ρ_0 [27].

Example 8. Let $\Omega=\mathcal{M}_N$, $k=2$, $p_1=p_2=1/2$, $G_1(\rho)=(\rho+2\rho_1)/3$, and $G_2(\rho)=(\rho+2\rho_2)/3$, where we choose both projectors $\rho_1=|1\rangle\langle 1|$ and $\rho_1=|2\rangle\langle 2|$ to be orthogonal. Since both homotheties G_i are contractions (with the Lipschitz constants 1/3), this QIFS is hyperbolic and a unique invariant measure $\tilde{\mu}$ exists. In analogy with the IFS discussed in Example 1 we see that the support of $\tilde{\mu}$ covers the Cantor set at the line joining both projectors ρ_1 and ρ_2. However, this is nothing but a rather sophisticated representation of the maximally mixed two-level state $\rho_*=(\rho_1+\rho_2)/2$, which follows from the symmetry of the Cantor set and may be formally verified by performing the integration prescribed by Eq. (12).

C. Completely positive maps and unitary QIFSs

From the mathematical point of view it may be sufficient to require that the map Λ is *positive*, that is, it transforms a positive operator into another positive operator. From the physical point of view it is desirable to require a stronger condition of complete positivity related to a possible coupling of the quantum system under consideration with an environment. A map Λ is *completely positive* (*CP map*), if the extended map $\Lambda\otimes\mathbb{I}$ is positive for any extension of the initial Hilbert space, $\mathcal{H}_N\to\mathcal{H}_N\otimes\mathcal{H}_E$, which describes coupling to the environment [34,35].

It is well known that each trace-preserving CP map Λ (sometimes called *quantum channel*) can be represented (not uniquely) in the following *Stinespring-Kraus form*:

$$\rho'=\Lambda_K(\rho)=\sum_{j=1}^{k} V_j\rho V_j^\dagger \quad \text{with} \quad \sum_{j=1}^{k} V_j^\dagger V_j=\mathbb{I}, \tag{16}$$

where linear operators V_j $(j=1,\ldots,k)$ are called *Kraus operators* [34,36]. For any quantum channel acting in an N-dimensional Hilbert space, the number of operators k need not exceed N^2 [37]. Each quantum channel can be treated (but not necessarily uniquely) as a pure or mixed state homogenous QIFS. Conversely, for each homogenous QIFS, formula (11) defines a quantum channel.

If, additionally, $\Sigma_{j=1}^k V_jV_j^\dagger=\mathbb{I}$ holds, then $\Lambda(\mathbb{I}/N)=\mathbb{I}/N$, and the map Λ is called *unital*. It is the case if all Kraus operators are normal, $V_jV_j^\dagger=V_j^\dagger V_j$ $(j=1,\ldots,k)$, however, this condition is not necessary. A unital trace-preserving CP map is called *bistochastic*. An example of a bistochastic channel is given by *random external fields* [38] defined by

$$\rho'=\Lambda_U(\rho)=\sum_{i=1}^{k} p_i\; U_i\rho U_i^\dagger, \tag{17}$$

where U_i, $i=1,2,\ldots,k$, are *unitary* operators and the vector of non-negative probabilities is normalized, i.e., $\Sigma_{i=1}^k p_i=1$. The Stinespring-Kraus form (16) can be reproduced setting $V_i=\sqrt{p_i}U_i$. Note that the random external fields (17) may be regarded as a homogenous QIFS of the first kind (with constant probabilities) with k unitary maps $G_i(\rho)=U_i\rho U_i^\dagger$ $(i=1,\ldots,k)$. In particular, Example 7 belongs to this class. In the sequel such QIFSs will be called *unitary*. For a unitary QIFS not only ρ_* is an invariant state of Λ_U, but also the measure δ_{ρ_*} is invariant for the Markov operator P_U induced by this QIFS.

