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Interfacial nonequilibrium and Bénard-Marangoni instability of a liquid-vapor system

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We study Bénard-Marangoni instability in a system formed by a horizontal liquid layer and its overlying vapor. The liquid is lying on a hot rigid plate and the vapor is bounded by a cold parallel plate. A pump maintains a reduced pressure in the vapor layer and evacuates the vapor. This investigation is undertaken within the classical quasisteady approximation for both the vapor and the liquid phases. The two layers are separated by a deformable interface. Temporarily frozen temperature and velocity distributions are employed at each instant for the stability analysis, limited to infinitesimal disturbances (linear regime). We use irreversible thermodynamics to model the phase change under interfacial nonequilibrium. Within this description, the interface appears as a barrier for transport of both heat and mass. Hence, in contrast with previous studies, we consider the possibility of a temperature jump across the interface, as recently measured experimentally. The stability analysis shows that the interfacial resistances to heat and mass transfer have a destabilizing influence compared to an interface that is in thermodynamic equilibrium. The role of the fluctuations in the vapor phase on the onset of instability is discussed. The conditions to reduce the system to a one phase model are also established. Finally, the influence of the evaporation parameters and of the presence of an inert gas on the marginal stability curves is discussed.

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I. INTRODUCTION

Evaporative convection [1] is of particular interest in chemical engineering because of its importance in heat exchangers, distillation columns, and drying technologies. The driving mechanism of evaporation can be provided by the heating of a liquid layer from below or by maintaining a small vapor pressure gradient in a nonsaturated gas when the liquid is open to ambient air. During evaporation, an essential mechanism is latent heat consumption that leads to intensive cooling of the liquid-vapor interface. This is equivalent to consider a liquid layer cooled from above. The reduction of temperature near the free surface makes the liquid layer more unstable. In turn, the onset of instability leads to an enhancement of evaporation rate.

To model interfacial heat and mass transfer, it is conventional to use an interfacial no-slip condition together with the interfacial thermal and chemical potential equilibrium condition [2,3]. This is referred to as the interfacial equilibrium assumption. Interfacial chemical potential equilibrium means that the temperature of the liquid is at its saturation value with respect to the pressure in the vapor. The two last assumptions are nevertheless questionable. Indeed, more generally, a kinetic relation like the well-known Hertz-Knudsen law [4] should be used in place of the chemical potential equilibrium condition. The Hertz-Knudsen relation predicts a mass flux through the interface proportional to the difference between the pressure of the vapor and its saturation value corresponding to the temperature in the liquid. It is only when the kinetic evaporation coefficient is infinite that interfacial equilibrium is recovered [5]. The interfacial nonequilibrium effect is especially important for a liquid undergoing rapid evaporation at reduced pressure [6–8]. Moreover, a temperature discontinuity at the liquid-vapor interface has been measured at reduced pressure by Shankar and Deshpande [9]. Recently, Fang and Ward [10] have experimentally confirmed the existence of such a temperature discontinuity at the interface during steady-state liquid evaporation. Huang and Joseph [11] have proposed interfacial relations that account for this interfacial jump of temperature but their relations have been postulated without firm kinetic or thermodynamic bases and, in addition, they disregarded Hertz-Knudsen's relation.

A more systematic and general thermodynamical modeling of heat and mass transfer through interfaces during phase changes under conditions of interfacial nonequilibrium was proposed by Bedeaux [12]. In this approach, based on classical irreversible thermodynamics, the vapor-liquid interface is viewed as a separate phase in local equilibrium. It has the advantage to yield interfacial relations that naturally cope with the possibility of a jump of temperature across the interface and that generalize the Hertz-Knudsen relation. Moreover, this theory is in agreement with the kinetic theory [13].

The purpose of this work is to study the role of such interfacial nonequilibrium effects on Bénard-Marangoni instability, during evaporation of a liquid into its overlaying vapor; this will be achieved within the framework of Bedeaux's theory. It is worth valuable to study the onset of convection and to compare these conditions with the experimental results because it gives more insight on the validity of the thermodynamical model as a predictive tool.

The structure of the paper is the following. In Sec. II the set of governing equations is established, and some physical parameters for water and its vapor are introduced. Section III

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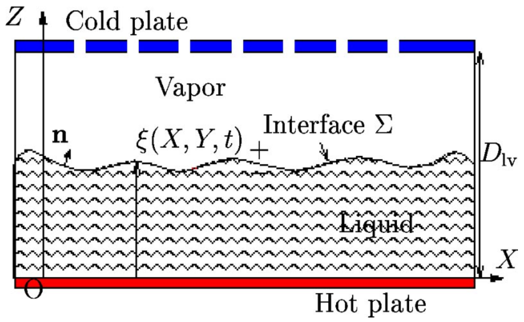

FIG. 1. Configuration.

deals with the study of some basic flow configurations under the classical quasisteady assumption for both the vapor and the liquid phases. A linear stability analysis is presented in Sec. IV. The particular case of no-flow heat flux evaporation in presence of an inert gas is investigated to examine the influence of gas phase diffusion on interfacial nonequilibrium situations. Finally, conclusions are drawn in Sec. V.

II. GOVERNING EQUATIONS

A. Configuration

Consider the general setup sketched in Fig. 1. The liquid is lying on a hot horizontal plate and its vapor is bounded by a parallel cold plate. Let D_{lv} be the distance between the two plates. The two horizontal layers are unbounded in the horizontal $(\mathbf{e}_X,\mathbf{e}_Y)$ directions. The vertical coordinate $\mathbf{e}_Z$ is taken to be directed from the liquid into the vapor, opposite to the uniform gravity acceleration $\mathbf{g}$. The hot lower plate is impermeable but vapor can be evacuated through the cold upper plate by a pump that maintains a reduced pressure in the vapor layer. It is convenient to use the orthonormal frame $(O,\mathbf{e}_X,\mathbf{e}_Y,\mathbf{e}_Z)$ with the origin O at the bottom plate, (X,Y,Z) are the Cartesian coordinates and t the time. We use subscripts l and v for liquid and vapor, while subscripts u and b design the upper and bottom surfaces, respectively. The equation of the liquid-vapor interface Σ is given by $Z-\xi(X,Y,t)=0$. The unit vector $\mathbf{n}$ normal to the interface Σ, and directed towards the vapor, is given by $\mathbf{n}=(-\boldsymbol{\nabla}_1\xi+\mathbf{e}_Z)/N$ where $\boldsymbol{\nabla}_1=(\partial_X,\partial_Y)$ and $N=\sqrt{1+|\boldsymbol{\nabla}_1\xi|^2}$. Let $\mathbf{t}_1=(\mathbf{e}_X+\partial_X\xi\mathbf{e}_Z)/N_X$ and $\mathbf{t}_2=(\mathbf{e}_Y+\partial_Y\xi\mathbf{e}_Z)/N_Y$ be unit vectors tangent to the interface, with $N_X=\sqrt{1+(\partial_X\xi)^2}$ and $N_Y=\sqrt{1+(\partial_Y\xi)^2}$. The interface normal velocity is given by $\mathbf{v}_\Sigma\cdot\mathbf{n}=\partial_t\xi/N$. Some quantities may be discontinuous across the liquid-vapor interface. We denote by a_+, a_- the values of any quantity a at the interface Σ, respectively, in the vapor and the liquid phases with $[[a]]=a_+-a_-$ being the jump of the quantity a through the interface. The velocity vector is $\mathbf{v}_{\mathrm{k}}=(u_{\mathrm{k}},v_{\mathrm{k}},w_{\mathrm{k}})$, $v_{n\mathrm{k}}=\mathbf{v}_{\mathrm{k}}\cdot\mathbf{n}$, and $\mathbf{v}_{\parallel\mathrm{k}}=\mathbf{v}_{\mathrm{k}}-v_{n\mathrm{k}}\mathbf{n}$ $(k=l,v)$.

The governing equations are the mass, momentum, and energy balance equations in both the liquid and vapor phases, and at the liquid-vapor interface.

B. Phase equations and boundary conditions

The vapor is assumed to be a gas at low Mach number so that its *thermodynamic pressure* P_{v} may be taken as constant [14]. Density is then a function of temperature only. The liquid, a Newtonian one, is assumed to have the same property. Moreover, one considers small variations of temperature across the liquid and vapor phases so that the *Boussinesq approximation* is valid for both the vapor and the liquid phases [14]. The vapor and liquid balance equations then reduce to the following equations.

(a) Vapor phase equations:

$$\boldsymbol{\nabla}\cdot\mathbf{v}_{\mathrm{v}}=0, \tag{1a}$$

$$\rho_{\mathrm{v}}\left(\frac{\partial\mathbf{v}_{\mathrm{v}}}{\partial t}+\mathbf{v}_{\mathrm{v}}\cdot\boldsymbol{\nabla}\mathbf{v}_{\mathrm{v}}\right)=-\boldsymbol{\nabla}p_{\mathrm{v}}+\eta_{\mathrm{v}}\Delta\mathbf{v}_{\mathrm{v}}-\rho_{\mathrm{v}}g[1-\alpha_{T\mathrm{v}}(T_{\mathrm{v}}-T_{\mathrm{r}})]\mathbf{e}_Z, \tag{1b}$$

$$\frac{\partial T_{\mathrm{v}}}{\partial t}+\mathbf{v}_{\mathrm{v}}\cdot\boldsymbol{\nabla}T_{\mathrm{v}}=\kappa_{\mathrm{v}}\Delta T_{\mathrm{v}}. \tag{1c}$$

Here T_{v} and p_{v} denote the temperature and *dynamical pressure* fields, while ρ_{v} is the constant density taken at a given reference temperature T_{r}. The coefficients α_T, η, and κ are, respectively, the volumetric expansion coefficient, the dynamic viscosity, and the heat diffusivity, $\Delta=\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ is the Laplacian operator.

(b) Liquid phase equations:

$$\boldsymbol{\nabla}\cdot\mathbf{v}_{\mathrm{l}}=0, \tag{2a}$$

$$\rho_{\mathrm{l}}\left(\frac{\partial\mathbf{v}_{\mathrm{l}}}{\partial t}+\mathbf{v}_{\mathrm{l}}\cdot\boldsymbol{\nabla}\mathbf{v}_{\mathrm{l}}\right)=-\boldsymbol{\nabla}p_{\mathrm{l}}+\eta_{\mathrm{l}}\Delta\mathbf{v}_{\mathrm{l}}-\rho_{\mathrm{l}}g[1-\alpha_{T\mathrm{l}}(T_{\mathrm{l}}-T_{\mathrm{r}})]\mathbf{e}_Z, \tag{2b}$$

$$\frac{\partial T_{\mathrm{l}}}{\partial t}+\mathbf{v}_{\mathrm{l}}\cdot\boldsymbol{\nabla}T_{\mathrm{l}}=\kappa_{\mathrm{l}}\Delta T_{\mathrm{l}}, \tag{2c}$$

where ρ_{l} is assumed to be constant.

Within the Boussinesq approximation, we have $|T_{\mathrm{v}}-T_{\mathrm{r}}|\ll 1/\alpha_{T\mathrm{v}}$ and $|T_{\mathrm{l}}-T_{\mathrm{r}}|\ll 1/\alpha_{T\mathrm{l}}$.

