Criticality versus q in the $(2+1)$-dimensional Z_q clock model

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(Received 11 December 2002; revised manuscript received 15 May 2003; published 9 October 2003)

Using Monte Carlo simulations we have studied the $d=3$ Z_q clock model in two different representations, the phase representation and the loop-gas/dumbbell-gas representation. We find that for $q\geqslant 5$ the critical exponents α and ν for the specific heat and the correlation length, respectively, take on values corresponding to the case $q\rightarrow$ infinity, the XY model. Hence in terms of critical properties the limiting behavior is reached already at $q=5$.

DOI: 10.1103/PhysRevE.68.046107 PACS number(s): 05.50.+q, 11.15.Ha, 64.60.Cn, 71.10.Hf

Matter coupled-gauge field theories in $2+1$ dimensions have come under renewed scrutiny in the context of condensed matter physics in the past decade, as effective theories of strongly correlated system [1]. Concepts such as confinement-deconfinement transitions, associated with the proliferation and recombination of topological defects of gauge fields, enter for instance in attempts at providing a theoretical foundation for breakdown of Fermi-liquid theory in more than one dimension. A large variety of such gauge-field theories have been proposed, and one model of particular interest is the compact Abelian Higgs model [1–6]. This model consists of a compact gauge field coupled minimally to a bosonic scalar field with the *gauge charge q*. In a particular limit the dual of this model reduces to a loop-gas representation of the global Z_q model [5,6]. This identification has been the motivation for the present work, for a detailed account of the q dependence of the full theory we refer to Refs. [5,6].

The spin Z_q model is a simple planar-spin model, where the direction of the spin is parametrized by a phase. This phase is restricted to the values $2\pi n/q$ with $n\in\mathbb{Z}$, and is defined by the following action

$$S=-\beta\sum_{\langle i,j\rangle}\cos\left(\frac{2\pi}{q}(n_i-n_j)\right). \tag{1}$$

The state is specified by the integer variables $n_i\in[0,1,\ldots,q-1]$. Special cases include $q=2$ which is the Ising model, $q=3$ which is the three-state Potts model, and the limit $q\rightarrow\infty$ which corresponds to the XY model. In addition, it is easy to see that for $q=4$ the partition function $Z(2\beta,4)=Z(\beta,2)Z(\beta,2)$. The aim of the present paper is to determine how the critical properties interpolate between the well-known Ising $(q=2)$ and XY $(q\rightarrow\infty)$ limits. We have done this by measuring the exponent combination $(1+\alpha)/\nu$ as a function of q.

In $d=2$ the model has a quite peculiar phase structure, with an intermediate *incompletely ordered phase* (IOP), where the system shows behavior similar to the critical Kosterlitz Thouless phase. Upon further cooling, the system will order completely into one of the q completely ordered states [7,8]. In $d=3$ the Z_q model does not have an IOP, but there are generalizations of the model which do [8–10].

A related case is that of a globally U(1) symmetric theory which is perturbed by a weak crystal field. Using renormalization group (RG) theory and duality arguments, it has been shown that for $q\geqslant 5$ the crystal field is an irrelevant perturbation, whereas for $q\leqslant 4$ the XY fixed point is rendered unstable [11].

It is important to emphasize that we have focused on the properties of the Z_q model *at* the critical point. For $T<T_c$, the discrete nature of the model will always be apparent. An interesting RG study of the Z_6 model shows how the couplings of the model flow towards a fixed point which is ultimately different from the three-dimensional (3D) XY fixed point in the $T\rightarrow 0$ limit [12,13].