Although a unitary QIFS consists of isometries, the operator Λ_U need not preserve the standard distances between any two mixed states. For the Hilbert-Schmidt metric we have

$$D_{HS}(\Lambda_U(\rho_1),\Lambda_U(\rho_2))\leqslant D_{HS}(\rho_1,\rho_2). \tag{18}$$

In fact this statement is true for any bistochastic channels as shown by Uhlmann [39], but it is false for arbitrary CP maps, since the Hilbert-Schmidt metric is not monotone [40]. On the other hand, Λ_U is a contraction for the Bures distance (Riemannian) and the trace distance (not Riemannian), which are monotone and do not grow under the action of any CP map [41,28]. Choosing for ρ_2 the maximally mixed state $\rho_*=\mathbb{I}/N$, which is invariant with respect to Λ_U for every unitary QIFS, we see in particular that the distance of any state ρ_1 to ρ_* does not increase in time. Similarly, the von Neumann entropy given by $H(\rho)=-\mathrm{tr}(\rho\ln\rho)$ for $\rho\in\mathcal{M}_N$ does not decrease during time evolution (17). On the other hand, the inequality in Eq. (18) is weak, and in some cases the distance may remain constant. The question, under which conditions this inequality is strong, is related to the problem, for which unitary QIFSs the maximally mixed state ρ_* is a unique invariant state of Λ_U. This is not the case, if all operators U_i commute, since then all density matrices diagonal in the eigenbase of U_i are invariant. Such a situation may occur also in subspaces of smaller dimension. To describe such a case we shall call unitary matrices of the same size *common block-diagonal*, if they are block diagonal in the same basis and with the same blocks. The uniqueness of the invariant state of a unitary QIFS is then characterized by the following proposition, the proof of which is provided in the Appendix.

Proposition 2. Let us assume that all probabilities p_i $(i=1,\ldots,k)$ are strictly positive. Then the maximally mixed state ρ_* is not a unique invariant state for the operator Λ_U if and only if unitary operators U_i $(i=1,\ldots,k)$ are common block diagonal.

It follows from the proof of this proposition that in this case there exists $\rho\neq\rho_*$ such that δ_ρ is an invariant measure for the operator P_U induced by the QIFS.

To show an application of Proposition 2 consider a two-level quantum system, called *qubit*, which may be used to carry a piece of quantum information. Let us assume that it is subjected to a random noise, described by the following map:

$$\rho\to\rho'=\Lambda_U(\rho)=(1-p)\rho+\frac{p}{3}[\sigma_1\rho\sigma_1+\sigma_2\rho\sigma_2+\sigma_3\rho\sigma_3]. \tag{19}$$

This bistochastic map, defined by the unitary Pauli matrices σ_i, is called *depolarizing quantum channel* [42], and the parameter p plays the role of the probability of error. This map transforms any vector inside the Bloch ball toward the center, so the length of the polarization vector decreases. In the formalism of QIFSs this quantum channel is equivalent to the following example.

Example 9. $\Omega=\mathcal{P}_2$, $k=4$, $U_1=\mathbb{I}$, $U_2=\sigma_1$, $U_3=\sigma_2$, $U_4=\sigma_3$, $p_1=1-p$, and $p_2=p_3=p_4=p/3>0$. Since the Pauli matrices are not common block diagonal, the maximally mixed state ρ_* is a unique invariant state of the CP map (19) associated with this unitary QIFS.

To introduce an example of QIFS arising from atomic physics, consider a two-level atom in a constant magnetic field B_z subjected to a sequence of resonant pulses of electromagnetic wave. The length of each wave pulse is equal to its period T and it interacts with the atom by the periodic Hamiltonian $V(t)=V(t+T)$. Let us assume that each pulse occurs randomly with probability p. Thus, the evolution operator transforms any initial pure state by the operator

$$U_1=\exp(-iH_0T/\hbar) \tag{20}$$

in the absence of the pulse, or by the operator

$$U_2=\hat{C}\,\exp\left[-\frac{i}{\hbar}\left(H_0T+\int_0^T V(t)\mathrm{d}t\right)\right] \tag{21}$$

in the presence of the pulse. The unperturbed Hamiltonian H_0 is proportional to B_zJ_z (J_z is the z component of the angular momentum operator) and $\hat{C}$ denotes the chronological operator. Thus, this random system may be described by the following unitary QIFS.

