Fluctuations of the winding number of a directed polymer in a random medium

Éric Brunet
Laboratoire de Physique Statistique, École Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05, France
(Received 18 June 2003; published 1 October 2003)

For a directed polymer in a random medium lying on an infinite cylinder that is in $1+1$ dimensions with finite width and periodic boundary conditions on the transverse direction, the winding number is simply the algebraic number of turns the polymer does around the cylinder. This paper presents exact expressions of the fluctuations of this winding number due to, first, the thermal noise of the system and, second, the different realizations of the disorder in the medium.

DOI: 10.1103/PhysRevE.68.041101 PACS number(s): 05.40.−a, 36.20.−r, 64.60.Cn, 71.55.Jv

INTRODUCTION

A directed polymer in a random medium is one of the most simple non-trivial disordered system and is, as such, of special theoretical importance. Indeed, several exact results on directed polymers with strong disorder have been obtained [1–5], so that general approximation schemes developed to tackle more complicated disordered systems such as spin glasses could be tested within the directed polymer context. The directed polymer is also relevant in the context of nonequilibrium phenomena, as it is related, through simple changes of variables, to growth models governed by the Kardar-Parisi-Zhang (KPZ) equation [6,7] and to nonturbulent flows such as the asymmetric exclusion process (ASEP) model [2,7].

The objective of this paper is to study the winding of a directed polymer lying on the surface of a cylinder. The algebraic number of turns W the polymer does around the cylinder is a random variable which depends on the realization of the disorder and which fluctuates because of the thermal noise. The statistics of the winding number of a polymer in an homogeneous medium in $2+1$ dimensions around a cylinder goes back to the work of Spitzer [8] and is relevant for the physics of vortices in type II superconductors [9–11]. In physical situations, the system is, however, usually disordered; the effect of columnar defects has been studied analytically [11], and the winding number around a cylinder of a polymer in a random medium with point-like disorder in $2+1$ dimension has been explored numerically [9,10]. When there is an attractive interaction between the polymer and the winding center, the polymer can be confined around the cylinder and the system can be regarded as a polymer in the $1+1$ dimension with periodic boundary conditions. In that situation, the present work gives exact expressions for the statistics of the winding number.

The directed polymer on a cylinder is also related to the classical limit of strongly correlated fermions in one dimension with disorder (Luttinger liquids): the x position of the directed polymer corresponds to the phase of the fermions, and the phase has, of course, periodic boundary conditions. The winding number of the polymer corresponds to the density of fermions. Disorder, while periodic in both cases, does not have exactly the same correlations, but the models are sufficiently similar to hope for some universality [12,13].

A first result of the present paper states that the thermal fluctuations of the winding number are simply equal to what one would obtain for a directed polymer in a homogeneous medium; disorder is simply averaged out. A second result concerns the thermal-averaged winding number $\overline{W}$. Because of the randomness of the medium, this quantity is not zero and the expression of its variance $\langle \overline{W}^2 \rangle$ averaged over disorder is obtained.

The present paper is organized as follows: Section I is a brief recall of how the directed polymer in a random medium can be mapped to a quantum mechanical problem of interacting bosons using the replica method, and how this quantum mechanical problem can be solved with the Bethe ansatz [1,14,2–4]. In Sec. II the winding number is introduced and defined and the two main results of this paper are stated in Eqs. (19) and (21). Section III gives the main lines of the derivation, and, finally, technical points are developed in the three appendixes.

I. DEFINITION, NOTATIONS, AND FREE ENERGY OF A DIRECTED POLYMER

Let us consider a directed polymer in $1+1$ dimensions where the dimension in which the polymer is directed (the "time" dimension) is taken to be very large and the transverse dimension (the "space" dimension) has width 1 and periodic boundary conditions. As it is directed, the polymer can be described by a single-valued function $y(t)$ and the partition function of a directed polymer of length t ending at position x is given by

$$Z(x,t) = \int_{y(t)=x} \mathcal{D}y(s) \times \exp\left(-\int_0^t ds \left[\frac{1}{2} \left(\frac{dy}{ds} \right)^2 + \eta(y(s),s) \right] \right), \quad (1)$$

where $\eta(x,t)$ is the contribution by the random medium to the energy of the system. Disorder in the medium is assumed to be characterized by an uncorrelated Gaussian noise of variance γ:

$$\langle \eta(x,t) \rangle = 0,$$
$$\langle \eta(x,t)\, \eta(x',t') \rangle = \gamma \delta(x-x')\, \delta(t-t'), \quad (2)$$

where the brackets $\langle\ \rangle$ represent the average over disorder.