(c) Wall boundary conditions: At the rigid, impermeable liquid-hot plate, $Z=0$, with imposed temperature T_{b} one has

$$u_{\mathrm{l}}=v_{\mathrm{l}}=w_{\mathrm{l}}=0, \tag{3a}$$

$$T_{\mathrm{l}}=T_{\mathrm{b}}. \tag{3b}$$

At the rigid permeable vapor-cold plate, $Z=D_{\mathrm{lv}}$ with imposed temperature T_{u} and vertical velocity w_{u}, one has

$$w_{\mathrm{v}}=w_{\mathrm{u}}, \tag{4a}$$

$$u_{\mathrm{v}}=v_{\mathrm{v}}=0, \tag{4b}$$

$$T_{\mathrm{v}}=T_{\mathrm{u}}. \tag{4c}$$

(d) Initial conditions: At $t=0$ the depth of the liquid layer is given by D_{l} and the initial temperature and velocity profiles are supposed to be independent of the (X,Y) coordinates and to be known.

C. Interfacial equations

The set of equations describing the liquid-vapor evaporation must be complemented by interfacial relations. These expressions will be either interfacial balance equations or constitutive relations. Here, we use Bedeaux's [12] description of heat and mass transfer through interfaces to model the phase changes under interfacial nonequilibrium conditions.

Let $\mathcal{J}=\rho_{\rm l}(\mathbf{v}_{\rm l}\cdot\mathbf{n}-\mathbf{v}_{\Sigma}\cdot\mathbf{n})$ be the interfacial mass flux relative to the motion of the interface and $J_{\rm q\pm}=-\lambda_{\rm k}\boldsymbol{\nabla}T_{\rm k\pm}\cdot\mathbf{n}$ the normal component of the heat flux at the interface with $\lambda_{\rm k}$ the conductivity of phase k. The only material property of the interface which is taken into account in this study is the surface tension. We consider that the *slow evaporation approximation* is valid, i.e., we neglect the viscous dissipation and molecular kinetic energy in the energy balance and the vapor recoil term in the normal momentum balance. Then the interfacial balance of mass, momentum, and energy equations are [12]

$$\mathcal{J}=\rho_{\rm v}(\mathbf{v}_{\rm v}\cdot\mathbf{n}-\mathbf{v}_{\Sigma}\cdot\mathbf{n}), \tag{5a}$$

$$0=-[[p]]\mathbf{n}-\gamma C\mathbf{n}+[[\mathbf{n}\cdot\boldsymbol{\tau}]]+\boldsymbol{\nabla}_{\parallel}\gamma, \tag{5b}$$

$$[[J_{\rm q}]]=-[[h]]\mathcal{J}, \tag{5c}$$

where γ is the surface tension, $\boldsymbol{\tau}$ is viscous stress tensor, $C=\boldsymbol{\nabla}\cdot\mathbf{n}$ is the curvature of the interface, and h is the specific enthalpy. The surface tension γ is only a function of the interfacial temperature which is chosen to be equal to the liquid side temperature T_- [12]. In the linear approximation, one has

$$\gamma=\gamma_{\rm r}-\gamma_{\rm r}^{s}(T_--T_{\rm r}), \tag{6}$$

with $\gamma_{\rm r}$ and $\gamma_{\rm r}^{s}$ reference values evaluated at $T=T_{\rm r}$.

We now look at the restrictions placed by the second law of thermodynamics stating that entropy production σ is positive definite. Designating by s the entropy density and by $\mathbf{J}_s$ its entropy flux, the entropy production is defined by [15]

$$\sigma=\frac{\partial\rho s}{\partial t}+\boldsymbol{\nabla}\cdot(\rho s\mathbf{v}+\mathbf{J}_s), \tag{7}$$

where the fields are distribution functions because they contain singular contributions at the liquid-vapor interface. Under the slow evaporation approximation and interfacial no-slip condition, it is checked [16] that the interfacial entropy production σ_s is

$$\sigma_s=J_{\rm q+}[[1/T]]-\mathcal{J}F, \tag{8}$$

with F the thermodynamical force corresponding to the thermodynamic flux $\mathcal{J}$. The quantity F is given by $F=T_-[[-h_+/T+\mu/T]]$, where μ is the chemical potential. The associated interfacial constitutive relations are then

$$J_{\rm q+}=l_{11}[[1/T]]-l_{12}F, \tag{9a}$$

$$\mathcal{J}=l_{21}[[1/T]]-l_{22}F, \tag{9b}$$

where l_{ij} are the phenomenological coefficients with $l_{12}=l_{21}/T_{\rm r}$ according to Onsager-Casimir reciprocity relations [15]. The coefficients l_{11} and l_{22} are positive. These interfacial constitutive relations generalize the usual interfacial equilibrium relations $T_+=T_-$ and $\mu_+=\mu_-$ that are recovered if we assume that the coefficients l_{11} and l_{22} are infinite. When the constitutive coefficients are finite, the interface appears as a barrier for the transport of both heat and mass. The interfacial coefficients are function of the interfacial temperature T_-. However, at a first approximation, they can be considered as constant; in analogy with the heat conductivity and the viscosity, we will consider the value of the interfacial coefficients the one corresponding to the equilibrium temperature $T_{\rm r}$.

Here, we consider evaporation close to a global equilibrium state $(P_{\rm r},T_{\rm r})$ taken as the reference state. Then the pressure $P_{\rm r}$ and the temperature $T_{\rm r}$ must verify the relation $\mathcal{P}(T_{\rm r})=P_{\rm r}$, where $\mathcal{P}$ is the liquid-vapor saturation law. Close to this equilibrium state, $[[h]]$ is assumed constant and equal to the heat of evaporation at equilibrium $L_{\rm lv}(T_{\rm r})$. Moreover, $[[1/T]]\approx-[[T]]/T_{\rm r}^2$ and $F\approx R_*T_{\rm r}\ln[P_+/\mathcal{P}(T_-)]$ as explicitly shown in Ref. [17], R_* is the ideal gas constant divided by the vapor molar mass. When evaporation takes place close to a global equilibrium with $l_{11}=\infty$, relation (9b) reduces in the linear approximation to the Hertz-Knudsen relation

$$\mathcal{J}=\frac{\beta}{\sqrt{2\pi R_*T_-}}[\mathcal{P}(T_-)-P_+],$$

with the accommodation coefficient β given by $\beta=l_{22}R_*T_{\rm r}\sqrt{2\pi R_*T_{\rm r}}/P_{\rm r}$. On the other hand, if, for the saturation law, we use the Clausius-Clapeyron equation

$$\mathcal{P}(T)=P_{\rm r}\exp\left[-\frac{L_{\rm lv}(T_{\rm r})}{R_*}\left(\frac{1}{T}-\frac{1}{T_{\rm r}}\right)\right]$$

then $F=-L_{\rm lv}(T_{\rm r})(T_--T_{\rm sat})/T_{\rm r}$, where the saturation temperature $T_{\rm sat}$ is defined by $\mathcal{P}(T_{\rm sat})=P_+$. Under these conditions the set of equations (9) can be rewritten as

$$J_{\rm q+}=-\frac{l_{11}}{T_{\rm r}^2}[[T]]+\frac{l_{12}L_{\rm lv}(T_{\rm r})}{T_{\rm r}}(T_--T_{\rm sat}), \tag{10a}$$

$$\mathcal{J}=-\frac{l_{21}}{T_{\rm r}^2}[[T]]+\frac{l_{22}L_{\rm lv}(T_{\rm r})}{T_{\rm r}}(T_--T_{\rm sat}), \tag{10b}$$

where the saturation temperature is a constant because the thermodynamic vapor pressure is assumed constant. It is then appropriate to choose this pressure as the reference state $P_{\rm r}=P_+$, so that $T_{\rm sat}=T_{\rm r}$.

The linearized law (10b) was proved to be satisfied for the study of dissipative systems when the temperature is continuous across the interface provided that the temperature remains close to the reference temperature (cf. Ref. [8]). There is no reason not to use our generalized linearized eqs. (10a) and (10b) in presence of a temperature jump at the interface.

TABLE I. Bulk dimensionless parameters.

Liquid Prandtl number	$\mathrm{Pr}_{\mathrm{l}}=\frac{\eta_{\mathrm{l}}}{\rho_{\mathrm{l}}\kappa_{\mathrm{l}}}$	Froude number	$\mathrm{Fr}_{\mathrm{l}}=\frac{\kappa_{\mathrm{l}}^{2}}{gD_{\mathrm{l}}^{3}}$
Density ratio	$\rho=\frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{l}}}$	Rayleigh number	$\mathrm{Ra}_{\mathrm{l}}=\frac{\rho_{\mathrm{l}}g\alpha_{T\mathrm{l}}D_{\mathrm{l}}^{3}(T_{\mathrm{b}}-T_{\mathrm{sat}})}{\eta_{\mathrm{l}}\kappa_{\mathrm{l}}}$
Dynamic viscosity ratio	$\eta=\frac{\eta_{\mathrm{v}}}{\eta_{\mathrm{l}}}$	Depth ratio	$D=\frac{D_{\mathrm{lv}}-D_{\mathrm{l}}}{D_{\mathrm{l}}}$
Diffusivity ratio	$\kappa=\frac{\kappa_{\mathrm{v}}}{\kappa_{\mathrm{l}}}$	Thermal ratio	$T=\frac{T_{\mathrm{u}}-T_{\mathrm{sat}}}{T_{\mathrm{b}}-T_{\mathrm{sat}}}$
Conductivity ratio	$\lambda=\frac{\lambda_{\mathrm{v}}}{\lambda_{\mathrm{l}}}$		$W_{\mathrm{u}}=\frac{w_{\mathrm{u}}D_{\mathrm{l}}}{\kappa_{\mathrm{l}}}$
Volumetric expansion ratio	$\alpha_T=\frac{\alpha_{T\mathrm{v}}}{\alpha_{T\mathrm{l}}}$	Capacity ratio	$C_p=\frac{\lambda}{\rho\kappa}$

D. Dimensionless equations

We choose D_{l}, $D_{\mathrm{l}}^{2}/\kappa_{\mathrm{l}}$, $\kappa_{\mathrm{l}}/D_{\mathrm{l}}$, $\eta_{\mathrm{l}}\kappa_{\mathrm{l}}/D_{\mathrm{l}}^{2}$, and $T_{\mathrm{b}}-T_{\mathrm{sat}}$ to scale the length, time, velocity, pressure, and temperature, respectively. In particular, the temperatures T_{l} and T_{v} are made dimensionless by $T_{\mathrm{l}}\rightarrow(T_{\mathrm{l}}-T_{\mathrm{sat}})/(T_{\mathrm{b}}-T_{\mathrm{sat}})$ and $T_{\mathrm{v}}\rightarrow(T_{\mathrm{v}}-T_{\mathrm{sat}})/(T_{\mathrm{b}}-T_{\mathrm{sat}})$. Interfacial mass flux $\mathcal{J}$, heat flux, viscosity tensor, and surface tension are scaled by $\rho_{\mathrm{l}}\kappa_{\mathrm{l}}/D_{\mathrm{l}}$, $\lambda_{\mathrm{l}}(T_{\mathrm{b}}-T_{\mathrm{sat}})/D_{\mathrm{l}}$, $\eta_{\mathrm{l}}\kappa_{\mathrm{l}}/D_{\mathrm{l}}^{2}$, and γ_{r}, respectively. As a consequence, the dimensionless equations are expressed as follows.