Example 10. $\Omega=\mathcal{P}_2$, $k=2$, $p_1=1-p$, and $p_2=p$, the Floquet operators U_1 [Eq. (20)] and U_2 [Eq. (21)] as specified above. The maximally mixed state $\rho_*=\mathbb{I}/2$, corresponding to the center of the Bloch ball, is the invariant state of the Markov operator given by Eq. (17). For a generic perturbation V, the matrices U_1 and U_2 are not common block diagonal, and so ρ_* is the unique invariant state for operator (17) related to the QIFS.

The QIFSs arise in a natural way if considering a quantum system acting on $\mathcal{H}_N$ coupled with an *ancilla*: a state in an auxiliary m-dimensional Hilbert space $\mathcal{H}_m$, which describes the environment. Initially, the composite state describing the system and the environment is in the product form, $\sigma=\rho_A\otimes\rho_*^B$, where $\rho_*^B:=\mathbb{I}_m/m$ is the maximally mixed state, but the global unitary evolution couples two subsystems together. A unitary matrix U of size Nm acting on the space $\mathcal{H}_N\otimes\mathcal{H}_m$ may be represented in its Schmidt decomposition form as $U=\sum_{i=1}^{K}\sqrt{q_i}V_i^A\otimes V_i^B$, where the number of terms is determined by the size of the smaller space, $K=\min\{N^2,m^2\}$; the operators V_i^A and V_i^B act on $\mathcal{H}_N$ and $\mathcal{H}_m$, respectively, and the Schmidt coefficients are normalized as $\sum_{i=1}^{K}q_i=1$. Restricting our attention to the system A one needs to trace out the variables of the environment B which leads to the following quantum channel (and to the respective homogenous QIFS):

$$\rho_A'=\Lambda(\rho_A)=\mathrm{tr}_B(U\sigma U^\dagger)=\sum_{i=1}^{K}q_iV_i^A\rho_AV_i^{A\dagger}. \tag{22}$$

Since for $\rho_*^A:=\mathbb{I}_N/N$ we have $\Lambda(\rho_*^A)=\mathrm{tr}_B(U(\rho_*^A\otimes\rho_*^B)U^\dagger)=\rho_*^A$, the CP map Λ is bistochastic.

IV. QUANTUM-CLASSICAL CORRESPONDENCE

To investigate various aspects of the semiclassical limit of the quantum theory it is interesting to compare a given discrete classical dynamical system generated by $f:\Omega\to\Omega$ with a family of the corresponding quantum maps, usually defined as $F_N:\mathcal{H}_N\to\mathcal{H}_N$ with an integer N. Several alternative methods of quantization of classical maps in compact phase space have been applied to construct quantum maps corresponding to the baker map on the torus [43,44], the Arnold cat map [45] and other automorphisms on the torus [46], the periodically kicked top [47], and the baker map on the sphere [48].

To specify in which manner the classical and the quantum maps are related, it is convenient to introduce a set of coherent states $|y\rangle\in\mathcal{H}_N$, indexed by classical points y of the phase space Ω. (For more properties of coherent states and a general definition consult the book by Perelomov [49].) They satisfy the resolution of identity formula: $\int_\Omega|y\rangle\langle y|dy=\mathbb{I}$, and allow us to represent any state ρ by its Husimi representation, $H(y)=\langle y|\rho|y\rangle$ $(y\in\Omega)$. Quantization of a classical map f, which leads to a family of quantum maps F_N is called *regular*, if for almost all classical points x the classical and the quantum images are connected in the sense that the normalized Husimi distribution of state $F_N|y\rangle$ integrated over a finite vicinity of the point $f(y)$ tends to unity in the limit $N\to\infty$ [50]. Another method of linking a classical map with a family of quantum maps is based on the *Egorov property*, which relates the classical and the quantum expectation values [51,52].

