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Instabilities in the two-dimensional cubic nonlinear Schrödinger equation

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The two-dimensional cubic nonlinear Schrödinger equation (NLS) can be used as a model of phenomena in physical systems ranging from waves on deep water to pulses in optical fibers. In this paper, we establish that every one-dimensional traveling wave solution of NLS with linear phase is unstable with respect to some infinitesimal perturbation with two-dimensional structure. If the coefficients of the linear dispersion terms have the same sign (elliptic case), then the only unstable perturbations have transverse wavelength longer than a well-defined cutoff. If the coefficients of the linear dispersion terms have opposite signs (hyperbolic case), then there is no such cutoff and as the wavelength decreases, the maximum growth rate approaches a well-defined limit.

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I. INTRODUCTION

The two-dimensional cubic nonlinear Schrödinger equation (NLS) is given by

$$i\psi_t+\alpha\psi_{xx}+\beta\psi_{yy}+\gamma|\psi|^2\psi=0, \tag{1}$$

where $\psi=\psi(x,y,t)$ is a complex-valued function, and α, β, and γ are real constants. All quantities are presented in unitless form. Among many other situations, NLS arises as an approximate model for the evolution of a nearly monochromatic wave of small amplitude in pulse propagation along optical fibers [1] where $\alpha\beta>0$, in gravity waves on deep water [2,3] where $\alpha\beta<0$ and in Langmuir waves in a plasma [4] where $\alpha\beta>0$. As a description of a superfluid [5], NLS is known as the Gross-Pitaevskii equation [6,7] with $\alpha\beta>0$. Sulem and Sulem [8] examine NLS in detail.

NLS admits a large class of one-dimensional traveling wave solutions of the form

$$\psi(x,y,t)=\phi(ax+by-st)e^{i\lambda t+ia\bar{s}x+ib\bar{s}y+i\eta}, \tag{2}$$

where ϕ is a real-valued function, $\bar{s}=s/(2\alpha a^2+2\beta b^2)$, and a, b, s, λ, and η are real parameters. By making use of the symmetries of NLS [8], all solutions of this form can be considered by studying the simplified form

$$\psi(x,y,t)=\phi(z)e^{i\kappa x+i\lambda t}, \tag{3}$$

where ϕ is a real-valued function, $z=x-2\alpha\kappa t$, and κ and λ are real parameters.

If $\alpha\gamma>0$, then NLS admits the following two solutions of form (3):

$$\phi(z)=\sqrt{2\frac{\alpha}{\gamma}}\,k\,\mathrm{cn}(z,k)\ \text{ with }\ \lambda=\alpha(2k^2-1-\kappa^2), \tag{4}$$

$$\phi(z)=\sqrt{2\frac{\alpha}{\gamma}}\,\mathrm{dn}(z,k)\ \text{ with }\ \lambda=\alpha(2-k^2-\kappa^2). \tag{5}$$

If $\alpha\gamma<0$, then NLS admits the following solution:

$$\phi(z)=\sqrt{-2\frac{\alpha}{\gamma}}\,k\,\mathrm{sn}(z,k)\ \text{ with }\ \lambda=-\alpha(1+k^2+\kappa^2). \tag{6}$$

Here $k\in[0,1]$ is a free parameter known as the elliptic modulus and $\mathrm{cn}(\cdot,k)$, $\mathrm{dn}(\cdot,k)$, and $\mathrm{sn}(\cdot,k)$ are Jacobi elliptic functions. Byrd and Friedman [9] provide a complete review of elliptic functions. If $k<1$, then each function $\phi(z)$ is periodic. As $k\rightarrow1$, the period of each increases without bound, and $\phi(z)$ limits to an appropriate hyperbolic function, which we call a "solitary wave."

These solutions, plus the "Stokes' wave" (plane wave)

$$\psi(x,t)=Ae^{i\kappa x-i(\alpha\kappa^2-\gamma|A|^2)t}, \tag{7}$$

contain the entire class of bounded traveling wave solutions of NLS with linear phase [10]. Davey and Stewartson [3] establish that a Stokes' wave is unstable unless either $\alpha\beta\gamma=0$ or $\alpha\beta>0$ and $\alpha\gamma<0$. In the remainder of this paper, we concentrate on Eqs. (4)–(6), and on their instabilities.