It is a well known result [1,14,2–4] that this system can be successfully mapped to a quantum mechanical problem using the replica method; indeed, if we define

$$\psi(\vec{x};t)=\psi(x_1,\ldots,x_n;t)=\frac{\langle Z(x_1,t)\cdots Z(x_n,t)\rangle}{\langle Z(t)\rangle^n}, \quad (3)$$

where

$$Z(t)=\int dx Z(x,t) \quad (4)$$

is the full partition function, then Eqs. (1) and (2) imply

$$\frac{\partial\psi}{\partial t}=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2\psi}{\partial x_i^2}+\gamma\sum_{i<j}\delta(x_i-x_j)\psi, \quad (5)$$

with periodic boundary conditions on all the space variables x_i:

$$\psi(\ldots,x_i=0,\ldots;t)=\psi(\ldots,x_i=1,\ldots;t). \quad (6)$$

The normalization by $\langle Z\rangle^n$ in Eq. (3) is just a simple way to get rid of a low-scale divergence introduced by the continuous description (1) of the system. In other words, without this normalization, there would be a trivial extra term in Eq. (5) involving the lattice size of an underlying discrete formulation of the problem.

For an infinitely long polymer, that is, in the large t limit, the amplitude of $\psi(\vec{x};t)$ is given by the fastest growing mode of Eqs. (5) and (6). In quantum mechanical language, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{\ln\psi(\vec{x};t)}{t}&=\lim_{t\to\infty}\frac{\ln\int dx_1\cdots dx_n\psi(\vec{x};t)}{t}\\&=\lim_{t\to\infty}\frac{\ln\langle Z(t)^n\rangle-n\ln\langle Z(t)\rangle}{t}\\&=-E(n,\gamma),\end{aligned} \quad (7)$$

where $E(n,\gamma)$ is the ground-state energy of the Hamiltonian

$$\mathcal{H}=-\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}-\gamma\sum_{i<j}\delta(x_i-x_j), \quad (8)$$

which describes n particles with *attractive* δ interactions on a ring of size 1.

The same Hamiltonian with a negative value of γ (that is, with a *repulsive* delta interaction) has been much studied to determine the spectrum of a gas of bosons [15–20]. In that context, using the Bethe ansatz [21], it was shown that the ground-state energy of Eq. (8) can be written as

$$E(\eta,\gamma)=-\frac{1}{2}\sum_{\alpha=1}^{n}\lambda_\alpha^2, \quad (9)$$

where the $\{\lambda_\alpha\}$ are solutions of

$$e^{\lambda_\alpha}=\prod_{\substack{1\le\beta\le n\\ \beta\neq\alpha}}\frac{\lambda_\alpha-\lambda_\beta+\gamma}{\lambda_\alpha-\lambda_\beta-\gamma} \quad \text{with} \quad \lim_{\gamma\to 0}\lambda_\alpha=0. \quad (10)$$

Of course, this expression obtained for $\gamma<0$ remains valid in the directed polymer context where $\gamma>0$.

In the quantum mechanical problem (8), the ground state energy $E(n,\gamma)$ is well defined only for integral n; after all, n is the number of particles. However, for the directed polymer, $\langle Z^n\rangle$, which is related to $E(n,\gamma)$ through (7), can be defined for arbitrary values of n. The small-n limit is of special importance here: as the directed polymer is a disordered system, the free energy is a random variable and $\langle Z^n\rangle$ is the generating function of this free energy. Indeed, we have

$$\frac{\ln\langle Z^n\rangle}{t}=n\frac{\langle\ln Z\rangle}{t}+\frac{n^2}{2}\frac{\langle\ln^2 Z\rangle_c}{t}+\frac{n^3}{6}\frac{\langle\ln^3 Z\rangle_c}{t}+O(n^4), \quad (11)$$

where $\langle\ln Z\rangle/t$, $\langle\ln^2 Z\rangle_c/t=(\langle\ln^2 Z\rangle-\langle\ln Z\rangle^2)/t$, etc., are the cumulants of the free energy per unit length of the directed polymer. Thus, if we can generalize Eqs. (9) and (10) to arbitrary values of n, the expansion of $E(n,\gamma)$ for small n gives, using Eq. (7), the distribution of the free energy of the directed polymer [22].

This method was used [1] for the directed polymer on a space of infinite width in the x direction. The Bethe ansatz equations are then much simpler than Eq. (10) and one obtains [23,1], when n is an integer, $E(n,\gamma)=\gamma^2(n-n^3)/24$. This result was used to argue that only the two cumulants $\langle\ln Z\rangle/t$ and $\langle\ln^3 Z\rangle_c/t$ do not vanish in the large t limit and that, therefore, the fluctuations of $\ln Z$ scale like $t^{1/3}$ [24–26].