In a similar way we may construct QIFSs related to certain classical IFSs. More precisely, a sequence of pure states QIFS $\mathcal{F}_N=\{\mathcal{P}_N;F_{i,N},p_{i,N}:i=1,\ldots,k\}$ induced by two sets of linear maps $V_{i,N},W_{i,N}:\mathcal{H}_N\to\mathcal{H}_N$ $(i=1,\ldots,k)$ [see Eqs. (5) and (6)] is a *quantization* of a classical IFS $\mathcal{F}_{\mathrm{Cl}}=\{\Omega;F_i,p_i:1,\ldots,k\}$, when the functions $F_{i,N}$ are quantum maps obtained by the quantization of the classical maps f_i and the probabilities $p_{i,N}$ computed at coherent states $|y\rangle$ fulfill

$$p_{i,N}(|y\rangle\langle y|)=\|W_{i,N}(|y\rangle)\|^2\xrightarrow{N\to\infty}p_i(y)\quad\text{for } y\in\Omega \text{ and } i=1,\ldots,k. \tag{23}$$

To illustrate the procedure let us consider random rotations on the sphere, performed along x and z axes. This special case of Example 4 may be easily quantized with the help of the components J_i $(i=x,y,z)$ of the angular momentum operator J, satisfying the standard commutation relations, $[J_i,J_j]=\epsilon_{ijk}J_k$. The size of the Hilbert space is determined by the quantum number j as $N=2j+1$.

Example 11. $k=2$, random rotations are given as the following.

(a) classical, $\mathcal{F}_{\rm Cl}=\{\Omega=S^2,\ f_1=R_z(\theta_1),\ f_2=R_x(\theta_2),\ p_1=p_2=1/2\}$. The Lebesgue measure on the sphere in an invariant measure for this IFS.

(b) quantum, $\mathcal{F}_N=\{\Omega=\mathcal{P}_N,\ F_1=e^{i\theta_1 J_z},\ F_2=e^{i\theta_2 J_x},\ p_1=p_2=1/2\}$. Since both unitary operators are not common block diagonal, due to Proposition 2, the maximally mixed state ρ_* is a unique invariant state for operator (11) related to the QIFS $\mathcal{F}_N$.

A quantization of an IFS of the second kind is given by the following modification of the previous example.

Example 12. $k=2$, random rotations on the sphere with varying probabilities depending on the latitude θ computed with respect to the z axis.

The spaces and functions are as in Example 11, but (a) classical IFS $\mathcal{F}_{\rm Cl}$, $p_1=(1+\cos\theta)/2$ and $p_2=(1-\cos\theta)/2$; (b) quantum IFS $\mathcal{F}_N$, $p_1=1/2+\langle J_z\rangle/2j$ and $p_2=1/2-\langle J_z\rangle/2j$ with $N=2j+1$. Interestingly, this modification influences the number of invariant states of the IFS. Since p_2 vanish at the north pole ($\theta=0$) of the classical sphere S^2, this point is invariant with respect to $\mathcal{F}_{\rm Cl}$. Similarly, the corresponding quantum state $|j,j\rangle$ localized at the pole is invariant with respect to operator (11) related to the QIFS $\mathcal{F}_N$.