Zakharov and Rubenchik [11] establish that Eqs. (4) and (5) with $k=1$ are unstable with respect to long-wave transverse perturbations. Pelinovsky [12] reviews the stability of solitary wave solutions of NLS with $\alpha\beta<0$ and $\alpha\gamma>0$, and presents an analytical expression for the growth rate of the instability near a cutoff. Extensive reviews of the stability of solitary wave solutions are given in Refs. [13–15]. Martin, Yuen and Saffman [16] examine numerically the stability of the spatially periodic solution given in Eq. (5) for a range of parameters. Infeld and Ziemkiewicz [17] establish that all nonlinear (or "nontrivial") phase solutions are unstable with respect to long-wavelength perturbations. Aleshkevich *et al.* [18] and Kartashov *et al.* [19] numerically examine the stability of Eqs. (4)–(6) in NLS and establish some of the results obtained here.

We present four main results in this paper. First, every one-dimensional traveling wave with linear phase is unstable with respect to some infinitesimal perturbation with two-dimensional structure. For *all* choices of the parameters, there are unstable perturbations with long transverse wave-

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length. This result was originally presented in Ref. [17]. Our work generalizes the method and results of Ref. [11].

Second, if $\alpha\beta > 0$, then the only unstable perturbations have transverse wavelength longer than a well-defined cutoff.

Third, if $\alpha\beta < 0$, then there is no such cutoff. There are unstable perturbations with arbitrarily short wavelengths in both the transverse and longitudinal directions.

Fourth, for $\alpha\beta < 0$, the unstable perturbations with short wavelength have transverse wave numbers that are confined to narrower and narrower intervals as the transverse wave number grows without bound. In these unstable intervals, as the transverse wave number grows without bound, the maximum growth rate approaches a well-defined limit. As $k \to 1$, this limiting growth rate tends to zero if $\alpha\gamma > 0$, and to a finite nonzero limit if $\alpha\gamma < 0$.

II. STABILITY ANALYSIS

We consider perturbed solutions, $\psi_p = \psi_p(x,y,t)$, with the following structure:

$$\psi_p = [\phi(z) + \epsilon u(x,y,t) + i\epsilon v(x,y,t) + O(\epsilon^2)]e^{i\kappa x + i\lambda t}, \tag{8}$$

where $u(x,y,t)$ and $v(x,y,t)$ are real-valued functions, ϵ is a small real parameter, $z = x - 2\alpha\kappa t$, and $\phi(z)e^{i\kappa x + i\lambda t}$ is one of the solutions presented in the preceding section. Substituting Eq. (8) into Eq. (1), linearizing and separating into real and imaginary parts gives

$$-(\alpha\kappa^2 + \lambda)u + 3\gamma\phi^2 u + \beta u_{yy} + \alpha u_{xx} = v_t, \tag{9a}$$

$$-(\alpha\kappa^2 + \lambda)v + \gamma\phi^2 v + \beta v_{yy} + \alpha v_{xx} = -u_t. \tag{9b}$$

Without loss of generality, assume that $u(x,y,t)$ and $v(x,y,t)$ have the forms

$$u(x,y,t) = U(z,\rho)e^{i\rho y - \Omega t} + \text{c.c.}, \tag{10a}$$

$$v(x,y,t) = V(z,\rho)e^{i\rho y - \Omega t} + \text{c.c.}, \tag{10b}$$

where ρ is a real constant, Ω is a complex constant, U and V are complex-valued functions, and c.c. denotes complex conjugate. This leads to

$$(\alpha\kappa^2 + \lambda)U - 3\gamma\phi^2 U + \beta\rho^2 U - \alpha\partial_z^2 U = \Omega V, \tag{11a}$$

$$(\alpha\kappa^2 + \lambda)V - \gamma\phi^2 V + \beta\rho^2 V - \alpha\partial_z^2 V = -\Omega U. \tag{11b}$$

These are the central equations in this paper. We assume that U and V are periodic with the same period as ϕ. More general boundary conditions are discussed in Ref. [20]. Instability occurs if Eq. (11) admits a periodic solution with $\mathrm{Re}(\Omega) < 0$. Without loss of generality, for the remainder of this paper we assume $\kappa = 0$ by redefining λ.

In Secs. III and IV, we examine Eq. (11) using small-ρ and large-ρ asymptotic analyses, respectively. In Sec. V, we present results from a numerical study in which Eq. (11) was solved for a wide range of ρ values.