Equation (1) is straightforwardly reformulated as a model of interacting ensemble links which either form closed loops or originate and terminate at point charges. We start with the partition function

$$Z(\beta,q)=\sum_{\{n_i\}}\exp\left[\beta\sum_i\left(\sum_{\hat{\mu}}\cos\left(\frac{2\pi}{q}\Delta_{\hat{\mu}}n_i\right)\right)\right]. \tag{2}$$

In Eq. (2), $\Delta_{\hat{\mu}}$ denotes the difference operator defined by $\Delta_{\hat{\mu}}F(\mathbf{x})=F(\mathbf{x}+\hat{\mu})-F(\mathbf{x})$, and $\mathbf{\Delta}$ without indices is the vector operator analogous to $\mathbf{\nabla}$. The first step is to replace the cosine with a quadratic potential, this is the Villain approximation [14]. Next, we promote the integers n_i to real-valued phase variables θ_i, at the expense of introducing an auxiliary integer field Q, which through the Poisson summation formula [15] restricts the θ_i variables to the discrete values allowed by original theory. The resulting partition function is then given by

$$Z_V(\beta,q)=\Xi[\beta]\int D\theta\sum_{\{\mathbf{k},Q\}}\exp\left[-\sum_i\left(\frac{\beta_V}{2}(\Delta\theta_i-2\pi\mathbf{k})^2+iq\theta Q\right)\right]. \tag{3}$$

In Eq. (3), $\{\mathbf{k}\}$ is an integer *link field* living on the links of the *original* lattice and $\{Q\}$ is a scalar field living on the *sites* of the same lattice. The prefactor $\Xi[\beta]$ and effective coupling $\beta_V=\beta_V(\beta)$ must be retained to get results which agree

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with Eq. (2) on a *quantitative* level [15], however they do *not* affect the critical properties and from now on we will assume $\beta_V = \beta$, $\Xi[\beta]=1$, and omit the V index on the partition function.

In Eq. (3), the Q field explicitly accounts for the discrete nature of the Z_q model. Setting $Q\equiv 0$, we recover the Villain representation of the XY model. Due to this similarity, the remaining analysis follows well-known steps [16], which we briefly include for completeness. A Hubbard-Stratonovich decoupling of the quadratic expression in Eq. (3) is performed by introducing an auxiliary field $\mathbf{v}$, thus bringing the partition function onto the form

$$Z(\beta,q)=\int D\mathbf{v}D\theta \sum_{\{\mathbf{k},Q\}} \exp\left[-\sum_i \left(\frac{1}{2\beta}\mathbf{v}^2+i\mathbf{v}\cdot(\mathbf{\Delta}\theta_i-2\pi\mathbf{k})+iq\theta Q\right)\right]. \tag{4}$$

In Eq. (4) the $\{\mathbf{k}\}$ summation can be performed, thereby restricting the velocity field $\mathbf{v}$ to integer values denoted by $\mathbf{l}$. In the term coupling $\Delta\theta$ and $\mathbf{l}$, a partial integration can be performed, such that θ only appears in the combination $i\theta(\mathbf{\Delta}\cdot\mathbf{l}-qQ)$, and from this we get the constraint

$$(\mathbf{\Delta}\cdot\mathbf{l}-qQ)=0. \tag{5}$$

At this stage the transformation to a loop gas is complete, and the partition function is given by

$$Z(\beta,q)=\sum_{\{\mathbf{l},Q\}} \delta_{\mathbf{\Delta l},qQ}\exp\left[\frac{-1}{2\beta}\sum_i \mathbf{l}^2\right]. \tag{6}$$

This is a theory consisting of the field $\{\mathbf{l}\}$ living on the *links* of the lattice, and the field $\{Q\}$ which lives on the *sites*. The field $\{Q\}$ is subject to the constraint $\Sigma_{\mathbf{x}}Q=0$, i.e., overall charge neutrality, whereas the field $\{\mathbf{l}\}$ must satisfy the local constraint $\mathbf{\Delta}\cdot\mathbf{l}=qQ$ on all lattice points. The latter constraint means that every $+Q/-Q$ pair must be joined by q occupied links, in addition we can have $\{\mathbf{l}\}$ excitations which are not nucleated to any $+Q/-Q$ pairs; these must form *closed* loops. Figure 1 shows a typical configuration for the $q=2$ model.