(a) Vapor phase equations:

$$\boldsymbol{\nabla}\cdot\mathbf{v}_{\mathrm{v}}=0, \tag{11a}$$

$$\rho\left(\frac{\partial\mathbf{v}_{\mathrm{v}}}{\partial t}+\mathbf{v}_{\mathrm{v}}\cdot\boldsymbol{\nabla}\mathbf{v}_{\mathrm{v}}\right)=\mathrm{Pr}_{\mathrm{l}}(-\boldsymbol{\nabla}p_{\mathrm{v}}+\eta\Delta\mathbf{v}_{\mathrm{v}})+\rho\left(\alpha_T\mathrm{Pr}_{\mathrm{l}}\mathrm{Ra}_{\mathrm{l}}T_{\mathrm{v}}-\frac{1}{\mathrm{Fr}_{\mathrm{l}}}\right)\mathbf{e}_Z, \tag{11b}$$

$$\frac{\partial T_{\mathrm{v}}}{\partial t}+\mathbf{v}_{\mathrm{v}}\cdot\boldsymbol{\nabla}T_{\mathrm{v}}=\kappa\Delta T_{\mathrm{v}}. \tag{11c}$$

(b) Liquid phase equations:

$$\boldsymbol{\nabla}\cdot\mathbf{v}_{\mathrm{l}}=0, \tag{12a}$$

$$\frac{\partial\mathbf{v}_{\mathrm{l}}}{\partial t}+\mathbf{v}_{\mathrm{l}}\cdot\boldsymbol{\nabla}\mathbf{v}_{\mathrm{l}}=\mathrm{Pr}_{\mathrm{l}}(-\boldsymbol{\nabla}p_{\mathrm{l}}+\Delta\mathbf{v}_{\mathrm{l}})+\left(\mathrm{Pr}_{\mathrm{l}}\mathrm{Ra}_{\mathrm{l}}T_{\mathrm{l}}-\frac{1}{\mathrm{Fr}_{\mathrm{l}}}\right)\mathbf{e}_Z, \tag{12b}$$

$$\frac{\partial T_{\mathrm{l}}}{\partial t}+\mathbf{v}_{\mathrm{l}}\cdot\boldsymbol{\nabla}T_{\mathrm{l}}=\Delta T_{\mathrm{l}}. \tag{12c}$$

(c) Wall boundary conditions.

At the liquid-hot plate $Z=0$:

$$u_{\mathrm{l}}=v_{\mathrm{l}}=w_{\mathrm{l}}=0, \tag{13a}$$

$$T_{\mathrm{l}}=1. \tag{13b}$$

At the vapor-cold plate $Z=1+D$:

$$w_{\mathrm{v}}=W_{\mathrm{u}}, \tag{14a}$$

$$u_{\mathrm{v}}=v_{\mathrm{v}}=0, \tag{14b}$$

$$T_{\mathrm{v}}=T. \tag{14c}$$

At the interface $Z=\xi$:

$$\mathcal{J}=\rho(\mathbf{v}_{\mathrm{v}}\cdot\mathbf{n}-\mathbf{v}_{\Sigma}\cdot\mathbf{n}), \tag{15a}$$

$$\mathrm{Cr}(p_{-}-p_{+})=(1-\mathrm{MaCr}T_{-})C+\mathrm{Cr}[\mathbf{n}\cdot\boldsymbol{\tau}_{-}\cdot\mathbf{n}-\eta\mathbf{n}\cdot\boldsymbol{\tau}_{+}\cdot\mathbf{n}], \tag{15b}$$

$$(\mathbf{n}\cdot\boldsymbol{\tau}_{-})\cdot\mathbf{t}=\eta(\mathbf{n}\cdot\boldsymbol{\tau}_{+})\cdot\mathbf{t}-\mathrm{Ma}\boldsymbol{\nabla}_{\parallel}T_{-}, \tag{15c}$$

$$J_{\mathrm{q}-}=J_{\mathrm{q}+}+\frac{1}{Ku}\mathcal{J}, \tag{15d}$$

and

$$T_{+}=(1-H_c)T_{-}-H_rJ_{\mathrm{q}+}, \tag{16a}$$

$$\frac{1}{H_j}T_{-}=H_cJ_{\mathrm{q}+}+\frac{1}{\mathrm{Ku}}\mathcal{J}, \tag{16b}$$

$$\mathbf{v}_{\parallel+}=\mathbf{v}_{\parallel-}, \tag{16c}$$

with $\mathbf{t}=(\mathbf{t}_1,\mathbf{t}_2)$, $\mathcal{J}=\mathbf{v}_{\mathrm{l}}\cdot\mathbf{n}-\mathbf{v}_{\Sigma}\cdot\mathbf{n}$, $J_{\mathrm{q}-}=-\boldsymbol{\nabla}T_{\mathrm{l}}\cdot\mathbf{n}$, and $J_{\mathrm{q}+}=-\lambda\boldsymbol{\nabla}T_{\mathrm{v}}\cdot\mathbf{n}$.

(d) Interfacial kinematic condition:

$$\mathbf{v}_{\Sigma}\cdot\mathbf{n}=\frac{\partial_t\xi}{\sqrt{1+|\boldsymbol{\nabla}_1\xi|^2}}. \tag{17}$$

(e) Initial conditions:

TABLE II. Interfacial dimensionless parameters.

Crispation number	$\mathrm{Cr}=\frac{\eta_l\kappa_l}{\gamma_r D_l}$		$H_r=\frac{T_r^2\lambda_l}{l_{11}D_l}$
Marangoni number	$\mathrm{Ma}=\frac{\gamma_r^s D_l(T_b-T_{sat})}{\eta_l\kappa_l}$	Transfer parameters	$k_j=\frac{l_{22}L_{lv}(T_r)^2 D_l}{T_r\lambda_l}$
Kutateladze number	$\mathrm{Ku}=\frac{C_{pl}(T_b-T_{sat})}{L_{lv}(T_r)}$		$k_h=\frac{l_{12}}{l_{22}L_{lv}(T_r)}$

$$\xi=1,\quad \text{temperature and velocity profiles independent of the } (X,Y) \text{ coordinates.} \tag{18}$$

The dimensionless parameters that appear in the above equations are defined in Tables I and II.

For water at 45 °C, $\kappa=120$, $\mathrm{Pr}_l=1.74$, $C_p=0.5$, $\eta=4.10^{-2}$, $\lambda=3.5\ 10^{-2}$, $\alpha_T=2.7\ 10^{-2}$, $\rho=6\times10^{-4}$. For an initial liquid layer depth $D_l=6$ mm, it is found that $\mathrm{Cr}=1.3\times10^{-7}$ which means that the crispation number is very small. Moreover, $\mathrm{Ku}/\mathrm{Ma}=8.2\times10^{-8}$ so that the Kutateladze number is very small as regard to the Marangoni number.

The values of the interfacial transfer coefficients at 0 °C have been determined by Bedeaux and Kjelstrup [17] from the experiments of Fang and Ward [10] on water evaporation into its own vapor. In these experiences, no convection is observed. Moreover, from steady experimental situations, one can only find two of the three independent interfacial transfer parameters H_r, k_h, and k_j. Therefore, following Ref. [17], we assume that the dimensionless coupling coefficient k_h will be given *a priori*. Two cases are investigated: $k_h=0$, i.e., no coupling and $k_h=0.18$ as derived from the kinetic theory of gases. We have also used the scaling coefficients proposed in Ref. [17] to obtain reasonable estimates of these coefficients at the reference temperature $T_r=45\,°\mathrm{C}$. The corresponding values for the interfacial dimensionless transfer parameters are reported in Table III.

The values of these coefficients are still acceptable for a fluid whose interface is moving because, according to the thermodynamical model, the interfacial coefficients are independent of the displacement of the fluid-gas interface but depend only on the temperature. Moreover, as proved in Sec. IV, the qualitative conclusions of our analysis are rather insensitive to the values of these interfacial parameters.

For further purpose, we introduce two extra interfacial parameters H_c and H_j given by $H_c=k_hk_jH_r$ and $H_j=[1-H_c^2/(k_jH_r)]^{-1}k_j^{-1}$. The parameter H_c is independent of the liquid layer depth D_l. It is of the order of one or smaller.

TABLE III. Interfacial dimensionless transfer parameter values ($D_l=6$ mm, $T_r=45\,°\mathrm{C}$).

k_h	0.18	0
H_r	4.2	3.3
k_j	4.8	21

Parameters H_r and H_j are proportional to $1/D_l$ so that their values are larger when the liquid depth is smaller.

From an experimental point of view, the temperatures T_b, T_u, and T_{sat}, the distance D_{lv} between the two plates, initial liquid layer depth D_l, and the top cold plate velocity w_u are parameters of the problem. It is convenient [18] to define two new dimensionless numbers α and Λ related to Ra and Ma by

$$\mathrm{Ra}=(\mathrm{Ra}_0)\,\alpha_g\Lambda, \tag{19a}$$

$$\mathrm{Ma}=(\mathrm{Ma}_0)(1-\alpha_g)\Lambda, \tag{19b}$$

where α_g is related to α by

$$\frac{1}{\alpha_g}-1=\frac{g_0}{g}\left(\frac{1}{\alpha}-1\right), \tag{20}$$

and where Ra_0, Ma_0 are two arbitrary numbers. Here we choose Ra_0 as the critical Rayleigh number for pure buoyancy instability and Ma_0 as the critical Marangoni number for pure thermocapillarity instability, g_0 is the acceleration at the earth surface due to gravity. The quantity α is a measure of the liquid depth parameter D_l since $1/\alpha-1=(\mathrm{Ra}_0/\mathrm{Ma}_0)\gamma_r^s/(\rho_l g_0\alpha_{Tl}D_l^2)$, while Λ is related to T_b-T_{sat}. It is convenient to take the acceleration g due to gravity as a variable parameter. Under microgravity ($g=0$), the parameter α_g is null whatever the value of the initial liquid depth D_l and cannot therefore be used to measure the depth of the liquid layer. In contrast the parameter α is not zero even for $g=0$ and is therefore the appropriate candidate for evaluating the liquid depth. The dimensionless parameters T, Ku, Fr, Cr, H_j and H_r are linked to α and Λ via coefficients independent of the liquid depth, for instance,

$$\mathrm{Ku}=\mathrm{Ku}_n\mathrm{Ma}\sqrt{1/\alpha-1}\sqrt{\mathrm{Ma}_0/\mathrm{Ra}_0} \tag{21}$$

with $\mathrm{Ku}_n=(\sqrt{\rho_l g_0\alpha_{Tl}}C_{pl}\eta_l\kappa_l)/(\gamma_r^{s\,3/2}L_{lv})$. The new dimensionless parameters α and Λ are preferred to the classical Rayleigh and Marangoni numbers for the stability study. Indeed, they are directly related to the relevant experimental data, namely, the temperature drop T_b-T_{sat} and the liquid depth parameter D_l.

III. QUASISTEADY BASIC SOLUTION

When solving the set of equations (11)–(18), one finds a basic one-dimensional unsteady solution independent of the (X,Y) coordinates. Here, we shall work in the frame of the *quasisteady assumption* for which the time derivatives in the partial differential equations both in the liquid and vapor phases are neglected but not in the interfacial kinematic condition (17). This assumption is satisfied after a short time, during which the liquid depth remains constant, because of the large value of the evaporation heat [8].