The above examples of unitary QIFS dealt with simple regular maps — rotations on the sphere. However, an IFS may also be constructed out of nonlinear maps, which may lead to deterministic chaotic dynamics. For instance, one may consider the map describing a periodically kicked top. It consists of a linear rotation with respect to the x axis by angle α and a nonlinear rotation with respect to the z axis by an angle depending on the z component. In a compact notation,
the classical top reads $T_{\rm Cl}(\alpha,\beta):=R_z(z\beta)R_x(\alpha)$, while its quantum counterpart, acting in the $N=(2j+1)$-dimensional Hilbert space can be defined by $T_Q(\alpha,\beta):=\exp(-i\beta J_z^2/2j)\exp(-i\alpha J_x)$ [47]. This quantum map is one of the most important toy models often studied in research on quantum chaos [53]. A certain modification of this model, in which the kicking strength parameter β was chosen randomly out of two values, was proposed and investigated by Scharf and Sundaram [54]. This random system may be put into the QIFSs formalism:

Example 13. Randomly kicked top: (a) classical, $\mathcal{F}_{\rm Cl}=\{\Omega=S^2,\ f_1=T_{\rm Cl}(\alpha,\beta),\ f_2=T_{\rm Cl}(\alpha,\beta+\Delta),\ p_1=p_2=1/2\}$; (b) quantum, $\mathcal{F}_N=\{\Omega=\mathcal{P}_N,\ F_1=T_Q(\alpha,\beta),\ f_2=T_Q(\alpha,\beta+\Delta),\ p_1=p_2=1/2\}$. For positive α and Δ both unitary operators are not block diagonal, so the maximally mixed state ρ_* is a unique invariant state for operator (11) related to this unitary QIFS. Our numerical results obtained for $\alpha=\pi/4$, $\beta=2$, and $\Delta=0.05$ suggest that the trajectory of any pure coherent state converges to the equilibrium exponentially fast.

To discuss a quantum analog of an IFS with a fractal invariant measure, consider the classical IFS presented in Example 3. The classical phase space Ω can be identified with the torus. For pedagogical purpose, let us rename both variables x,y into q,p, representing canonically coupled position and momentum. We shall work in $N=3L$-dimensional Hilbert space. Let $|j\rangle_q$ with $j=1,\dots,N$ be the eigenstates of the position operator, and similarly $|l\rangle_p$ with $l=1,\dots,N$ be the eigenstates of the momentum operator. Both bases are related by $|l\rangle_p=\Sigma_{j=1}^N W_{lj}|j\rangle_q$, where the matrix W is the N point discrete Fourier transformation with $W_{lj}=(1/\sqrt{N})e^{-2\pi i l j/N}$ for $l,j=1,\dots,N$. The classical map f_1 in Eq. (3), representing a threefold contraction in the x direction, corresponds to the transformation G_1 of the density operator given by

$$G_1(\rho):=\sum_{i,j=1}^{L}|i\rangle_q\left(\sum_{m,n=0}^{2}\langle 3i+m|_q\rho|3j+n\rangle_q\right)\langle j|_q. \tag{24}$$

In a similar way, the quantum map G_2 corresponding to f_2 is defined by

$$G_2(\rho):=\sum_{i,j=2L+1}^{3L}|i\rangle_q\left(\sum_{m,n=0}^{2}\langle 3i+m|_q\rho|3j+n\rangle_q\right)\langle j|_q. \tag{25}$$

The maps G_3 and G_4 are obtained in analogy to G_1 and G_2, using the eigenstates of the momentum operator $|k\rangle_p$,

$$G_3(\rho):=\sum_{k,l=1}^{L}|k\rangle_p\left(\sum_{m,n=0}^{2}\langle 3k+m|_p\rho|3l+n\rangle_p\right)\langle l|_p, \tag{26}$$

$$G_4(\rho):=\sum_{k,l=2L+1}^{3L}|k\rangle_p\left(\sum_{m,n=0}^{2}\langle 3k+m|_p\rho|3l+n\rangle_p\right)\langle l|_p. \tag{27}$$

The random system defined below may be considered as a QIFS related to the IFS introduced in Example 3:

Example 14. Quantum tartan: $\mathcal{F}_N=\{\Omega=\mathcal{P}_N,\ k=4,\ G_1,G_2,G_3,G_4;\ p_1=p_2=p_3=p_4=1/4\}$.