In the compact Abelian Higgs model considered in Refs. [6,5], the fields $\{\mathbf{l}\}$ and $\{Q\}$ represent *vortices* and *monopoles*, i.e., they are the *topological excitations* of the matter field and gauge field, respectively. That interpretation does *not* apply in the current case, but the interpretation of the $\{Q\}$ field is that it maintains the discrete properties of the original theory, Eq. (1). With $Q\equiv 0$ (the $q\to\infty$ limit), Eq. (6) reduces to a loop gas with steric repulsion, this is a well-known model with an *inverted XY* transition [17]. Note that the special case $q=1$ effectively represents no constraint. In this case, the theory [Eq. (6)] is noninteracting, and sustains no phase transition. For all $q\geqslant 2$, Eq. (6) has a phase transition between a phase filled with link segments for $\beta>\beta_c$ and a vacuum phase which does not contain link excitations.

We have performed Monte Carlo simulations of the Z_q model, using both a phase representation, Eq. (1), as well as the loop-gas/dumbbell-gas (LDG) representation, Eq. (6). The phase representation is simulated as a conventional spin simulation. In the LDG representation, the fundamental Monte Carlo moves are represented by alternating attempts of inserting a closed loop excitation of the $\mathbf{l}$ field and a dumbbell configuration consisting of a $+Q/-Q$ pair connected with an occupied q-valued link (the vertical link to the left in Fig. 1 is an example of an elementary dumbbell excitations). For $q=2$ the (vacuum) excitations of a loop or a $+Q/-Q$ pair have the same energy, while for $q>2$ the elementary dumbbell excitations are more expensive than the elementary loop excitations, and their relative importance diminishes with increasing q.

The main goal has been to determine how the critical properties change with q. The central quantity we have considered is the connected third-order moment of the action [6]

$$\langle (S-\langle S\rangle)^3\rangle \propto |\beta-\beta_c|^{1+\alpha}, \tag{7}$$

which recently has been demonstrated to yield surprisingly good scaling results compared to second moments [6]. When approaching the critical point, the correlation length ξ di-

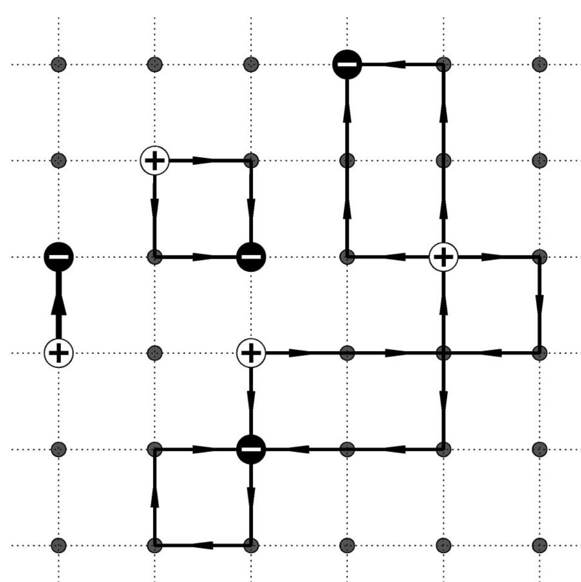

FIG. 1. A typical LDG configuration for the $q=2$ (Ising) model. Multiply connected links, like the vertical along the left edge have much lower entropy than loop/dumbbell combinations, and hence give a relatively small contribution to the partition function.

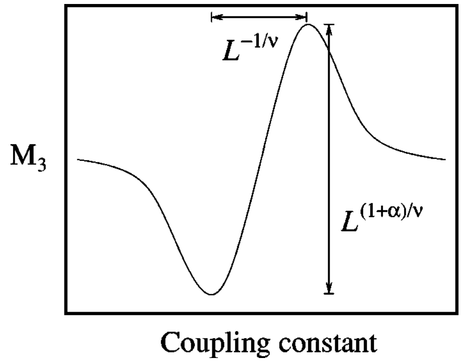

FIG. 2. Schematic figure showing third moment of action, and how data are extracted for FSS analysis. For further details of this method see Ref. [6].

verges as $\xi \propto |\beta-\beta_c|^{-\nu}$. Therefore, in a finite system of linear extent L we find that the third-order moment in Eq. (7) scales with L as

$$\langle (S-\langle S \rangle)^3 \rangle \propto L^{(1+\alpha)/\nu}. \quad (8)$$

The main advantages of the third-order moment in Eq. (7) are that (1) good quality scaling is achieved for practical system sizes even for models with $\alpha<0$, e.g., the 3D XY model, and (2) one set of measurements gives *both* the combinations $(1+\alpha)/\nu$ and $-1/\nu$ *independently* [6], although it is more difficult to achieve high precision on the latter. A schematic figure of $\langle (S-\langle S \rangle)^3 \rangle$ as a function of coupling constant is shown in Fig. 2 and Figs. 3 and 4 show finite-size scaling (FSS) of the peak to peak value.