The incompressibility condition leads to a null velocity in the liquid phase, which is therefore in a purely conductive state, and a constant velocity in the vapor phase. As a consequence, the interfacial balance of mass reads as $W_{\mathrm{u}}=(1/\rho-1)\mathcal{J}$. The value of the velocity W_{u} at the top cold plate must assume the particular value given by the solution of the quasisteady problem.

Within the quasisteady assumption, the temperature profiles are

$$T_{\mathrm{l}}=1-\frac{1-T_-}{\xi}Z, \tag{22a}$$

$$T_{\mathrm{v}}=T+(T_+-T)\frac{1-\exp\left[-\dfrac{C_p\mathcal{J}}{\lambda}(1+D-Z)\right]}{1-\exp\left[-\dfrac{C_p\mathcal{J}}{\lambda}(1+D-\xi)\right]}, \tag{22b}$$

after using the approximation $(1/\rho)-1\approx 1/\rho$. In addition,

$$J_{\mathrm{q}-}=\frac{1-T_-}{\xi}, \tag{23a}$$

$$J_{\mathrm{q}+}=\frac{\lambda(T_+-T)}{1+D-\xi}f\left(\frac{C_p\mathcal{J}(1+D-\xi)}{\lambda}\right), \tag{23b}$$

with $f(x)=x/(\exp x-1)$.

As a consequence, the interfacial relations (15d), (16a), and (16b) constitute a set of three equations for the three unknowns $\mathcal{J}$, T_-, and T_+, parametrized by the liquid layer depth $\xi(t)$. The variation of the liquid layer depth is given by the interfacial kinematic condition $\mathcal{J}=-d\xi/dt$.

A. One-sided model

If we assume that the heat flux term $J_{\mathrm{q}+}$ is negligible in relations (15d) and (16b), i.e.,

$$J_{\mathrm{q}+}\ll\mathcal{J}/\mathrm{Ku} \quad\text{and}\quad J_{\mathrm{q}+}\ll\mathcal{J}/(\mathrm{Ku}H_c), \tag{24}$$

we obtain a liquid phase problem that is uncoupled from the vapor phase problem and whose solution is the same as that given by Burelbach *et al.* [8], namely,

$$\mathcal{J}=\frac{\mathrm{Ku}}{\xi+H_j}, \tag{25a}$$

$$T_-=\frac{H_j}{\xi+H_j}, \tag{25b}$$

$$\xi(t)=-H_j+\sqrt{(1+H_j)^2-2(\mathrm{Ku})t}. \tag{25c}$$

The temperature field T_+ is then obtained by using relation (16b) in which the small heat flux term $J_{\mathrm{q}+}$ is not negligible and evaluated via relation (23b). The vapor phase solution is then given by

$$J_{\mathrm{q}+}=\frac{\dfrac{(1-H_c)H_j}{\xi+H_j}-T}{1+H_rHf(x)}Hf(x), \tag{26a}$$

$$T_+=T_--\frac{\dfrac{H_cH_j}{\xi+H_j}+H_r\left(\dfrac{H_j}{\xi+H_j}-T\right)\dfrac{Hf(x)}{\xi}}{1+H_rHf(x)}, \tag{26b}$$

with

$$x=\frac{C_p\mathrm{Ku}}{H}\frac{1}{\xi+H_j},\quad H=\frac{DH_0}{1+D-\xi},\quad H_0=\frac{\lambda}{D}.$$

According to the interfacial mass flux and heat flux expressions (25a) and (26a) and assumption (24), we conclude that the one-sided model is satisfactory at time $t=0$ for small values of H_0T and H_0H_j, i.e., when the thermal ratio $(\lambda/D)T$ is not too large and the initial vapor layer depth $D_{\mathrm{lv}}-D_{\mathrm{l}}$ not too small. We have calculated that $D_{\mathrm{lv}}-D_{\mathrm{l}}$ must be larger than 0.13 mm for $k_h=0.18$. Moreover, if the one-sided model is satisfactory at time $t=0$, it remains also satisfactory as evaporation proceeds because the quantity H decreases with time so that the values HT and HH_j remain small. Concerning the interfacial equilibrium situation ($H_j\to 0$), it should be stressed that when the diffusion mass flux $\mathcal{J}$ becomes infinite at the end of the interfacial evaporation process, the quasisteady approximation becomes wrong according to Burelbach *et al.* [8].

B. No-flow heat conduction solution

A particular situation of interest is when the saturation temperature T_{sat} (or T) is chosen such that the velocity imposed at the cold top plate is zero. Then, there will be no evaporation. This is the so-called *no-flow heat conduction* solution. We obtain in this case a time-independent solution $\xi(t)=1$ with

$$T=-\frac{1+[H_r+(-1+H_c)H_cH_j]H_0}{1+H_cH_j}\frac{1}{H_0}, \tag{27a}$$

$$T_-=\frac{H_cH_j}{1+H_cH_j}, \tag{27b}$$

$$T_+=T_--\frac{H_r+H_c^2H_j}{1+H_cH_j}. \tag{27c}$$

With the no-flow saturation temperature given by Eq. (27a), we find that TH_0 is of the order 1, so that the one-sided model is not valid. Moreover, for an imposed saturation temperature close but larger (smaller) than the no-flow saturation temperature, the liquid will completely evaporate (condense).

C. Small diffusion mass flux approximation

Another approximation besides the one-sided model is the *small diffusion mass flux* approximation defined by $\mathcal{J} \ll H/C_p$, i.e., $C_p\mathcal{J}(1+D-\xi)/\lambda \ll 1$. Under this assumption, relations (22b) and (23b) can be reduced and the problem to be solved becomes a two phase coupled linear problem whose solution reads as

$$\frac{\mathcal{J}}{\mathrm{Ku}}=\frac{1+H[H_r+H_cH_j(-1+H_c+T)+T\xi]}{H_t}, \tag{28a}$$

$$T_-=H_j\{1+H[H_r-(-1+H_c)T\xi]\}/H_t, \tag{28b}$$

$$T_+=T_-+\{HH_jH_r(-1+T)-H_cH_j+H[(-1+H_c)H_cH_j +H_r]T\xi\}/H_t, \tag{28c}$$

with $H_t=H_j+\xi+HH_jH_r+H[(-1+H_c)^2H_j+H_r]\xi$. The domain of validity of this approximation includes the no-flow heat flux solution and is complementary to the one-sided approximation because no small values of H_0T and H_0H_j are allowed. As the quantity H is usually small, one has

$$\mathcal{J}\approx\frac{\mathrm{Ku}}{\xi+H_j}[1+HT(H_cH_j+\xi)],$$

so that the interface velocity increases as evaporation proceeds and the validity condition $\mathcal{J}\ll H/C_p$ can be easily checked. In particular, for interfacial equilibrium, the small diffusion mass flux approximation is not valid at the end of the evaporation process because the diffusion mass flux $\mathcal{J}$ becomes infinite, as mentioned above.

The evolution of the liquid-vapor interface $\xi(t)$ that we derive in this section, and, in particular, the law $t^{1/2}$ of relation (25c), is only valid when the liquid phase is in the state of conductive flow.

After having determined the basic quasisteady solution of the liquid-vapor evaporation problem, we will study, in the following section, its stability with respect to infinitesimally small disturbances in order to determine the onset of instability.

IV. STABILITY ANALYSIS

A. Linearized equations

To study the linear stability of the quasisteady solution, let us reformulate the relevant linearized balance equations. Let $G'=G-G^{\mathrm{b}}$ be the perturbation of a general quantity G with the superscript b referring to the basic quasisteady solution. For simplicity the superscript $'$ will be omitted in the following of this section. We assume that the basic solution is frozen during the stability analysis, i.e., that the perturbations evolve more rapidly than the basic state. The linearized dimensionless equations for the perturbed quantities are the following.

(a)Vapor phase equations:

$$\boldsymbol{\nabla}\cdot\mathbf{v}_{\mathrm{v}}=0, \tag{29a}$$

$$\frac{\rho}{\mathrm{Pr}_{\mathrm{l}}}\left(\frac{\partial\mathbf{v}_{\mathrm{v}}}{\partial t}+w_{\mathrm{v}}^{\mathrm{b}}\frac{\partial\mathbf{v}_{\mathrm{v}}}{\partial Z}\right)=-\boldsymbol{\nabla}p_{\mathrm{v}}+\eta\Delta\mathbf{v}_{\mathrm{v}}+\alpha_T\rho\mathrm{Ra}_{\mathrm{l}}T_{\mathrm{v}}\mathbf{e}_Z, \tag{29b}$$

$$\frac{\partial T_{\mathrm{v}}}{\partial t}+w_{\mathrm{v}}^{\mathrm{b}}\frac{\partial T_{\mathrm{v}}}{\partial Z}-\frac{1}{\lambda}J_{\mathrm{qv}}^{\mathrm{b}}w_{\mathrm{v}}=\kappa\Delta T_{\mathrm{v}}. \tag{29c}$$

(b) Liquid phase equations:

$$\boldsymbol{\nabla}\cdot\mathbf{v}_{\mathrm{l}}=0, \tag{30a}$$

$$\frac{\partial\mathbf{v}_{\mathrm{l}}}{\partial t}=\mathrm{Pr}_{\mathrm{l}}(-\boldsymbol{\nabla}p_{\mathrm{l}}+\Delta\mathbf{v}_{\mathrm{l}}+\mathrm{Ra}_{\mathrm{l}}T_{\mathrm{l}}\mathbf{e}_Z) \tag{30b}$$

$$\frac{\partial T_{\mathrm{l}}}{\partial t}-J_{\mathrm{ql}}^{\mathrm{b}}w_{\mathrm{l}}=\Delta T_{\mathrm{l}}. \tag{30c}$$

(c) Wall boundary conditions:

$$\text{At } Z=0,\quad w_{\mathrm{l}}=u_{\mathrm{l}}=v_{\mathrm{l}}=T_{\mathrm{l}}=0. \tag{31a}$$

$$\text{At } Z=1+D,\quad w_{\mathrm{v}}=u_{\mathrm{v}}=v_{\mathrm{v}}=T_{\mathrm{v}}=0. \tag{31b}$$