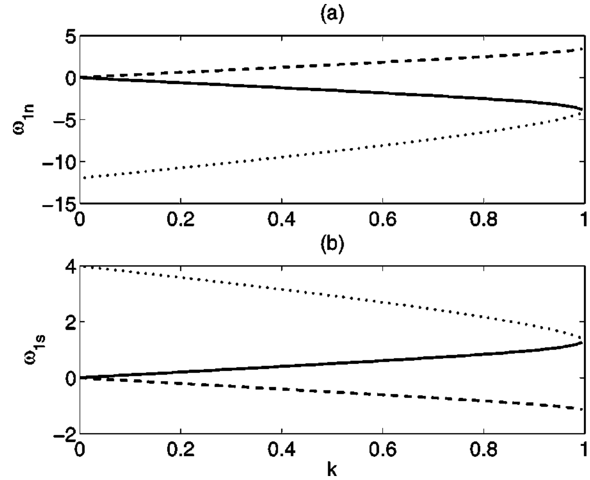

FIG. 1. Plots of ω_{1n} and ω_{1s} vs k in (a) and (b), respectively. The solid line corresponds to Eq. (4), the dotted line corresponds to Eq. (5), and the dashed line corresponds to Eq. (6).

III. SMALL-ρ LIMIT

Generalizing the work in Ref. [11], we assume that for fixed k and for fixed small ρ, Eq. (11) admits solutions of the form

$$U \sim u_0(z) + \rho u_1(z) + \rho^2 u_2(z) + \cdots, \tag{12a}$$

$$V \sim v_0(z) + \rho v_1(z) + \rho^2 v_2(z) + \cdots, \tag{12b}$$

$$\Omega^2 \sim \rho^2 \omega_1 + \rho^3 \omega_2 + \cdots, \tag{12c}$$

where ω_j are complex constants and u_j and v_j are complex-valued periodic functions with the same period as ϕ.

This assumption leads to the "neck" mode

$$U_n(z,\rho) = O(\rho), \tag{13a}$$

$$V_n(z,\rho) = \phi + O(\rho), \tag{13b}$$

$$\Omega_n^2 = -\alpha\beta\rho^2\omega_{1n} + O(\rho^3) \tag{13c}$$

and the "snake" mode

$$U_s(z,\rho) = \frac{d\phi}{dz} + O(\rho), \tag{14a}$$

$$V_s(z,\rho) = O(\rho), \tag{14b}$$

$$\Omega_s^2 = -\alpha\beta\rho^2\omega_{1s} + O(\rho^3), \tag{14c}$$

where ω_{1n} and ω_{1s} are functions of the elliptic modulus of the unperturbed solution. Complicated but exact expressions for ω_{1n} and ω_{1s} are derived in Ref. [20]. The final results are presented in Fig. 1, where we plot ω_{1n} and ω_{1s} (which determine the growth rates) versus k for Eqs. (4)–(6).

These plots establish that $\omega_{1n} < 0$ for Eqs. (4) and (5) and $\omega_{1s} < 0$ for Eq. (6). Therefore, if $\alpha\beta > 0$, Eqs. (4) and (5) are unstable with respect to long-wave transverse perturbations corresponding to the neck mode and Eq. (6) is unstable with respect to long-wave transverse perturbations corresponding to the snake mode.

These plots also establish that $\omega_{1s}>0$ for Eqs. (4) and (5) and $\omega_{1n}>0$ for Eq. (6). Therefore, if $\alpha\beta<0$, Eqs. (4) and (5) are unstable with respect to the snake mode and Eqs. (6) is unstable with respect to the neck mode.

It follows from these results that a linear phase solution of NLS is unstable to a growing neck mode if $\beta\gamma>0$, and to a growing snake mode if $\beta\gamma<0$.

IV. LARGE-ρ LIMIT

If $\alpha\beta>0$ and ρ is chosen to be large enough to satisfy

$$\rho^2>5\left|\frac{\alpha}{\beta}\right|, \tag{15}$$

then the two operators on the left hand side of Eq. (11) have the same sign, so $\Omega^2<0$. Therefore, there is no large-ρ instability if $\alpha\beta>0$.