We have considered systems of size $L\times L\times L$ with $L=8,10,12,16,20,24,32,40,48$, and up to 2×10^7 sweeps over the lattice. In addition to the $q=4$ and $q=5$ presented in Figs. 3 and 4, we have also studied the q values q

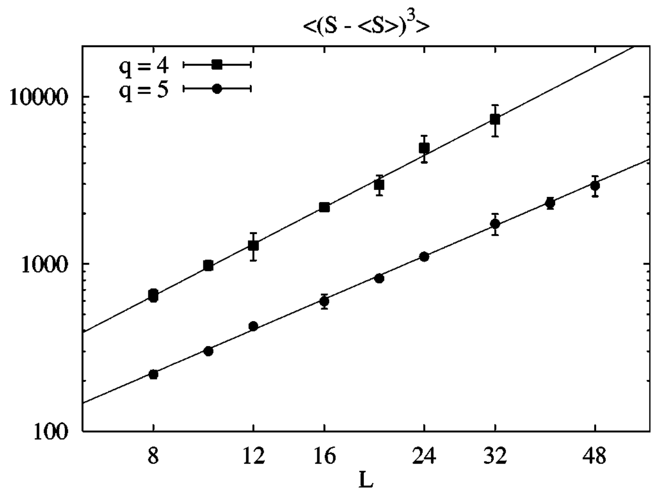

FIG. 3. This figure shows the scaling of $\langle (S-\langle S \rangle)^3 \rangle$ for $q=4$ (■) and $q=5$ (●); the results are obtained using the phase representation, Eq. (2). The $q=4$ results show Z_2 scaling with $(1+\alpha)/\nu=1.76\pm 0.05$, and the $q=5$ curve shows XY scaling with $(1+\alpha)/\nu=1.46\pm 0.03$.

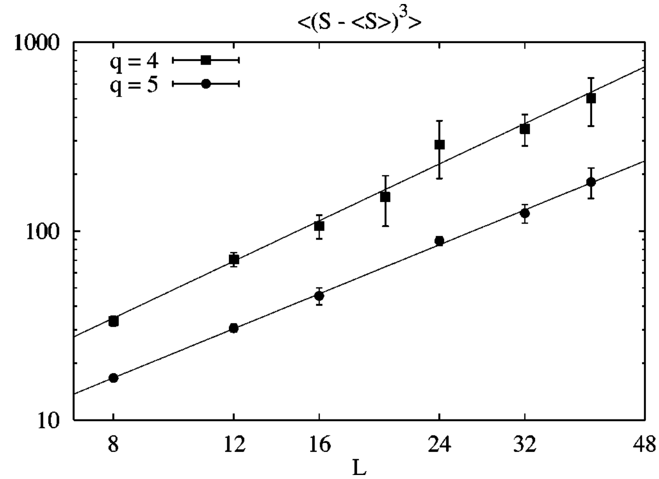

FIG. 4. This figure is similar to Fig. 3, but the results are obtained using representation (6). The $q=4$ results show Z_2 scaling with $(1+\alpha)/\nu=1.70\pm 0.05$, and the $q=5$ results scale with $(1+\alpha)/\nu=1.47\pm 0.06$, i.e., qualitatively similar to the results in Fig. 3.

$=6,8,12,16$, and 24, Ref. [6] shows results of $q=2$ simulations of Eq. (6). We find that the combination $(1+\alpha)/\nu$ changes abruptly from the Z_2 value of 1.763 [18] to the XY value of 1.467 [19] when increasing q from $q=4$ to $q=5$. A further increase of q beyond $q=5$ does not affect the value of $(1+\alpha)/\nu$, as shown in Fig. 5.