When space has finite width, however, it is easy to see that the free energy is an extensive function and that all its cumulants scale like t. In two previous papers [3,4] we solved the Bethe ansatz equations (10) and computed the three first terms of the small n expansion of $E(n,\gamma)$. Up to the order n^2, the result is

$$E(n,\gamma)=n\left(\frac{\gamma}{2}+\frac{\gamma^2}{24}\right)-\frac{n^2\gamma^{3/2}}{4\sqrt{2}}\int_0^{+\infty}d\lambda\,\frac{\lambda^2 e^{-\lambda^2/2}}{\tan\dfrac{\lambda\sqrt{\gamma}}{2\sqrt{2}}}+O(n^3), \quad (12)$$

so that, using Eq. (7),

$$\lim_{t\to\infty}\frac{\langle\ln Z\rangle-\ln\langle Z\rangle}{t}=-\left(\frac{\gamma}{2}+\frac{\gamma^2}{24}\right), \quad (13)$$

$$\lim_{t\to\infty}\frac{\langle \ln^2 Z\rangle_c}{t}$$

$$= \frac{\gamma^{3/2}}{2\sqrt{2}}\int_0^{+\infty} d\lambda\, \frac{\lambda^2 e^{-\lambda^2/2}}{\tanh\dfrac{\lambda\sqrt{\gamma}}{2\sqrt{2}}}$$

$$= \gamma + \frac{\gamma^2}{12} - \frac{\gamma^3}{360} + \frac{\gamma^4}{5040} + O(\gamma^5) \quad \text{for small } \gamma$$

$$= \frac{\sqrt{\pi}\gamma^{3/2}}{4} + 4\zeta(3) + O\left(\frac{1}{\gamma}\right) \quad \text{for large } \gamma, \quad (14)$$

where $\zeta(3)=\Sigma k^{-3}\approx 1.20206$.

II. WINDING NUMBER OF THE DIRECTED POLYMER

An important topological property of a directed polymer is its winding number W, that is the algebraic number of full turns the polymer makes around the cylinder on which it lays. One way to define this winding number is to increase W by one for each ''time'' t where the x coordinate of the polymer goes from 1^- to 0^+ and decrease W by one when x goes from 0^+ to 1^-. Another way is to unroll the x coordinate and set $W=\int \dot{x}dt$. Of course, the differences between those two definitions smear out in the large t limit.

As for any quantity in a disordered system at finite temperature, the winding number W fluctuates for two distinct reasons. One is the thermal fluctuations: for a given realization of the disorder and at finite temperature, the directed polymer fluctuates around the path with the lowest energy, and those fluctuations may change the winding number of the polymer. The other source of fluctuations is the quenched disorder on the medium.

In this work, a horizontal bar is used to denote the thermal average, which is the average computed over all the possible directed polymers counted with their Boltzmann weights. The cumulants are noted with an extra c subscript: $\overline{W}$ is the thermal average of W, and $\overline{(W^k)}_c$ the kth thermal cumulant of W, with $\overline{(W^2)}_c=\overline{W^2}-\overline{W}^2$, $\overline{(W^3)}_c=\overline{W^3}-3\overline{W}\,\overline{W^2}+2\overline{W}^3$, etc. These thermal averages and cumulants are calculated for a given, fixed, realization of the disorder and usually depend on that realization.

The average and cumulants of a quantity $\mathcal{Q}$ computed over all the realizations of the disorder are written with brackets: $\langle\mathcal{Q}\rangle$ is the average of $\mathcal{Q}$ computed over all realizations of the disorder, and $\langle\mathcal{Q}^k\rangle_c$ is the kth disorder cumulant of $\mathcal{Q}$.

It is worth noting that, for a given realization of the disorder, the thermal average $\overline{W}$ of the winding number is *not* zero; the disorder breaks the symmetry and may favor one orientation over the other. However, $\overline{W}$ is an extensive quantity and, if we imagine that we cut an extremely long polymer in many very long sections, all the sections are nearly independent and $\overline{W}$ may be regarded as the sum of uncorrelated random variables. Therefore,

$$\lim_{t\to\infty}\frac{\overline{W}}{t}=\lim_{t\to\infty}\frac{\langle\overline{W}\rangle}{t}=0. \quad (15)$$

This property that $\overline{W}$ approaches $\langle\overline{W}\rangle$ in the large t limit is known as ''auto averaging.'' Likewise, all the thermal cumulants of W (which are also extensive quantities) share the same property,

$$\lim_{t\to\infty}\frac{\overline{(W^k)}_c}{t}=\lim_{t\to\infty}\frac{\langle\overline{(W^k)}_c\rangle}{t}. \quad (16)$$

Those cumulants, which characterize the thermal fluctuations of a directed polymer's winding number, depend on the realization of the disorder only when the length t of the polymer is finite.

Other quantities of interest are the disorder cumulants of the thermal average of the winding number of the polymer. Indeed, the quantity $\overline{W}$ depends on the realization of the disorder, and its fluctuations are characterized by another series of cumulants:

$$\lim_{t\to\infty}\frac{\langle\overline{W}^k\rangle_c}{t}, \quad (17)$$

with $\langle\overline{W}^2\rangle_c=\langle\overline{W}^2\rangle-\langle\overline{W}\rangle^2$, etc. Actually, we might be interested in computing many quantities characterizing the winding number, such as

$$\lim_{t\to\infty}\frac{\langle(\overline{W^2})^2\rangle-\langle\overline{W^2}\rangle^2}{t}, \quad (18)$$

which represents the fluctuations due to the disorder of the thermal-mean square of the winding number, per unit length.