An invariant states for the map Λ induced by this QIFS are illustrated in Fig. 1 for $N=3^4$, $N=3^5$, and $N=3^6$. Invariant quantum state ρ_* is shown in the generalized Husimi representation

$$H_\rho(p,q)=\frac{1}{2\pi}\frac{\langle q,p|\rho|q,p\rangle}{\langle q,p|q,p\rangle}, \tag{28}$$

based on the set of coherent states on the torus $|q,p\rangle=Y^{Np-N/2}X^{Nq-N/2}|\kappa\rangle$. The reference state $|\kappa\rangle$ is chosen as an arbitrary state localized in (1/2,1/2)

$$\langle n|\kappa\rangle=(2/N)^{-1/4}e^{-\pi(n-N/2)^2/N-i\pi n}, \tag{29}$$

while X denotes the operators of shift in position $X|j\rangle=|j+1\rangle$ with an identification $|j+N\rangle=|j\rangle$ for $j=1,\dots,N$. Similarly, Y shifts the momentum eigenstates, $Y|l\rangle=|l+1\rangle$ and $|l+N\rangle=|l\rangle$ for $l=1,\dots,N$. The quantum state $|q,p\rangle$ is well localized in the vicinity of the classical point (q,p) on the torus [55]. This representation of quantum states corresponding to the classical system on the torus was used in the analysis of an irreversible quantum baker map [56].

The larger value of N, the finer structure of the invariant state ρ_* is visible in the phase space. In the semiclassical limit $N\to\infty$ (which means $\hbar\to 0$), the invariant state ρ_*

tends to be localized at the fractal support of the invariant measure of the classical IFS, shown for comparison in Fig. 1(d). Strictly speaking, for any finite N, the Husimi distribution of ρ_* does not possess fractal character, since self-similarity has to terminate at a length scale comparable with $\sqrt{\hbar}$. In other words, quantum effects are responsible for smearing out the fractal structure of the classical invariant measure. However, the classical fractal structures may be approximated with an arbitrary accuracy by quantum objects in the semiclassical limit [57].

V. CONCLUDING REMARKS

Classical iterated function systems display several interesting mathematical properties and may be applied in various problems from different branches of physics. In this work we have generalized the formalism of IFSs introducing the concept of QIFSs. Quantum iterated function systems may be defined in the space of pure states on a finite-dimensional Hilbert space $\mathcal{H}_N$, or more generally, in the space of density operators acting on $\mathcal{H}_N$. As their classical analogs, QIFSs allow a certain degree of stochasticity, in the sense that at each step of time evolution the choice of one of the prescribed quantum maps is random.

This formalism may be applied to describe several problems of quantum mechanics, including nonunitary dynamics, processes of decoherence, and quantum measurements. In fact, the large class of quantum channels, called random external fields may serve directly as examples of a QIFS. Furthermore, for several classical IFSs one may construct the corresponding QIFSs and analyze the relations between them. As shown in the last example, one may focus on the fractal properties of invariant measures of some classical IFSs and study their quantum counterpart. Thus, the concept of QIFS allows one to investigate the semiclassical limit of random quantum systems.

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APPENDIX: PROOF OF PROPOSITION 3

We start from the following lemma.

Lemma. Let $U=(U_{nm})_{n,m=1,\ldots,N}$ be an N-dimensional unitary matrix. Assume that there exist two nonempty sets of indices A and B such that $A\cup B=I:=\{1,\ldots,N\}$ and $A\cap B=\emptyset$. Then, $U_{nm}=0$ for $n\in A$ and $m\in B$ implies $U_{nm}=0$ for $n\in B$ and $m\in A$.