That the Z_q model is in the XY universality class for $q \geqslant 5$ must imply that *at the critical point* the discrete structure is rendered irrelevant for these q values. To investigate this point further, we have implemented a simple real-space RG procedure, which attempts to probe for what values of q the discrete nature of Z_q model is relevant at the critical point. We denote the untransformed phases and fields as θ_0. The renormalized phase at level $n+1$ is given by the *block spin* construction,

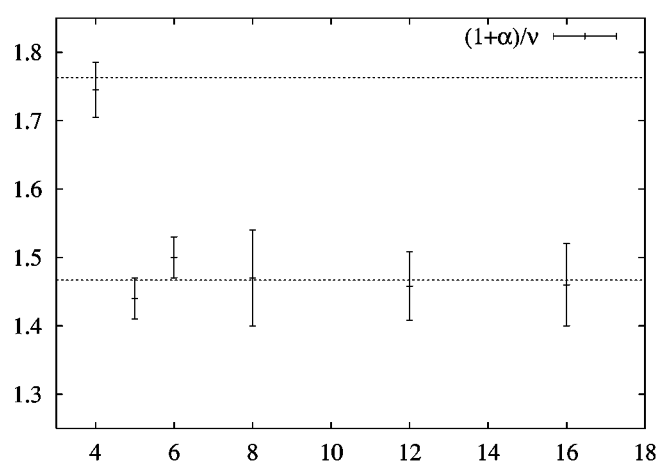

FIG. 5. The exponent combination $(1+\alpha)/\nu$ versus q. Note how it changes value abruptly as q is increased from $q=4$ to $q=5$. The dashed lines are the Ising (Z_2) and XY (Z_∞) values of 1.763 and 1.467, respectively.

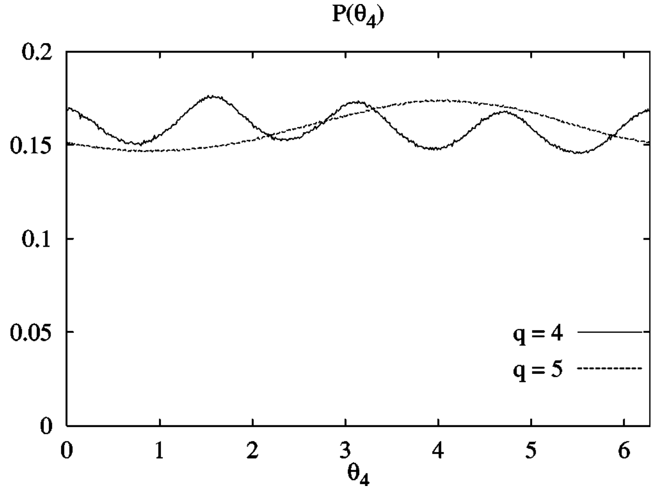

FIG. 6. Histograms of θ after four rescalings of the critical state. For $q=4$, the distribution shows clear signs of a discrete background, whereas for $q=5$ this is not the case. The slow variation in the $q=5$ histogram is *not* commensurable with a wavelength of $2\pi/5$, and probably only due to insufficient sampling.

$$\theta_{n+1}=\arctan\left(\frac{\sum_k \sin\theta_n(k)}{\sum_k \cos\theta_n(k)}\right), \tag{9}$$

where the sum over k in Eq. (9) is over the eight spins in a $2\times2\times2$ cube. For $q=2$, this transformation is clearly trivial, since adding a number of phases 0 and π will still give 0 or π. However, for $q>2$ the effective q^* will increase with n, and for $n\rightarrow\infty$ the resulting block spins can take *any* direction.

We next investigate whether the system flows towards an infinite value of q^* or not under such a RG transformation. This is tantamount to asking whether the discrete structure is rendered irrelevant or not on long length scales. To this end, at each iteration step n, we have recorded *histograms* $h_n(\theta)$ of the phase distributions on the lattice and monitored the manner in which this histogram flows under rescaling. By purely visual inspection, we find that for $q=4$ the discrete nature of the Z_q model persists, whereas for $q=5$ it is washed away, this is illustrated in Fig. 6.