A first result of the present paper is

$$\lim_{t\to\infty}\frac{\langle\overline{(W^2)}_c\rangle}{t}=1 \quad \text{and} \quad \lim_{t\to\infty}\frac{\langle\overline{(W^k)}_c\rangle}{t}=0 \quad \text{for } k\neq 2. \quad (19)$$

In other words, thermal fluctuations of the winding numbers are Gaussian and independent of the disorder γ. For an infinitely long polymer, the thermal fluctuations of the winding number of the polymer behave as if the directed polymer was simply doing a random walk in a disorder-less environment.

A second result of the present paper is

$$\lim_{t\to\infty}\frac{\langle\overline{W}^2\rangle_c}{t}=\lim_{n\to 0}\left[\frac{2}{n^2}\left(\frac{\partial E(n,\gamma)}{\partial\gamma}-\frac{2}{\gamma}E(n,\gamma)\right)+\frac{1}{n}-1\right]$$

$$= -1+\left(\frac{2}{\gamma}-\frac{\partial}{\partial\gamma}\right)\lim_{t\to\infty}\frac{\langle\ln^2 Z\rangle_c}{t}, \quad (20)$$

where $E(n,\gamma)$ is the ground state energy of the quantum problem computed in [3,4] and given in Eq. (12). Therefore,

$$\lim_{t\to\infty}\frac{\langle \overline{W^2}\rangle_c}{t}=\frac{\sqrt{\gamma}}{\sqrt{2}}\left(\int_0^{+\infty}d\lambda\,\frac{\lambda^2 e^{-\lambda^2/2}}{\tanh\frac{\lambda\sqrt{\gamma}}{2\sqrt{2}}}-\frac{1}{4}\int_0^{+\infty}d\lambda\,\frac{\lambda^4 e^{-\lambda^2/2}}{\tanh\frac{\lambda\sqrt{\gamma}}{2\sqrt{2}}}\right)-1$$
$$=\frac{\sqrt{\pi\gamma}}{8}-1+\frac{8\zeta(3)}{\gamma}+O(\gamma^{-2})\quad\text{for large }\gamma$$
$$=\frac{\gamma^2}{360}-\frac{\gamma^3}{2520}+\frac{\gamma^4}{16800}+O(\gamma^5)\quad\text{for small }\gamma \tag{21}$$

[where $\zeta(3)=\Sigma k^{-3}\approx 1.202\,06$].

The expression (1) of the directed polymer's free energy is written with dimensionless variables. If we explicitly put back physical constants and use the following expression instead of (1):

$$Z(x,t)=\int_{y(t)=x}\mathcal{D}y(s)\exp\left\{-\beta\int_0^t ds\times\left[\frac{\kappa}{2}\left(\frac{dy}{ds}\right)^2+\eta(y(s),s)\right]\right\}, \tag{22}$$

where $\beta=(k_BT)^{-1}$ is the inverse of temperature, κ is the rigidity modulus of the line and where the spatial dimension x has finite width w and periodic boundary conditions, then Eqs. (19) and (21) become

$$\lim_{t\to\infty}\frac{\langle\overline{(W^2)_c}\rangle}{t}=\frac{1}{\beta\kappa w^2},$$

$$\lim_{t\to\infty}\frac{\langle\overline{(W^k)_c}\rangle}{t}=0\quad\text{for }k\neq 2,$$

$$\lim_{t\to\infty}\frac{\langle\overline{W^2}\rangle_c}{t}=\frac{1}{\beta\kappa w^2}F(\beta^3\kappa w\gamma), \tag{23}$$

where $F(\gamma)$ is the scaling function given in Eq. (21). We obtain the following expansions:

$$\lim_{t\to\infty}\frac{\langle\overline{W^2}\rangle_c}{t}\approx\frac{\sqrt{\pi\beta\gamma}}{8\sqrt{\kappa}w^{3/2}}\quad\text{at low temperature}$$
$$\approx\frac{\beta^5\kappa\gamma^2}{360}\quad\text{at high temperature.} \tag{24}$$

III. DERIVATION OF EQS. (19) AND (20)

A. Equivalence to a quantum mechanical problem

To obtain both results (19) and (20), we define a new partition function $Z_z(x,t)$, the purpose of which is to count the winding number of the polymer:

$$Z_z(x,t)=\int_{y(t)=x}\mathcal{D}y(s)\times e^{-[\text{energy of path } y(s)]+z(\text{winding number of that path})}. \tag{25}$$

The sum is made over all the directed polymers ending in x, and the "energy of a path" is the same as in Eq. (1).