(d) Interface ($Z=\xi^{\mathrm{b}}$):

$$\mathcal{J}=\rho(w_+-w_\Sigma), \tag{32a}$$

$$\mathrm{Cr}(p_--p_+)=2\mathrm{Cr}\left[\frac{\partial w_{\mathrm{l}}}{\partial Z_-}-\eta\frac{\partial w_{\mathrm{v}}}{\partial Z_+}\right]+\mathrm{Cr}\chi_0^{\mathrm{b}}\xi -(1-\mathrm{MaCr}T_-^{\mathrm{b}})\Delta_1\xi, \tag{32b}$$

$$\frac{\partial\mathbf{v}_{1\mathrm{l}}}{\partial Z_-}+\boldsymbol{\nabla}_1w_-=\eta\left(\frac{\partial\mathbf{v}_{1\mathrm{v}}}{\partial Z_+}+\boldsymbol{\nabla}_1w_+\right)-\mathrm{Ma}(\boldsymbol{\nabla}_1T_--J_{\mathrm{q}-}^{\mathrm{b}}\boldsymbol{\nabla}_1\xi), \tag{32c}$$

$$J_{\mathrm{q}-}=J_{\mathrm{q}+}+\frac{1}{\mathrm{Ku}}\mathcal{J}+\chi_1^{\mathrm{b}}\xi, \tag{32d}$$

and

$$T_+=(1-H_c)T_--H_rJ_{\mathrm{q}+}+\chi_2^{\mathrm{b}}\xi, \tag{33a}$$

$$\frac{1}{H_j}T_-=H_cJ_{\mathrm{q}+}+\frac{1}{\mathrm{Ku}}\mathcal{J}+\chi_3^{\mathrm{b}}\xi, \tag{33b}$$

$$\mathbf{v}_{1+}=\mathbf{v}_{1-}-w_{\mathrm{v}}^{\mathrm{b}}\boldsymbol{\nabla}_1\xi, \tag{33c}$$

with

$$\Delta_1=\partial_X^2+\partial_Y^2,\quad \mathbf{v}_1=(u,v,0),\quad w_\Sigma=\frac{\partial\xi}{\partial t},\quad \mathcal{J}=w_--w_\Sigma,$$

$$J_{\mathrm{q}_-}=-\frac{\partial T_{\mathrm{l}}}{\partial Z_-},\quad J_{\mathrm{q}+}=-\lambda\frac{\partial T_{\mathrm{v}}}{\partial Z_+},$$

and

$$\chi_0^{\mathrm{b}}=-\frac{\partial p_{\mathrm{l}}^{\mathrm{b}}}{\partial Z_-}+\frac{\partial p_{\mathrm{v}}^{\mathrm{b}}}{\partial Z_+},\tag{34a}$$

$$\chi_1^{\mathrm{b}}=\frac{\partial^2 T_{\mathrm{l}}^{\mathrm{b}}}{\partial {Z^2}_-}-\lambda\frac{\partial^2 T_{\mathrm{v}}^{\mathrm{b}}}{\partial {Z^2}_+},\tag{34b}$$

$$\chi_2^{\mathrm{b}}=-(1-H_c)J_{\mathrm{q}-}^{\mathrm{b}}+\frac{1}{\lambda}J_{\mathrm{q}+}^{\mathrm{b}}+H_r\lambda\frac{\partial^2 T_{\mathrm{v}}^{\mathrm{b}}}{\partial {Z^2}_+},\tag{34c}$$

$$\chi_3^{\mathrm{b}}=\frac{1}{H_j}J_{\mathrm{q}-}^{\mathrm{b}}-H_c\lambda\frac{\partial^2 T_{\mathrm{v}}^{\mathrm{b}}}{\partial {Z^2}_+}.\tag{34d}$$

It is convenient to eliminate the horizontal components of the velocity in the linearized set of equations to obtain a vertical velocity-pressure-temperature (w,p,T) system. This is done by applying the divergence operator on the Navier-Stokes equations (29b) and (30b), the horizontal divergence operator on the no-slip boundary and interfacial conditions (31), (33c) and the Marangoni condition (32c). After use of the continuity equation, we obtain, in the bulk,

$$\Delta p_{\mathrm{v}}=\rho\alpha_T\mathrm{Ra}_{\mathrm{l}}\frac{\partial T_{\mathrm{v}}}{\partial Z},\tag{35a}$$

$$\mathrm{Pr}_{\mathrm{l}}^{-1}\left(\frac{\partial w_{\mathrm{v}}}{\partial t}+w_{\mathrm{v}}^{\mathrm{b}}\frac{\partial w_{\mathrm{v}}}{\partial Z}\right)=-\frac{1}{\rho}\frac{\partial p_{\mathrm{v}}}{\partial Z}+\frac{\eta}{\rho}\Delta w_{\mathrm{v}}+\alpha_T\mathrm{Ra}_{\mathrm{l}}T_{\mathrm{v}},\tag{35b}$$

$$\frac{\partial T_{\mathrm{v}}}{\partial t}+w_{\mathrm{v}}^{\mathrm{b}}\frac{\partial T_{\mathrm{v}}}{\partial Z}-\frac{1}{\lambda}J_{\mathrm{qv}}^{\mathrm{b}}w_{\mathrm{v}}=\kappa\Delta T_{\mathrm{v}},\tag{35c}$$

and

$$\Delta p_{\mathrm{l}}=\mathrm{Ra}_{\mathrm{l}}\frac{\partial T_{\mathrm{l}}}{\partial Z},\tag{36a}$$

$$\mathrm{Pr}_{\mathrm{l}}^{-1}\frac{\partial w_{\mathrm{l}}}{\partial t}=-\frac{\partial p_{\mathrm{l}}}{\partial Z}+\Delta w_{\mathrm{l}}+\mathrm{Ra}_{\mathrm{l}}T_{\mathrm{l}},\tag{36b}$$

$$\frac{\partial T_{\mathrm{l}}}{\partial t}-J_{\mathrm{ql}}^{\mathrm{b}}w_{\mathrm{l}}=\Delta T_{\mathrm{l}},\tag{36c}$$

while at the boundaries and the interface:

$$\text{At}\ \ Z=0,\quad \frac{\partial w_{\mathrm{l}}}{\partial Z}=w_{\mathrm{l}}=T_{\mathrm{l}}=0,\tag{37a}$$

$$\text{At}\ \ Z=1+D,\quad \frac{\partial w_{\mathrm{v}}}{\partial Z}=w_{\mathrm{v}}=T_{\mathrm{v}}=0,\tag{37b}$$

$$\text{At}\ \ Z=\xi^{\mathrm{b}},\quad \Delta_1 w_{\mathrm{l}-}-\frac{\partial^2 w_{\mathrm{l}}}{\partial {Z^2}_-}=\eta\left(\Delta_1 w_{\mathrm{v}+}-\frac{\partial^2 w_{\mathrm{v}}}{\partial {Z^2}_+}\right)-\mathrm{Ma}(\Delta_1 T_- -J_{\mathrm{q}-}^{\mathrm{b}}\Delta_1\xi),\tag{37c}$$

$$\frac{\partial w_{\mathrm{l}}}{\partial Z_-}=\frac{\partial w_{\mathrm{v}}}{\partial Z_+}-w_{\mathrm{v}}^{\mathrm{b}}\Delta_1\xi,\tag{37d}$$

plus the remaining interfacial conditions (32a),(32b),(32d), (33a), and (33b).

B. Normal mode technique

1. Differential system of equations

According to the normal-mode decomposition, we seek solutions of the form

$$(p,w,T)=[P(Z),W(Z),\Theta(Z)]\exp[ct+ikX],\tag{38}$$

$$\xi=\phi\exp[ct+ikX],$$

where k is the wave number and c the stability parameter. With the notation $'=d/dZ$, $''=d^2/dZ^2$, we obtain the following equations for the amplitudes $P(Z)$, $W(Z)$, $\Theta(Z)$, and ϕ, respectively:

$$W_{\mathrm{l}}''-\left(k^2+\frac{1}{\mathrm{Pr}_{\mathrm{l}}}c\right)W_{\mathrm{l}}-P_{\mathrm{l}}'+\mathrm{Ra}_{\mathrm{l}}\Theta_{\mathrm{l}}=0,\tag{39a}$$

$$P_{\mathrm{l}}''-k^2P_{\mathrm{l}}=\mathrm{Ra}_{\mathrm{l}}\Theta_{\mathrm{l}}',\tag{39b}$$

$$\Theta_{\mathrm{l}}''-(k^2+c)\Theta_{\mathrm{l}}+J_{\mathrm{ql}}^{\mathrm{b}}W_{\mathrm{l}}=0,\tag{39c}$$

$$W_{\mathrm{v}}''-\left(k^2+\frac{\rho}{\eta\mathrm{Pr}_{\mathrm{l}}}c\right)W_{\mathrm{v}}-\frac{\rho w_{\mathrm{v}}^{\mathrm{b}}}{\eta\mathrm{Pr}_{\mathrm{l}}}W_{\mathrm{v}}'-\frac{1}{\eta}P_{\mathrm{v}}'+\frac{\rho}{\eta}\alpha_T\mathrm{Ra}_{\mathrm{l}}\Theta_{\mathrm{v}}=0,\tag{39d}$$

$$P_{\mathrm{v}}''-k^2P_{\mathrm{v}}=\rho\mathrm{Ra}_{\mathrm{l}}\Theta_{\mathrm{v}}',\tag{39e}$$

$$\Theta_{\mathrm{v}}''-\left(k^2+\frac{1}{\kappa}c\right)\Theta_{\mathrm{v}}-\frac{w_{\mathrm{v}}^{\mathrm{b}}}{\kappa}\Theta_{\mathrm{v}}'+\frac{1}{\kappa\lambda}J_{\mathrm{qv}}^{\mathrm{b}}W_{\mathrm{v}}=0.\tag{39f}$$

The corresponding boundary conditions are

$$W_{\mathrm{l}}(0)=0,\tag{40a}$$

$$W_{\mathrm{l}}'(0)=0,\tag{40b}$$

$$\Theta_{\mathrm{l}}(0)=0,\tag{40c}$$

$$W_{\mathrm{v}}(1+D)=W_{\mathrm{v}}'(1+D)=0,\tag{40d}$$

$$\Theta_{\mathrm{v}}(1+D)=0,\tag{40e}$$

and at the interface $(Z=\xi^{\mathrm{b}})$:

$$\mathcal{J}=W_- -c\phi,\tag{41a}$$

$$W_- - c\phi = \rho(W_+ - c\phi), \tag{41b}$$

$$\mathrm{Cr}(P_- - P_+) = 2\,\mathrm{Cr}(W'_- - \eta W'_+) + [(1 - \mathrm{MaCr}T^{\mathrm{b}}_-)k^2 + \mathrm{Cr}\chi^{\mathrm{b}}_0]\phi, \tag{41c}$$

$$W''_- + k^2 W_- = \eta(W''_+ + k^2 W_+) - \mathrm{Ma}k^2(\Theta_- - J^{\mathrm{b}}_{\mathrm{q}-}\phi), \tag{41d}$$

$$\Theta'_- = \lambda\Theta'_+ - \frac{1}{\mathrm{Ku}}\mathcal{J} - \chi^{\mathrm{b}}_1\phi, \tag{41e}$$

and

$$W'_- = W'_+ + k^2 w^{\mathrm{b}}_{\mathrm{v}}\phi, \tag{42a}$$

$$\Theta_+ = (1 - H_c)\Theta_- + H_r\lambda\Theta'_+ + \chi^{\mathrm{b}}_2\phi, \tag{42b}$$

$$\frac{1}{H_j}\Theta_- = -H_c\lambda\Theta'_+ + \frac{1}{\mathrm{Ku}}\mathcal{J} + \chi^{\mathrm{b}}_3\phi. \tag{42c}$$

To investigate the role of the interfacial thermal and mass resistances on the liquid-vapor instability, we consider a configuration for which the interfacial deformation and gravity forces are negligible. We show in the following section that in this situation, the instability problem can be reduced to a one-sided liquid phase model with the vapor fluctuations taken into account through the introduction of an equivalent Biot number, the expression of which will be derived. To be specific, we consider a liquid layer of initial depth $D_{\mathrm{l}} = 6$ mm equal to the depth of the vapor phase, i.e., $D = 1$, $\alpha = 0.148$, and $T_{\mathrm{r}} = 45\,°\mathrm{C}$. In this case, H_0 is small and we are left with the only independent parameters Λ and T. Moreover, we suppose that the interface is *undeformable* ($\phi = 0$), since the crispation number is very small for a water layer of this depth, and taken for granted the *exchange of stability* hypothesis ($c = 0$).