To study this RG flow at a *quantitative* level, we have written the phase distribution $P_n(\theta_n)$ as a sum of harmonic functions

$$P_n(\theta_n)=a_{n,0}+\sum_k\left[a_{n,k}\cos\left(\frac{k2\pi}{q}\right)+b_{n,k}\sin\left(\frac{k2\pi}{q}\right)\right]. \tag{10}$$

Here, the coefficient $a_{n,k}$ in Eq. (10) denotes the kth Fourier-cosine component at RG level n. Clearly, the coefficient $a_{n,q}$ is the interesting component, we have studied how this coefficient flows under repeated rescaling. For $q=4$ this coefficient shows critical fixed point behavior, whereas for $q=5$ it flows to zero, even for T well below the critical temperature, this is shown in Fig. 7.

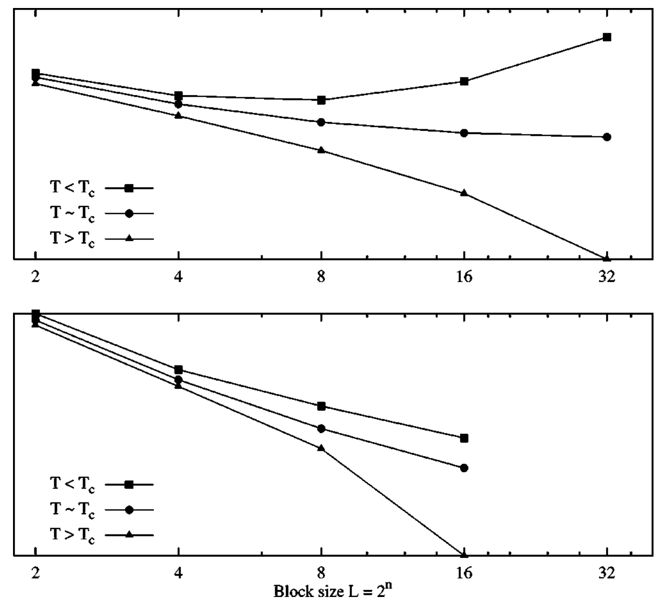

FIG. 7. The flow of the coefficient $a_{n,q}$ for $q=4$ and $q=5$. For $q=4$, we see that there is a fixed point at the critical point, whereas for $q=5$ we see that $a_{n,q}$ *flows to zero at the critical point*. In the figure $a_{n,5}$ flows to zero also for $T<T_c$; this is a finite-size effect. This coefficient will eventually flow to infinity for sufficiently large systems/low T.

Also the LDG representation, Eq. (6), gives a qualitative indication that for $q\geqslant5$ the discrete nature of the theory is irrelevant. In this representation, the discrete nature is represented solely by the Q excitations, so measurements of $\langle|Q|\rangle$ should give a quantitative indication of the the importance of the discrete structure. Measurements of $\langle|Q|\rangle$ at the critical point give $\langle|Q|\rangle\approx0.07, 5.9\times10^{-4}$, and 2.75×10^{-6} for $q=2,4$, and 5, respectively, whereas the link density $\langle|\mathbf{l}|\rangle\approx0.15$ for all q. Hence at $q=5$ the discrete Q excitations have been completely frozen out, and the tangle is essentially identical to the *pure-loop* tangle of the 3D XY model.

In summary, we have determined the critical exponent combination $(1+\alpha)/\nu$ in the $d=3$ Z_q spin model for $q\geqslant4$. Using two different representations we have found that for $q\geqslant5$, the combination $(1+\alpha)/\nu$ takes a value which is consistent with the value taken in the 3D XY model. Along with other more qualitative indicators this means that at the critical point, a discrete structure finer than $q=5$ is irrelevant at the critical point, and the long distance properties of the theory are determined by the larger symmetry group U(1). These results are in accordance with RG studies starting with a U(1) symmetric theory which is perturbed by a perturbation with Z_q symmetry.

We acknowledge support from the Norwegian Research Council through the Norwegian High Performance Computing Center (NOTUR). All computations were carried out on an Origin SGI3800. Martin Hasenbusch is acknowledged for valuable comments on an earlier version of the manuscript.

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