Clearly, $Z_z(t)=\int Z_z(x,t)dx$ is related to the winding number W by

$$Z_z(t)=Z_0(t)\overline{e^{zW}}. \tag{26}$$

If we define the winding number W as an integer that changes by ± 1 each time the directed polymer wraps around the domain by crossing the $x=0$ or $x=1$ boundary, then the boundary conditions for Z_z is

$$Z_z(0,t)=e^zZ_z(1,t). \tag{27}$$

Apart from that, the equations satisfied by $Z_z(x,t)$ are the same as the equations satisfied by $Z(x,t)$. In particular, if we define

$$\psi_{z_1,\ldots,z_n}(x_1,\ldots,x_n;t)=\frac{\langle Z_{z_1}(x_1,t)\cdots Z_{z_n}(x_n,t)\rangle}{\langle Z_0(t)\rangle^n}, \tag{28}$$

this new wave function ψ is also a solution of Eq. (5); only the boundary conditions are changed: instead of Eq. (6), the new conditions read

$$\psi_{z_1,\ldots,z_n}(x_1,\ldots,x_i=0,\ldots,x_n;t)=e^{z_i}\psi_{z_1,\ldots,z_n}(x_1,\ldots,x_i=1,\ldots,x_n;t). \tag{29}$$

Thus, as in Eq. (7), the long "time" t behavior of $Z_z(t)$ is given by

$$\lim_{t\to\infty}\frac{\ln\langle Z_{z_1}\cdots Z_{z_n}\rangle-n\ln\langle Z_0\rangle}{t}=-E(n,\gamma;z_1,\ldots,z_n), \tag{30}$$

where $E(n,\gamma;z_1,\ldots,z_n)$ is the ground state energy of the same Hamiltonian (8) as before, but with the new boundary conditions (29).

This new ground state energy E contains all the information on the winding number W. For instance, from Eq. (26), and by definition of the cumulants, we have, for $k>0$,

$$\overline{(W^k)_c}=\frac{\partial^k}{\partial z^k}\ln Z_z(t)\bigg|_{z=0}. \tag{31}$$

$\langle \ln Z_z \rangle$ is easily obtained from Eq. (30): we set all the $\{z_i\}$ to one single value z, make a small n expansion, and retain only the first order. We get

$$\lim_{t\to\infty} \frac{\langle \overline{(W^k)_c} \rangle}{t} = -\lim_{n\to 0} \frac{\partial^k}{\partial z^k} \frac{\partial}{\partial n} E(n,\gamma;z,\ldots,z)\Big|_{z=0}. \quad (32)$$

Getting $\overline{W}^2$ is more tricky. We would need $(\partial \ln Z_z/\partial z)^2$, but that quantity can only be obtained from Eq. (30) if the parameters $\{z_i\}$ take at least two different values. For example, we have

$$\lim_{n\to 0} \langle Z_{z_1} Z_z^{n-1} \rangle = \left\langle \frac{Z_0(1+z_1\overline{W}+O(z_1^2))}{Z_0(1+z\overline{W}+O(z^2))} \right\rangle,$$

$$= 1 + (z_1 - z)\langle \overline{W} \rangle - z z_1 \langle \overline{W}^2 \rangle + O(z_1^2) + O(z^2), \quad (33)$$

and

$$\lim_{n\to 0} \ln \langle Z_{z_1} Z_z^{n-1} \rangle = (z_1 - z)\langle \overline{W} \rangle - z z_1 (\langle \overline{W}^2 \rangle - \langle \overline{W} \rangle^2) + O(z_1^2) + O(\mathcal{Z}^2). \quad (34)$$

Therefore, putting all the pieces together,

$$\lim_{t\to\infty} \frac{\langle \overline{W}^2 \rangle - \langle \overline{W} \rangle^2}{t} = \lim_{n\to 0} \frac{\partial^2}{\partial z \partial z_1} E(n,\gamma;z_1,z,\ldots,z)\Big|_{\substack{z=0 \\ z_1=0}}. \quad (35)$$

Finally, to obtain the results announced, we need to compute $E(n,\gamma;z,\ldots,z)$ and, to the first order in z and z_1, $E(n,\gamma;z_1,z,\ldots,z)$.

B. Determination of $E(n,\gamma;z,\ldots,z)$

When all the parameters $\{z_i\}$ are equal to one single value z, the problem is easy: all the replica play a symmetric role, so that the ground state eigenvector $\psi_{z,\ldots,z}(x_1,\ldots,x_n)$ of the Hamiltonian (8) is a symmetric function of all the $\{x_i\}$. As shown in Appendix A, the standard Bethe ansatz derivation gives the result. Instead of Eqs. (9, 10), we get

$$E(n,\gamma;z,\ldots,z) = -\frac{1}{2}\sum_{\alpha=1}^{n} \lambda_\alpha^2, \quad (36)$$

where the $\{\lambda_\alpha\}$ are solutions of

$$e^{\lambda_\alpha + z} = \prod_{\substack{1\le\beta\le n \\ \beta\neq\alpha}} \frac{\lambda_\alpha - \lambda_\beta + \gamma}{\lambda_\alpha - \lambda_\beta - \gamma} \quad \text{with} \quad \lim_{\substack{\gamma\to 0 \\ z\to 0}} \lambda_\alpha = 0. \quad (37)$$

If we define

$$\tilde{\lambda}_\alpha = \lambda_\alpha + z \quad (38)$$

then the $\{\tilde{\lambda}_\alpha\}$ are clearly solutions of the standard Bethe ansatz equations (10). Using Eq. (36), we obtain

$$E(n,\gamma;z,\ldots,z) = E(n,\gamma) - \frac{n}{2}z^2. \quad (39)$$

$E(n,\gamma) = E(n,\gamma;0,\ldots,0)$ is the ground state energy (12) before introduction of the $\{z_i\}$. We have used $\Sigma\tilde{\lambda}_\alpha = 0$, which can be easily deduced [3,4] from Eq. (10).