If $\alpha\beta<0$ and ρ is large, then one can show that there is no instability unless $\Omega=O(1)$. Therefore, we assume

$$U\sim\zeta_1(\mu z)+\rho^{-2}\zeta_2(\mu z)+\cdots, \tag{16a}$$

$$V\sim\xi_1(\mu z)+\rho^{-2}\xi_2(\mu z)+\cdots, \tag{16b}$$

$$\Omega\sim w_1+\rho^{-2}w_2+\cdots, \tag{16c}$$

$$\alpha\mu^2=-\beta\rho^2+\nu+O(\rho^{-2}), \tag{16d}$$

where ν is a real constant, w_j are complex constants, and ζ_j and ξ_j are complex-valued periodic functions with the same period as ϕ.

Substituting Eq. (16) into Eq. (11), one finds at leading order

$$U\sim\zeta_{11}\sin(\mu z+z_0)+O(\rho^{-2}), \tag{17a}$$

$$V\sim\xi_{11}\sin(\mu z+z_0)+O(\rho^{-2}), \tag{17b}$$

where ζ_{11}, ξ_{11}, and z_0 are constants. Requiring U and V to have the same period as $\phi(z)$ forces μ to take on discrete values: $\mu=2\pi N/L$, where L is the period of $\phi(z)$, and N is an integer. To satisfy Eq. (16d), $N\gg1$. At the next order in ρ, solutions are periodic only if

$$w_1=\pm\sqrt{\gamma f-\lambda+\nu}\sqrt{\lambda-3\gamma f-\nu}, \tag{18}$$

where f is the Fourier coefficient [in $\sin(\mu z+z_0)$] of $\phi^2(z)\sin(\mu z+z_0)$ and ν is $O(1)$ but otherwise arbitrary. Minimizing the negative root in Eq. (18) with respect to ν leads to

$$w_{min}=-|\gamma f|, \tag{19}$$

when $\nu=(\lambda-2\gamma f)$. Then Eq. (16d) defines $\rho_{_N}$, the value of ρ at which the Nth unstable mode achieves its maximum growth rate:

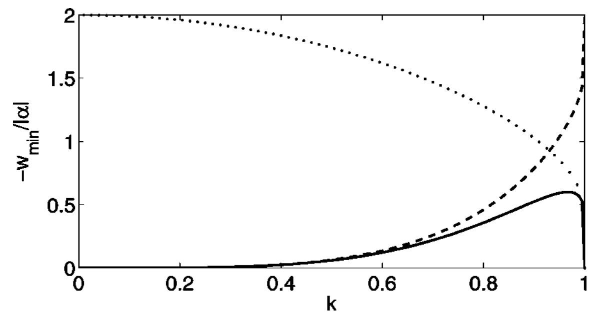

FIG. 2. Plots of $-w_{min}/|\alpha|$ vs k. The solid line corresponds to Eq. (4), the dotted line corresponds to Eq. (5), and the dashed line corresponds to Eq. (6).

$$-\beta\rho_{_N}^2=\alpha(2\pi N/L)^2+2\gamma f-\lambda. \tag{20}$$

We also find how far ρ can deviate from $\rho_{_N}$ before Ω^2 becomes negative

$$\delta\rho_{_N}\sim|\gamma|f/(2|\beta|\rho_{_N})=O(1/N) \quad \text{for} \quad N\gg1. \tag{21}$$

Analytic expressions for f corresponding to the solutions given in Eqs. (4)–(6) are not known. But, in the large-ρ limit, the Riemann-Lebesgue lemma [21] can be used to determine approximate expressions. As $\rho\to\infty$, for the solution given in Eq. (4),

$$f\sim\frac{1}{2K(k)}\left|\frac{\alpha}{\gamma}\right|\{E[\mathrm{am}(4K(k))]+4(k^2-1)K(k)\}, \tag{22}$$

for the solution given in Eq. (5),

$$f\sim\frac{1}{K(k)}\left|\frac{\alpha}{\gamma}\right|\{E[\mathrm{am}(2K(k))]\}, \tag{23}$$

and for the solution given in Eq. (6),

$$f\sim\frac{1}{2K(k)}\left|\frac{\alpha}{\gamma}\right|\{4K(k)-E[\mathrm{am}(4K(k))]\}. \tag{24}$$

In each of these expressions, $\mathrm{am}(\cdot)$ gives the Jacobi amplitude and $K(\cdot)$ and $E(\cdot)$ are the complete elliptic integrals of the first and second kind, respectively [9].