2. No-flow heat conduction approximation

We first study the stability of the no-flow heat conduction basic state because it is the much easier situation. Then $w^{\mathrm{b}}_{\mathrm{v}} = 0$ and $\xi = 1$. Moreover, as this basic state is still steady, the frozen assumption is not necessary.

Under microgravity conditions ($g = 0$), the solutions in the vapor phase have the following structure:

$$P_{\mathrm{v}} = \eta[b_1 f_6(Z) + b_2 f_5(Z)], \tag{43a}$$

$$W_{\mathrm{v}} = (b_3 + b_1 Z/2) f_6(Z) + (b_4 + b_2 Z/2) f_5(Z), \tag{43b}$$

$$\Theta_{\mathrm{v}} = \left[b_5 + \frac{1}{\kappa\lambda}J^{\mathrm{b}}_{\mathrm{qv}} f_7(Z)\right] f_6(Z) + \left[b_6 + \frac{1}{\kappa\lambda}J^{\mathrm{b}}_{\mathrm{qv}} f_8(Z)\right] f_5(Z), \tag{43c}$$

with six unknown coefficients b_i and

$$f_5(Z) = \cosh[k(Z - 1 - D)],$$

$$f_6(Z) = \sinh[k(Z - 1 - D)],$$

$$f_7(Z) = -[b_2 k^2 Z^2 + k(-b_1 + 4b_4 k)Z]/(8k^3),$$

$$f_8(Z) = -[b_1 k^2 Z^2 + k(-b_2 + 4b_3 k)Z]/(8k^3).$$

If the two interfacial liquid side quantities W_- and W'_- are known, then the four equations (40d), (41a), (41b), and (42a) give the values of the coefficients b_1, b_2, b_3, and b_4, while equations (40e) and (42b) lead to the values of the remaining two coefficients b_5 and b_6 if we admit that the quantity Θ_- is known. The remaining three interfacial conditions take the form

$$\Theta_- - \frac{H_j}{\mathrm{Ku}}W_- = -H_j H_c g T_{\mathrm{v}}, \tag{44a}$$

$$W''_- + k^2 W_- + \mathrm{Ma}k^2\Theta_- = -k^2\frac{\eta}{\rho} fW_{\mathrm{v}}(Dk)W_- - \frac{\eta}{D} dfW_{\mathrm{v}}(Dk)W'_-, \tag{44b}$$

$$\Theta'_- + \frac{1}{\mathrm{Ku}}W_- = gT_{\mathrm{v}}, \tag{44c}$$

with

$$gT_{\mathrm{v}} = \frac{J^{\mathrm{b}}_{\mathrm{qv}}D^2}{\kappa} dfT_{\mathrm{v}}(Dk)W'_- + \frac{J^{\mathrm{b}}_{\mathrm{qv}}D}{\kappa\rho} fT_{\mathrm{v}}(Dk)W_- - (1 - H_c)H_0 ftT_{\mathrm{v}}(Dk)\Theta_-, \tag{45a}$$

and

$$fW_{\mathrm{v}}(x) = 6f(x)/x^2,$$

$$dfW_{\mathrm{v}}(x) = 4f(x)/h(x),$$

$$fT_{\mathrm{v}}(x) = \frac{3\cosh[2x] - 3(1 + 2x^2) - 2x^3\tanh[x]}{(8/3)x^6(g(x)/f(x))},$$

$$dfT_{\mathrm{v}}(x) = \frac{\sinh[x]^2\tanh[x] - x^3}{(4/3)x^7(g(x)/f(x))}, \tag{45b}$$

$$ftT_{\mathrm{v}}(x) = 1/g(x),$$

$$f(x) = \frac{2x^4/3}{\cosh[2x] - 1 - 2x^2},$$

$$g(x) = \frac{(H_r\lambda/D)x + \tanh[x]}{x},$$

and

$$h(x) = \frac{4x^3/3}{\sinh[2x] - 2x}.$$

The problem is now equivalent to a liquid-side problem. This one-sided model can always be obtained since the two functions g and h are positive so that the quantities defined by the set (45b) exist. Note that the above one-sided model is not of the Pearson-like form [19] as derived by Burelbach *et al.* [8] because of the presence of the parameter W'_- in the set (44). However, we can overcome this problem by expressing W'_- as a function of W''_- and W_-.

This can be achieved as follows. First, observe that in the liquid, the field variables can be written as

$$p_1 = a\cosh[kZ] + b\sinh[kZ], \tag{46a}$$

$$w_1 = (c + aZ/2)\cosh[kZ] + (d + bZ/2)\sinh[kZ], \tag{46b}$$

$$\frac{T_1}{J_{\mathrm{ql}}^{\mathrm{b}}} = \left[f - \frac{b}{8k}Z^2 + \left(\frac{a}{8k^2} - \frac{d}{2k} \right) Z \right]\cosh[kZ] + \left[g - \frac{a}{8k}Z^2 + \left(\frac{b}{8k^2} - \frac{c}{2k} \right) Z \right]\sinh[kZ]. \tag{46c}$$

If we suppose that the solutions of the set (40a) and (40b), W_-, and W''_- are known, we can determine the coefficients a, b, c, and d and write W''_- as a function of W_- and W'_-. This leads to

$$W''_- = \frac{4f(k)}{h(k)} W'_- - (k^2 + 6f(k))W_-,$$

so that the quantity W'_- can now be written in terms of W_- and Θ_- by elimination of W''_- from Eq. (44b); the result is

$$\alpha_1 W'_- = \left(1 - \frac{\eta}{\rho D^2}\frac{f(Dk)}{f(k)} \right) \frac{3h(k)}{2} W_- - \mathrm{Ma}\frac{k^2 h(k)}{4f(k)}\Theta_-,$$

wherein $\alpha_1 = 1 + (\eta/D)[h(k)/4f(k)]dfW_{\mathrm{v}}(Dk)$.

Expression of gT_{v} reduces to

$$gT_{\mathrm{v}} = gtT_{\mathrm{v}}\Theta_- + gwT_{\mathrm{v}}W_-, \tag{47}$$

with

$$gtT_{\mathrm{v}} = -\alpha_1^{-1}\frac{k^2 h(k)}{4f(k)} dfT_{\mathrm{v}}(Dk)\frac{J_{\mathrm{qv}}^{\mathrm{b}} D^2}{\kappa}\mathrm{Ma} - (1 - H_c)\frac{\lambda}{D} ftT_{\mathrm{v}}(Dk), \tag{48a}$$

$$gwT_{\mathrm{v}} = \frac{J_{\mathrm{qv}}^{\mathrm{b}} D}{\kappa\rho} fT_{\mathrm{v}}(Dk) + \alpha_1^{-1}\frac{k^2 h(k)}{4f(k)} dfT_{\mathrm{v}}(Dk) \times \left(1 - \frac{\eta}{\rho D^2}\frac{f(Dk)}{f(k)} \right)\frac{3h(k)}{2}, \tag{48b}$$

so that Eq. (44a) reduces to

$$W_- = \mathrm{Pe}_r\theta_-,$$

after introducing the scaling $\theta = \Theta/J_{\mathrm{ql}}^{\mathrm{b}}$ for the liquid temperature and setting $\mathrm{Pe}_r = J_{\mathrm{ql}}^{\mathrm{b}}[1 + H_j H_c gtT_{\mathrm{v}}]/[1 - \mathrm{Ku}H_c gwT_{\mathrm{v}}](\mathrm{Ku}/H_j)$. These operations lead to a Pearson-like "reduced" model, expressed by

$$W_- = \mathrm{Pe}_r\theta_-, \tag{49a}$$

$$W''_- + k^2\mathrm{Ma}_r\theta_- = 0, \tag{49b}$$

$$\theta'_- + \mathrm{Bi}_r\theta_- = 0. \tag{49c}$$

with the new "reduced" Marangoni Ma_r and Biot numbers Bi_r given by

$$\mathrm{Ma}_r = J_{\mathrm{ql}}^{\mathrm{b}}\mathrm{Ma} + (1 + f_3)\mathrm{Pe}_r + \frac{\eta dfW_{\mathrm{v}}(Dk)}{Dk^2\alpha_1}\left[f_4\frac{3h(k)}{2}\mathrm{Pe}_r - \frac{k^2 h(k)}{4f(k)} J_{\mathrm{ql}}^{\mathrm{b}}\mathrm{Ma} \right], \tag{50a}$$

$$\mathrm{Bi}_r = \frac{\dfrac{1}{H_j} - (1 - H_c)gtT_{\mathrm{v}} - \dfrac{\mathrm{Ku}}{H_j}gwT_{\mathrm{v}}}{1 - \mathrm{Ku}H_c gwT_{\mathrm{v}}}, \tag{50b}$$

with $f_3 = (\eta/\rho)fW_{\mathrm{v}}(Dk)$ and $f_4 = 1 - [k^2/6f(k)]f_3$.

The relations (50a) and (50b) can still be simplified under the following hypotheses. Indeed, if η/D is small then $\alpha_1 \approx 1$ and $\mathrm{Ma}_r \approx J_{\mathrm{ql}}^{\mathrm{b}}\mathrm{Ma} + (1 + f_3)\mathrm{Pe}_r$. Moreover, the function f_3 is less than 100 provided that $Dk > 1.6$. The ratio $(1 + f_3)\mathrm{Pe}_r/(J_{\mathrm{ql}}^{\mathrm{b}}\mathrm{Ma})$ is of the order of $(1 + f_3)\mathrm{Ku}/(\mathrm{Ma}H_j)$ and is very small provided that H_j is not too small which implies $l'_{22} \ll 6\times 10^{-4}\ \mathrm{kg}^2\,\mathrm{J}^{-1}\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}$. This quantity is of the order of $10^{-8}\ \mathrm{kg}^2\,\mathrm{J}^{-1}\,\mathrm{m}^{-2}\mathrm{s}^{-1}$ [17] so that $\mathrm{Ma}_r \approx J_{\mathrm{ql}}^{\mathrm{b}}\mathrm{Ma}$. In this relation $J_{\mathrm{ql}}^{\mathrm{b}}$ is of the order of 1 and comes out because we have chosen $T_{\mathrm{b}} - T_{\mathrm{sat}}$ instead of $T_{\mathrm{b}} - T_-$ for the characteristic difference of temperature. The denominator in Biot's relation (50b) can be approximated by one since the Kutaleladze number is very small. Moreover, in the numerator of Eq. (50b), the third term is negligible compared to the second one as $\mathrm{Ku}/(\mathrm{Ma}H_j)$ is very small so that $\mathrm{Bi}_r \approx (1/H_j) - (1 - H_c)gtT_{\mathrm{v}}$. The first term in the expression (48a) of gtT_{v} can also be neglected compared to the second one and therefore one obtains the final simplified expressions

$$\mathrm{Bi}_r \approx \frac{1}{H_j} + \frac{(1 - H_c)^2}{H_r + \dfrac{\tanh[Dk]}{\lambda k}}, \tag{51a}$$

$$\mathrm{Ma}_r \approx J_{\mathrm{ql}}^{\mathrm{b}}\mathrm{Ma}. \tag{51b}$$

For $H_r = H_c = 0$, the second contribution in Bi_r is due to heat conduction in the gas only, while the first one is due to kinetic nonequilibrium effects at the interface [5]. Clearly, the relation (51a) is a generalization including heat and mass transfer barriers at the interface.