Using Eq. (32), the result (19) on the thermal cumulants of the winding number is then immediate. This method, based on a Bethe ansatz, is not the simplest way to obtain (19). Indeed, the result could be obtained using the *statistical tilt symmetry* [27,28] of the problem; we define the winding number W of a path $y(s)$ as being simply the unrolled coordinate:

$$W = \int_0^t ds \frac{dy}{ds}. \quad (40)$$

(This new definition is, of course, equivalent to the previous one in the large t limit.) The change of variable $y(s) = \tilde{y}(s) + zs$ in the definition (25) of Z_z gives then

$$Z_z(t) = e^{z^2/2} \int \mathcal{D}\tilde{y}(s) \times \exp\left\{ -\int_0^t ds \left[\frac{1}{2}\left(\frac{d\tilde{y}}{ds}\right)^2 + \eta(\tilde{y}(s)+zs,s) \right] \right\}. \quad (41)$$

Clearly $\eta(\tilde{y}(s)+zs,s)$ have the same statistical properties of $\eta(\tilde{y}(s),s)$ and one gets

$$\langle \ln Z_z(t) \rangle = \frac{z^2}{2} + \langle \ln Z_0(t) \rangle, \quad (42)$$

from which the result (19) is straightforward. The first derivation with the Bethe ansatz was included here as it demonstrates part of the method used to obtain the second result (20), which cannot be derived from a *statistical tilt symmetry* argument

C. Determination of $E(n,\gamma;z_1,z,\ldots,z)$

When the parameters $\{z_i\}$ are not identical, the wave function ψ is no longer a symmetric function of the $\{x_i\}$ and the problem is much more complicated. Therefore, the standard bosonic Bethe ansatz used in the previous case will not work. However, as shown in Appendix B, using a more general Bethe ansatz that was first introduced to deal with non-bosonic particles [29,30,31], we get the following result:

$$E(n,\gamma;z_1,z,\ldots,z) = -\frac{1}{2}\sum_{\alpha=1}^{n} \lambda_\alpha^2, \quad (43)$$

where the $\{\lambda_\alpha\}$ are solutions of

$$e^{\lambda_\alpha+\zeta_\alpha}=\prod_{\substack{1\leqslant\beta\leqslant n\\ \beta\neq\alpha}}\frac{\lambda_\alpha-\lambda_\beta+\gamma}{\lambda_\alpha-\lambda_\beta-\gamma}\quad\text{with}\quad\lim_{\substack{\gamma\to0\\ z\to0}}\lambda_\alpha=0, \tag{44}$$

and where the $\{\zeta_\alpha\}$ are such that

$$e^z\prod_{\beta=1}^{n}\left[e^{\zeta_\alpha}+\frac{\lambda_\alpha-\lambda_\beta}{\gamma}(e^z-e^{\zeta_\alpha})\right]=e^{z_1}\prod_{\beta=1}^{n}\left[e^z+\frac{\lambda_\alpha-\lambda_\beta}{\gamma}(e^z-e^{\zeta_\alpha})\right],$$

$$\lim_{\{z_i\}\to0}\zeta_\alpha=0. \tag{45}$$

When $z_1=z$, we recover $\zeta_\alpha=z$, the result of the previous section.

To determine the fluctuation $\langle\overline{W^2}\rangle$ of the winding number of the polymer, we only need to compute $E(n,\gamma;z_1,z,\ldots,z)$ to the second order in the $\{z_i\}$. From Eq. (45), we easily get

$$\zeta_\alpha=\frac{z_1+(n-1)z}{n}-\frac{n\lambda_\alpha-\Sigma_{k=1}^{n}\lambda_k}{\gamma n^3}(z_1-z)^2+O(\{z_i\}^3). \tag{46}$$

We define, for all α,

$$\tilde{\lambda}_\alpha=\lambda_\alpha+\zeta_\alpha,$$

$$=\lambda_\alpha\left(1-\frac{(z_1-z)^2}{\gamma n^2}\right)+\frac{z_1+(n-1)z}{n}+\frac{\Sigma_{k=1}^{n}\lambda_k}{\gamma n^3}(z_1-z)^2, \tag{47}$$

and

$$\tilde{\gamma}=\gamma\left(1-\frac{(z_1-z)^2}{\gamma n^2}\right). \tag{48}$$

Using those new variables into Eq. (44), we obtain the familiar Bethe ansatz equations:

$$e^{\tilde{\lambda}_\alpha}=\prod_{\substack{1\leqslant\beta\leqslant n\\ \beta\neq\alpha}}\frac{\tilde{\lambda}_\alpha-\tilde{\lambda}_\beta+\tilde{\gamma}}{\tilde{\lambda}_\alpha-\tilde{\lambda}_\beta-\tilde{\gamma}}+O(\{z_i\}^3), \tag{49}$$

so that [3,4], using the ground state energy $E(n,\gamma)$ given by Eq. (12),

$$\sum_{\alpha=1}^{n}\tilde{\lambda}_\alpha^2=-2E(n,\tilde{\gamma})+O(\{z_i\}^3)\quad\text{and}\quad\sum_{\alpha=1}^{n}\tilde{\lambda}_\alpha=O(\{z_i\}^3). \tag{50}$$