Proof of the lemma. We compute the number of elements of the set A:

$$|A|=\sum_{n\in A}\sum_{m\in I}|U_{nm}|^2=\sum_{n\in A}\sum_{m\in A}|U_{nm}|^2+\sum_{n\in A}\sum_{m\in B}|U_{nm}|^2$$

$$=\sum_{n\in A}\sum_{m\in A}|U_{nm}|^2=\sum_{n\in I}\sum_{m\in A}|U_{nm}|^2-\sum_{n\in B}\sum_{m\in A}|U_{nm}|^2$$

$$=|A|-\sum_{n\in B}\sum_{m\in A}|U_{nm}|^2,$$

and so $\Sigma_{n\in B}\Sigma_{m\in A}|U_{nm}|^2=0$, as required.

Now we turn to the proof of Proposition 3.

Let U_i $(i=1,\ldots,k)$ be block diagonal in the common base, and let dimension of the blocks be $\alpha_1,\ldots,\alpha_L$, where $\Sigma_{j=1}^{L}\alpha_j=N$. Define a diagonal density matrix as a direct sum

$$\rho:=\bigoplus_{j=1}^{L}\frac{\sigma_j}{\alpha_j}\mathbb{I}_{\alpha_j}, \tag{A1}$$

where $\Sigma_{j=1}^{L}\sigma_j=1$. Then, $U_i\rho U_i^\dagger=\rho$ for every $i=1,\ldots,k$. Hence, ρ is Λ_U invariant and δ_ρ is a P_U-invariant measure on $\mathcal{P}_N$ for an arbitrary choice of $(\sigma_j)_{j=1,\ldots,L}$.

Let ρ be an invariant state for Λ_U such that $\rho\neq\rho_*$. Then, ρ can be written in the form

$$\rho=\sum_{n=1}^{N}\sigma_n|\Psi_n\rangle\langle\Psi_n|, \tag{A2}$$

where $|\Psi_m\rangle\in\mathcal{P}_N$, $\langle\Psi_n|\Psi_m\rangle=\delta_{nm}$ $(n,m=1,\ldots,N)$, and $\sigma_1\leqslant\sigma_2\leqslant\cdots\leqslant\sigma_N$; $\sigma_1<1/N$. For $\gamma\in[0,1]$ the density operator $\rho_\gamma:=\gamma\rho+(1-\gamma)\rho_*=\Sigma_{n=1}^{N}\sigma_n'|\Psi_n\rangle\langle\Psi_n|$, where $\sigma_n':=\gamma\sigma_n+(1-\gamma)N^{-1}$ $(n=1,\ldots,N)$ is also an invariant state for Λ_U. Put $\gamma:=1/(1-\sigma_1 N)$. This choice implies $\sigma_1'=0$ and $\Sigma_{n=1}^{N}\sigma_n'=1$. Assume that $\sigma_n'=0$ for $n=1,\ldots,n'$ and $\sigma_n'>0$ for $n=n'+1,\ldots,N$, where $n'\geqslant1$. The equation $\Lambda_U(\rho_\gamma)=\rho_\gamma$ can be rewritten in the form

$$\sigma_n'=\sum_{i=1}^{k}p_i\sum_{m=1}^{N}|(U_i)_{nm}|^2\sigma_m', \tag{A3}$$

where $(U_i)_{nm}$ $(n,m=1,\ldots,N)$ are the elements of matrices U_i $(i=1,\ldots,k)$ in the basis $(|\Psi_n\rangle)_{n=1,\ldots,N}$.

For $n=1,\ldots,n'$ we get

$$0=\sum_{i=1}^{k}p_i\sum_{m=n'+1}^{N}|(U_i)_{nm}|^2\sigma_m'. \tag{A4}$$

Hence $(U_i)_{nm}=0$ for $n=1,\ldots,n'$ and $m=n'+1,\ldots,N$. Using the Lemma, we deduce that $(U_i)_{nm}=0$ for $n=n'+1,\ldots,N$ and $m=1,\ldots,n'$. Thus, U_i $(i=1,\ldots,k)$ are common block diagonal.

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