The set of six equations (49a)–(49c), and (40a)–(40c) has a nontrivial solution only if

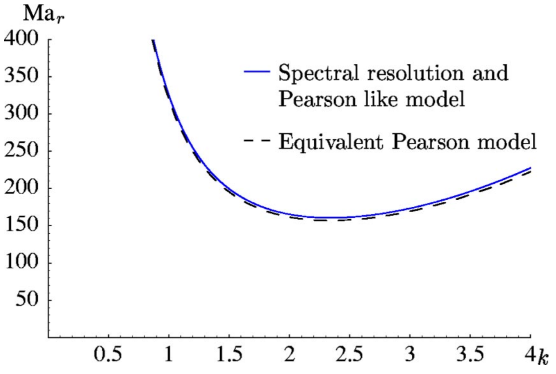

FIG. 2. Neutral stability curves: Marangoni versus wave number ($k_h = 0.18$).

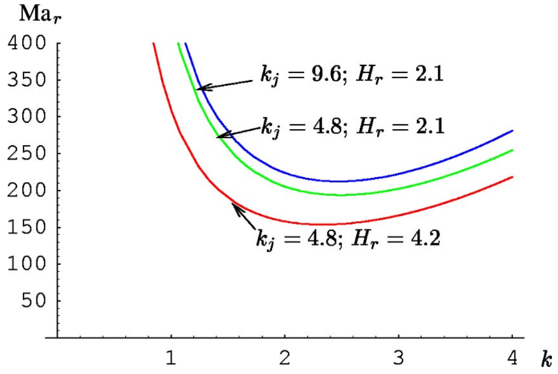

FIG. 3. Interfacial parameters variations ($k_h = 0.18$).

$$\mathrm{Ma}_r = f_{\mathrm{Ma}}(k)\left(1 + \frac{\tanh[k]}{k}\mathrm{Bi}_r - f_{\mathrm{Pe}}(k)\mathrm{Pe}_r\right), \tag{52}$$

with

$$f_{\mathrm{Ma}}(k) = 8\frac{\cosh[k](\cosh[k]\sinh[k]/k - 1)}{(\sinh[k]/k)^3 - \cosh[k]}$$

$$f_{\mathrm{Pe}}(k) = \frac{(7 - 4k^2 + 5\cosh[2k])\tanh[k]/k - 2(6 + k^2)}{16k^2(\cosh[k]\sinh[k]/k - 1)}.$$

The function $f_{\mathrm{Ma}}(k)$ is identical to the well-known Marangoni neutral stability curve for an adiabatically isolated upper boundary. It is minimum at 79.61 with a critical wave number $k = 1.99$. The function $f_{\mathrm{Ma}}(k)[1 + (\tanh[k]/k)\mathrm{Bi}_r]$ was derived by Pearson [19]. It has a minimum varying between 79.61 and ∞ with critical wave number values varying from $k = 1.99$–3.01 when Bi$_r$ number is increased from 0 to ∞.

The function $f_{\mathrm{Pe}}(k)$ decreases from 0.275 at $k = 0$ to 0 at $k = \infty$. For k between 2 and 3, $f_{\mathrm{Pe}} \approx 0.1$; the term $f_{\mathrm{Pe}}\mathrm{Pe}_r$ is then negligible according to the above choice of the coefficient l'_{22}, so that the Pearson relation remains valid with the above (51a)–(51b) choice of equivalent Biot and Marangoni numbers.

To conclude, we have shown that the stability of the liquid-vapor system can be described by Pearson's model [19] at the condition to introduce an equivalent Biot number given by (51a). This is the Biot number that would have been found from the *vapor conductive assumption*, i.e., by neglecting the vapor velocity fluctuations. In this case, the system is the most stable as the coefficients H_j and H_r are small. Using the numerical values of Table III, it is checked from Eq. (51a) that the interfacial thermal resistance provides a small correction to the Biot number if $k_h = 0.18$ which becomes negligible for $k_h = 0$. Therefore, it can be stated that the limiting mechanism is essentially interfacial-mass transfer. We have compared the neutral stability curves obtained, respectively, from a spectral tau numerical resolution of the linear two phase problem, the linear Pearson-like model (49) and the above equivalent Pearson model: Fig. 2 shows a very good agreement. This comparison validates the spectral Tau numerical code at least in absence of gravity.

As mentioned earlier, there remains an incertitude about the values of the phenomenological coefficients. To check the role of these parameters, we have computed (see Fig. 3) the neutral stability curve for an interfacial phenomenological coefficient H_r that is 50% less than the value reported in Table III and a value of k_j twice the value reported in Table III. Figure 3 shows that the results behave qualitatively in the same way so that the relative importance of the different mechanisms that contribute to the onset of the instability, appear to be not fundamentally influenced by the values of the interfacial transfer coefficients. Of course, it would be valuable to compare our results with experimental ones performed in presence of convection to obtain more realistic values of the coupling coefficient k_h and to validate our interfacial thermodynamical modeling.

3. Quasisteady flow

To have an insight about the role of the evaporation rate on stability, we shall investigate the stability of the quasisteady flow in the small diffusion mass flux limit. Interfacial thermal resistance will be neglected according to the results of Sec. IV B 2. As the evaporation rate $\mathcal{J}^b$ is generally small, the analysis will be limited to the first-order correction in $\mathcal{J}^b$. After some algebra, it is shown that the equivalent Biot and Marangoni numbers are

$$\mathrm{Bi}_r = \frac{1}{H_j} + \frac{k\lambda}{\tanh[k(1 + D - \xi)]} - \frac{C_p}{2}\mathcal{J}^b, \tag{53a}$$

$$\mathrm{Ma}_r = J_{\mathrm{ql}}^{\mathrm{b}}\mathrm{Ma}, \tag{53b}$$

with $\mathcal{J}^b = \mathrm{Ku}(1 + HT)/(H_j + \xi^b + HH_j)$. It follows that the neutral stability curve Ma$_r$ versus the wavenumber, with Ma$_r$ defined in terms of the temperature difference across the liquid layer, is not influenced by the quasisteady basic flow if we neglect the third contribution in Bi$_r$. In contrast (recalling that the liquid heat flux is given by $J_{\mathrm{ql}}^{\mathrm{b}} = [1 - (H_j/Ku)\mathcal{J}^b]/\xi^b$), the Marangoni number Ma expressed in terms of the difference $T_{\mathrm{b}} - T_{\mathrm{sat}}$ between the temperature at the lower plate and the saturation temperature at vapor pressure is greatly modified by the presence of the quasisteady basic flow.

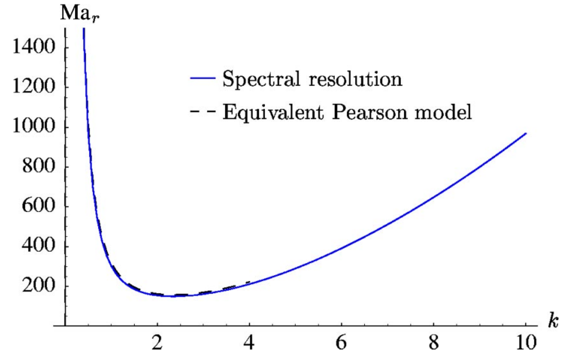

FIG. 4. Neutral stability curves: Marangoni versus wave number ($k_h = 0.18$, thermal ratio $T=1$, initial time $t_b=0$).

As shown in Fig. 4, the agreement between the results provided by a spectral Tau numerical resolution of the linear two phase problem and the above equivalent Pearson model is excellent at the initial time $t_b=0$.

If the temperature difference $T_{\rm b}-T_{\rm sat}$ is small enough then the system is initially stable. When evaporation proceeds, the neutral stability curve goes up so that the system is more and more stable. For sufficiently large values of the temperature difference $T_{\rm b}-T_{\rm sat}$, instability sets in at the initial time $t_b=0$.

4. No-flow heat conduction with an inert gas

To investigate the influence of the interfacial mass resistance compared to the diffusion of the vapor in the gas phase in presence of an inert gas, we reexamine the no-flow heat conduction approximation of Sec. IV B 2 in presence of an inert gas. As justified by the results of Sec. IV B 2, we neglect the interfacial thermal resistance.

This situation is easily described by introducing a new variable in the gas phase, namely, the vapor specific density $Y_{\rm v}$. This quantity verifies the following balance equation written in dimensional form

$$\frac{\partial Y_{\rm v}}{\partial t}+\mathbf{v}_{\rm g}\cdot\boldsymbol{\nabla}Y_{\rm v}=\mathcal{D}_{\rm g}\Delta Y_{\rm v}, \tag{54}$$

$\mathcal{D}_{\rm g}$ is the gas diffusivity. Subscript g will replace subscript v used in the previous sections. The relevant boundary condition is $\partial Y_{\rm v}/\partial Z(D_{\rm lv})=0$. The nonsolubility property of the inert gas in the liquid can be expressed by

$$\text{at } Z=\xi,\quad \mathcal{D}_{\rm g}\boldsymbol{\nabla}Y_{\rm v}+(1-Y_{\rm v})\mathcal{J}/\rho_{\rm g}=0. \tag{55}$$

By neglecting the interfacial thermal resistance, the interfacial phenomenological relations are

$$T_+=T_-, \tag{56a}$$

$$\mathcal{J}=l_{22}R_*T_{\rm r}\ln[\mathcal{P}(T_-)/P_{\rm v+}], \tag{56b}$$

with $P_{\rm v}$ the partial vapor pressure [12]. The liquid-vapor saturation curve writes as

$$\mathcal{P}(T)=P_{\rm r}\exp\left[-\frac{L_{\rm lv}(T_{\rm r})}{R_*}\left(\frac{1}{T}-\frac{1}{T_{\rm r}}\right)\right] \tag{57}$$

while, according to the ideal gas equation of state, the partial vapor pressure is given by

$$P_{\rm v}=P_{\rm g}\bigg/\left(1+\frac{W_{\rm v}}{W_{\rm a}}\frac{1-Y_{\rm v}}{Y_{\rm v}}\right), \tag{58}$$

wherein the global pressure $P_{\rm g}$ is a constant. The saturation temperature $T_{\rm sat}$ is defined by $\mathcal{P}(T_{\rm sat})=P_{\rm g}$ and is also a constant. Setting $r_w=W_{\rm a}/W_{\rm v}$ the ratio of molar weights, for temperatures close to the reference temperature $T_{\rm r}$, relation (56b) can be rewritten as

$$\mathcal{J}=l_{22}R_*T_{\rm r}\left[-\frac{L_{\rm lv}(T_{\rm r})}{R_*T_{\rm r}^2}(T_{\rm sat}-T_-)+\ln\left(\frac{1+(r_w-1)Y_{\rm v+}}{r_wY_{\rm v+}}\right)\right]. \tag{59}$$

In dimensionless form, Eqs. (54), (55), and (59) become

$$\frac{\partial Y_{\rm v}}{\partial t}+\mathbf{v}_{\rm g}\cdot\boldsymbol{\nabla}Y_{\rm v}=\frac{\kappa}{\rm Le}\Delta Y_{\rm v}, \tag{60}$$

$$\frac{\kappa}{\rm Le}\boldsymbol{\nabla}Y_{\rm v+}+(1-Y_{\rm v+})\mathcal{J}/\rho=0, \tag{61a}$$

$$\mathcal{J}=\frac{\rm Ku}{H_j}\left[T_-+\frac{\rm Ky}{\rm Ku}\ln\left(\frac{1+(r_w-1)Y_{\rm v+}}{r_wY_{\rm v+}}\right)\right], \tag{61b}$$

wherein we have introduced two new parameters: the Lewis number ${\rm Le}=\kappa_{\rm g}/\mathcal{D}_{\rm g}$ and ${\rm Ky}=C_{pl}R_*T_{\rm r}^2/L_{\rm lv}(T_{\rm r})^2$. Typically ${\rm Le}\approx 1$ and ${\rm Ky}=3\times10^{-2}$.