From there, using Eq. (47), one can write $\Sigma\lambda_\alpha^2$. We finally get

$$E(n,\gamma;z_1,z,\ldots,z)=E(n,\gamma)-\frac{1}{2}\frac{[z_1+(n-1)z]^2}{n}+\frac{1}{n^2}\left[\frac{2}{\gamma}E(n,\gamma)-\frac{\partial E(n,\gamma)}{\partial\gamma}\right](z_1-z)^2+O(\{z_i\}^3). \tag{51}$$

Then, finally, from Eq. (35), we get the announced result (20).

CONCLUSION

Using the replica method with the directed polymer, one obtains a bosonic quantum mechanical problem which can be solved by the Bethe ansatz. By extending this method and using a more general Bethe ansatz that was introduced to deal with nonbosonic particles [29], it has been shown how the different quantities characterizing the fluctuations of the directed polymer's winding number can be computed using new Bethe ansatz equations. Building upon a previous work [3,4], those equations were explicitly solved in two cases giving the results (19) and (20), (21). The second result is particularly interesting as it simply relates through Eq. (20) the fluctuations of the thermal-averaged winding number and the fluctuations of the free energy of the directed polymer. It would be interesting to understand this relation in a more direct way.

In principle, the method presented in the present paper should allow us to compute more cumulants of the winding number and, eventually, its complete probability distribution. For that, however, one needs, as a first step, to generalize Eq. (51) and write the expansion of $E(n,\gamma;z_1,z,\ldots,z)$ to higher orders in the $\{z_i\}$. Indeed, one can show that, for example,

$$\lim_{t\to\infty}\frac{\langle\overline{W^3}\,\overline{W}\rangle-3\langle\overline{W^2}\rangle\langle\overline{W^2}\rangle}{t}=\lim_{n\to0}\frac{\partial^4}{\partial z\,\partial z_1^3}E(n,\gamma;z_1,z,\ldots,z)\bigg|_{\substack{z=0\\ z_1=0}}. \tag{52}$$

Obtaining $E(n,\gamma;z_1,z,\ldots,z)$ to the fourth order in the $\{z_i\}$ is not, however, an easy task, as the trick used in Eq. (47) would not work at that order.

As a second step, to compute more complicated cumulants of the winding number such as $\langle\overline{W^4}\rangle-3\langle\overline{W^2}\rangle^2$, one needs to generalize Eqs. (43)–(45) to the case where the $\{z_i\}$ take at least four different values. Higher order cumulants would require, of course, the energy of the system with more different values of the $\{z_i\}$. A matrix approach such as the one developed in the present paper could lead to the result. Another possibility might be to try an approach similar to the ''nested Bethe ansatz'' method developed by Yang and Sutherland [29–31,1,5] to compute the ground-state energy of the system described by the Hamiltonian (8) with different types of particles and symmetry relations which depend on the type of particles. Their results are not directly applicable to

the directed polymer's winding number as all the particles have the same symmetry relations but different boundary conditions, but it might be worth investigating if the nested Bethe ansatz could be adapted.

ACKNOWLEDGMENTS

I would like to thank Pierre Le Doussal, who suggested this work, and Bernard Derrida for his interesting discussions.

APPENDIX A: BETHE ANSATZ EQUATIONS WHEN ALL THE z_i HAVE THE SAME VALUE z

When all the $\{z_i\}$ are equal to z, all the particles have symmetric roles and the standard bosonic Bethe ansatz leads to the result. To recall the standard derivation [15,17], we look for solutions of the following form:

$$\psi_{z,\ldots,z}(x_1,\ldots,x_n)=\sum_{\sigma} a(\sigma)e^{\sum_{i=1}^{n}\lambda_{\sigma(i)}x_{\tau(i)}}, \tag{A1}$$

where the sum is made over the $n!$ permutations σ of $\{1,\ldots,n\}$ and where τ is the permutation defined by

$$x_{\tau(1)}<x_{\tau(2)}<\cdots<x_{\tau(n)}. \tag{A2}$$

The $n!$ amplitudes $\{a(\sigma)\}$ and the n pseudo-wave-numbers $\{\lambda_\alpha\}$ are unknown variables to be determined.