Plots of $-w_{min}/|\alpha|$, a growth rate, versus k are given in Fig. 2. This argument establishes that all finite-period one-dimensional linear phase solutions are unstable with respect to arbitrarily short wavelength transverse perturbations if $\alpha\beta<0$, and that the growth rate of the instability remains bounded as $\rho\to\infty$.

Note that as $k\to1$, $\phi(z)$ in Eq. (6) approaches a hyperbolic tangent and the growth rate approaches that of the Stokes' wave with an amplitude of $\sqrt{-2\alpha/\gamma}$. This establishes that there are an infinite number of unstable branches if $\alpha\beta<0$ and $\alpha\gamma<0$.

Also note that as $k\to1$, $\phi(z)$ in both Eqs. (4) and (5) approaches a hyperbolic secant, and the corresponding growth rate limits to zero. This establishes that there is no large-ρ instability in the solitary wave limit if $\alpha\gamma>0$.

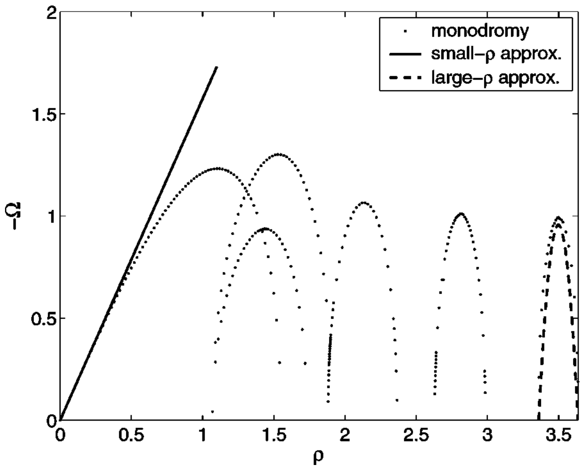

FIG. 3. Plots of $-\Omega$ vs ρ corresponding to Eq. (6) with $k=\sqrt{0.8}$ and $-\alpha=\beta=\gamma=1$. See text for a description.

V. MONODROMY

The system of equations in Eq. (11) is Hamiltonian in z, with periodic boundary conditions. The coordinates on the phase space are $p_1=dU/dz$, $p_2=-dV/dz$, $q_1=U$, and $q_2=V$. The Hamiltonian is

$$H=\frac{1}{2}(p_1^2-p_2^2)-\frac{1}{2\alpha}[\lambda+\beta\rho^2-3\gamma\phi^2(x)]q_1^2 + \frac{1}{2\alpha}(\lambda+\beta\rho^2-\gamma\phi^2)q_2^2+\frac{\Omega}{\alpha}q_1q_2. \tag{25}$$

Such a Hamiltonian system necessarily has a monodromy structure with invariants [22]. We used this structure to identify the periodic solutions of Eq. (11) by numerically integrating Eq. (11) over one period of ϕ. This numerical method is very similar to the numerical method used in Refs. [18,19].

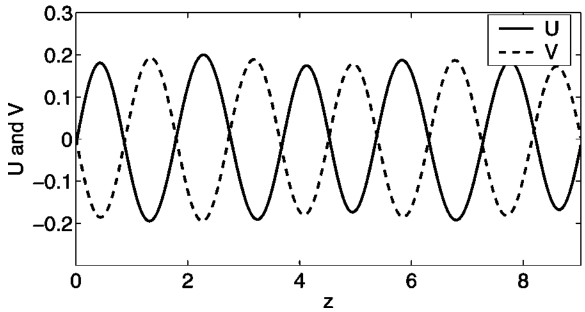

FIG. 4. Plots of U and V vs z corresponding to $\rho=3.5$, $\Omega=-0.99$, $N=5$.

The growth rates obtained from numerical simulations corresponding to Eq. (6) with $k=\sqrt{0.8}$ and $-\alpha=\beta=\gamma=1$ are included in Fig. 3 as dots. The line is obtained from the small-ρ results. The dashed curve is obtained from the large-ρ results with $N=5$. Each dotted curve corresponds to a different unstable mode. A plot of the spatial structure of the mode corresponding to $\rho=3.5$, $\Omega=-0.99$, and $N=5$ is given in Fig. 4.

Figure 3 demonstrates strong agreement between the numerical results and the small-ρ analysis when ρ is near zero. It also demonstrates agreement between the numerical results and the large-ρ analysis.

Figure 4 demonstrates that the U and V obtained numerically are similar in form to the U and V obtained in the large-ρ analysis.

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