The no-flow basic solution is know given by the temperature profiles

$$T_{\rm l}^{\rm b}=1-(1-T_-^{\rm b})Z, \tag{62a}$$

$$T_{\rm v}^{\rm b}=T+(T_+^{\rm b}-T)(1+D-Z)/D, \tag{62b}$$

$$T=-\frac{1}{H_0}-\left(1+\frac{1}{H_0}\right)\frac{\rm Ky}{\rm Ku}\ln\left(\frac{1+(r_w-1)Y_{\rm v}^{\rm b}}{r_wY_{\rm v}^{\rm b}}\right), \tag{62c}$$

$$T_-^{\rm b}=-\frac{\rm Ky}{\rm Ku}\ln\left(\frac{1+(r_w-1)Y_{\rm v}^{\rm b}}{r_wY_{\rm v}^{\rm b}}\right), \tag{62d}$$

$$T_+^{\rm b}=T_-^{\rm b}, \tag{62e}$$

where $Y_{\rm v}^{\rm b}$ is uniform and known. The linearized perturbed equations are

$$\frac{\partial Y_{\rm v}}{\partial t}=\frac{\kappa}{\rm Le}\Delta Y_{\rm v}, \tag{63}$$

with, at $Z=1$:

$$\frac{\kappa}{\rm Le}\boldsymbol{\nabla}Y_{\rm v}+(1-Y_{\rm v}^{\rm b})\mathcal{J}/\rho=0, \tag{64a}$$

$$\mathcal{J}=\frac{\mathrm{Ku}}{H_j}\left(T_- - \frac{\mathrm{Ky}}{\mathrm{Ku}}f_{Y_\mathrm{v}}Y_{\mathrm{v}+}\right), \tag{64b}$$

wherein $f_{Y_\mathrm{v}}=(1/Y_\mathrm{v}^\mathrm{b})/[1+(r_w-1)Y_\mathrm{v}^\mathrm{b}]$ varying from ∞ at $Y_\mathrm{v}^\mathrm{b}=0$ to $1/r_w$ at $Y_\mathrm{v}^\mathrm{b}=1$.

Referring to the normal mode technique, we write $Y_\mathrm{v}=Y\exp[ct+ikX]$ so that for an undeformable interface, the relevant equations are

$$Y''-\left(k^2+\frac{\mathrm{Le}}{\kappa}c\right)Y=0, \tag{65a}$$

$$\text{at } Z=1+D, \quad Y'=0, \tag{65b}$$

$$\text{at } Z=1, \quad \frac{\kappa}{\mathrm{Le}}Y'_+ +(1-Y_\mathrm{v}^\mathrm{b})\mathcal{J}/\rho=0, \tag{65c}$$

$$\frac{\mathrm{Ku}}{H_j}\left(T_- - \frac{Ky}{Ku}f_{Y_\mathrm{v}}Y_+\right)=\mathcal{J}. \tag{65d}$$

Assuming exchange of stability $(c=0)$, the solution of Eqs. (65a)–(65c) takes the form

$$Y=b_7\cosh[k(z-1-D)], \tag{66}$$

with $b_7=\mathrm{Le}(1-Y_\mathrm{v}^\mathrm{b})/(\kappa\rho k\sinh[Dk])\mathcal{J}$; the interfacial temperature is found to be given by

$$T_-=\frac{H_{jr}}{\mathrm{Ku}}\mathcal{J}, \tag{67}$$

wherein we have introduced an equivalent interfacial H_{jr} coefficient

$$H_{jr}=\frac{\mathrm{Ky}f_{Y_\mathrm{v}}\mathrm{Le}(1-Y_\mathrm{v}^\mathrm{b})}{\kappa\rho k\tanh[Dk]}+H_j. \tag{68}$$

This new coefficient is independent of the temperature difference $T_\mathrm{b}-T_\mathrm{sat}$ but depends on the wave number k. We are led to the same mathematical set of equations as in Sec. IV B 2, but with H_{jr} instead of H_j. The new value of the number Pe_r is now $\mathrm{Pe}_r=J_\mathrm{ql}{}^\mathrm{b}\mathrm{Ku}/H_{jr}$. As Pe_r is much smaller than Ku/H_{jr}, one has

$$\mathrm{Ma}_r\approx J_\mathrm{ql}{}^\mathrm{b}\mathrm{Ma}, \tag{69a}$$

$$\mathrm{Bi}_r\approx\mathrm{Pe}_r/(\mathrm{Ku}J_\mathrm{ql}{}^\mathrm{b}). \tag{69b}$$

We recover again Pearson's model at the condition to introduce the Biot number

$$\mathrm{Bi}_r=\frac{1}{H_{jr}}=\frac{1}{\dfrac{\mathrm{KyLe}}{\kappa\rho}\dfrac{f_{Y_\mathrm{v}}(1-Y_\mathrm{v}^\mathrm{b})}{k\tanh[Dk]}+H_j}. \tag{70}$$

In the denominator of relation (70), the interfacial nonequilibrium contribution is negligible as regard to the first mass diffusion term if $H_j\kappa\rho k\tanh[Dk]/\mathrm{Ky}f_{Y_\mathrm{v}}\mathrm{Le}(1-Y_\mathrm{v}^\mathrm{b})\ll 1$. For a liquid layer surmounted by a vapor-air layer of the same depth with $T_\mathrm{r}=45\,°\mathrm{C}$ and for $k=2$, this condition becomes $H_j\ll(1-Y_\mathrm{v}^\mathrm{b})/[4.6(1+0.6Y_\mathrm{v}^\mathrm{b})Y_\mathrm{v}^\mathrm{b}]$. This is satisfied if H_j is small enough and Y_v^b not too close to one for which we recover the liquid-vapor previous situation. Then interfacial mass resistance is negligible compared to the vapor diffusion mass resistance. For example, for $Y_\mathrm{v}^\mathrm{b}=0.2$, we need $H_j\ll 0.8$ and this condition is met by a 6-mm-thick liquid layer if the coupling coefficient k_h is zero (see Table III). If interfacial nonequilibrium is negligible, the equivalent Biot number is the inverse of a thermal resistance that is now due to the vapor diffusion which is the limiting process. Instability is more likely to occur than in the liquid-vapor system. In the above expression of the Biot number, we have neglected the gas phase conduction thermal resistance since this term is small with respect to the vapor diffusion resistance.

According to the Pearson-like model, the critical Marangoni number Ma_r expressed in terms of the difference of temperature across the liquid layer, is positive so that the liquid temperature gradient is negative at the onset of instability. Moreover, the basic liquid heat flux is given by

$$J_\mathrm{ql}{}^\mathrm{b}=1+\alpha_a/\mathrm{Ma}, \tag{71}$$

with

$$\alpha_a=\frac{\mathrm{Ky}}{\mathrm{Ku}_n\sqrt{1/\alpha-1}\sqrt{\mathrm{Ma}_0/\mathrm{Ra}_0}}\ln\left(\frac{1+(r_w-1)Y_\mathrm{v}^\mathrm{b}}{r_wY_\mathrm{v}^\mathrm{b}}\right).$$

It follows from relation (69a) that $\mathrm{Ma}=\mathrm{Ma}_r-\alpha_a$. As the coefficient α_a is usually larger than the critical value for Ma_r, the critical Marangoni number Ma defined by the difference between the temperature of the lower plate and the saturation temperature at vapor pressure is negative and $T_\mathrm{b}<T_\mathrm{sat}$. Moreover, from $T\approx-\mathrm{Ma}_r/(H\mathrm{Ma})$ and $H\ll 1$, it is seen that $T>0$ so that $T_\mathrm{u}>T_\mathrm{sat}$ and the gas temperature gradient is positive.

As no-flow heat conduction is a particular case, one cannot extend the results for the mass-diffusion-limited regime of Sec. IV B 4 to the quasisteady situation of Sec. IV B 3. Indeed, for the quasisteady situation with an inert gas, the diffusion equation is coupled with the gas momentum balance and therefore more investigation is needed. Ha and Lai [3] have studied Marangoni instability in a horizontal layer of finite depth that evaporates into an infinite depth gas layer composed of both vapor and inert gas. Their analysis assumes an initial transitory regime where the gas phase is quasisteady, but the liquid phase unsteady with a constant liquid layer depth. Moreover, their analysis is restricted to fluctuations in the gas phase that are unidirectional and similar to the basic solution, so that they are led to a liquid phase model. It is not clear to which extent such an assumption is justified. Indeed, in the no-flow situation presented in this section, we find an equivalent Biot number that depends on the wave number. Clearly, the role of mass diffusion under general vapor fluctuations remains an open problem.

V. CONCLUSIONS

The role of evaporation (in a two-phase liquid-vapor system) on Marangoni instability is examined. It is shown that the perturbation equations for the two-phase setting can be reduced to a single liquid phase model by the introduction of an equivalent Biot number.

It is seen that evaporation has a stabilizing influence since the transfer of heat and mass through the liquid-vapor interface is equivalent to a positive Biot number that tends to infinity in the interfacial equilibrium case. In contrast, interfacial nonequilibrium has a destabilizing influence as the Biot number decreases when interfacial nonequilibrium effects become more and more important. Even if the temperature field is discontinuous through the liquid-vapor interface, the interfacial thermal resistance has a small influence on the evaporation rate and on the stability. The regime is interfacial-mass-transfer limited.

The position of the liquid-vapor interface is not necessarily fixed but has been allowed to vary as $t^{1/2}$. Such a dependence is shown to be valid when the liquid is in the unperturbed reference state and it is therefore justified to use the same law to determine the threshold of instability. Above this critical point, motion sets in and the position of the interface will generally follow a more general dependence in the course of time. The determination of such a law requires a nonlinear analysis which is outside the scope of the present work but is presently under progress.

Onset of instability usually coincides with the beginning of the quasisteady regime but such regime should occur after a transitory regime that should be investigated in more detail to better apprehend the physics behind the evaporation process.

The role of diffusion in the gas phase has also been examined in the simple no-flow situation. We have found that the perturbation equations still reduce to a liquid phase model by the introduction of an equivalent Biot number. This Biot number includes the heat resistance of both vapor diffusion in the gas phase and interfacial nonequilibrium. We have shown that the interface can be considered at equilibrium under the condition that the vapor specific density is not too close to 1 and the liquid depth not too small. The regime is then mass-diffusion limited. More general situations, including finite velocities will to be investigated in the future.

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