We use this expression of ψ in $\mathcal{H}\psi=E\psi$, where $\mathcal{H}$ is the Hamiltonian (8). In the regions where all the $\{x_i\}$ are different, it is straightforward to get

$$E(n,\gamma;z,\ldots,z)=-\frac{1}{2}\sum_{\alpha=1}^{n}\lambda_\alpha^2, \tag{A3}$$

so that we only need to determine the $\{\lambda_\alpha\}$. At each crossing of two particles, we have to ensure the correct discontinuities in the derivatives of ψ to compensate for the δ functions in $\mathcal{H}$. This gives the following conditions, for all σ and all $1\leqslant k<n$:

$$a(\sigma\circ T_k)=\frac{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}-\gamma}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}+\gamma}a(\sigma), \tag{A4}$$

where T_k is the permutation that swaps k and $k+1$ and leaves all the other integers unchanged.

As any permutation σ can be written as a product of the elementary permutations T_k, one can use Eq. (A4) to write all the $\{a(\sigma)\}$ up to an arbitrary multiplicative factor. However, as the decomposition of a permutation σ as a product of T_k is not unique, one must check that the $(n-1)n!$ equations (A4) are self-consistent. The best way to do that is to write down explicitly the solution

$$a(\sigma)=\prod_{1\leqslant\alpha<\beta\leqslant n}\frac{\lambda_{\sigma(\alpha)}-\lambda_{\sigma(\beta)}+\gamma}{\lambda_{\sigma(\alpha)}-\lambda_{\sigma(\beta)}}. \tag{A5}$$

It is easily checked that this is indeed the solution of all Eqs. (A4).

So, for *any* set of values $\{\lambda_\alpha\}$, the wave function (A1) where the $\{a(\sigma)\}$ are given by (A5) is an eigenvector of the Hamiltonian (8). The values of the $\{\lambda_\alpha\}$ can then be obtained from the boundary conditions (29). One gets

$$a(\sigma)=e^{z+\lambda_{\sigma(1)}}a(\sigma\circ\mathcal{C}), \tag{A6}$$

where $\mathcal{C}$ is the circular permutation $\mathcal{C}(1)=2$, $\mathcal{C}(2)=3,\ldots,\mathcal{C}(n-1)=n$, $\mathcal{C}(n)=1$. Using Eq. (A5), we easily get the new Bethe ansatz equations. For all α,

$$e^{\lambda_\alpha+z}=\prod_{\substack{1\leqslant\beta\leqslant n\\ \beta\neq\alpha}}\frac{\lambda_\alpha-\lambda_\beta+\gamma}{\lambda_\alpha-\lambda_\beta-\gamma}. \tag{A7}$$

We are only interested in the ground-state solution. By continuity of this ground state, we get the last condition

$$\lim_{\substack{z\to 0\\ \gamma\to 0}}\lambda_\alpha=0. \tag{A8}$$

APPENDIX B: BETHE ANSATZ EQUATIONS WHEN ALL THE $\{z_i\}$ EXCEPT z_1 HAVE THE SAME VALUE z

When the parameters $\{z_i\}$ do not take the same value z, the computation of the energy E is more complicated; indeed the wave function ψ is no longer a symmetric function of the $\{x_i\}$ and there is no way that the standard Bethe ansatz (A1) might lead to the result.

However, in order to study the Hamiltonian (8) for fermionic particles or, more generally, for particles with arbitrary symmetries and anti-symmetries, a more general ansatz than (A1) has been proposed [29,30,1,5]: in Eqs. (A1) and (A2), the permutation τ is only introduced as a convenient way to get the coordinates $\{x_i\}$ of the n particles sorted from the leftmost particle to the rightmost in the expression of the wave function. An easy way to break the symmetry of ψ is to make the parameters $\{a(\sigma)\}$ explicitly dependent on the permutation τ:

$$\psi_{z_1,\ldots,z_n}(x_1,\ldots,x_n)=\sum_{\sigma} a(\tau,\sigma)e^{\sum_{i=1}^{n}\lambda_{\sigma(i)}x_{\tau(i)}}, \tag{B1}$$

where the permutation τ is, as before, defined by Eq. (A2).

As shown below, the solution to our problem with the unusual boundary conditions (29) can also be written using Eq. (B1). We first begin with the most general case where all the $\{z_i\}$ are different and, at some point, specialize to the simpler case where all the $\{z_i\}$ except z_1 are identical.

1. General Bethe Ansatz equations for arbitrary $\{z_i\}$

Using the Ansatz (B1) in $\mathcal{H}\psi=E\psi$ where $\mathcal{H}$ the Hamiltonian (8), we have, as usual,

$$E(n,\gamma;z_1,\ldots,z_n)=-\frac{1}{2}\sum_{\alpha=1}^{n}\lambda_\alpha^2. \tag{B2}$$

The new equations for the parameters $\{a(\tau,\sigma)\}$ are more complicated than (A4),

$$a(\tau,\sigma\circ T_k)=\frac{(\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)})a(\tau\circ T_k,\sigma)-\gamma a(\tau,\sigma)}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}+\gamma}, \tag{B3}$$

for any permutations τ and σ and for any integer $1\leqslant k<n$.

A convenient way to write the $(n!)^2$ parameters $\{a(\tau,\sigma)\}$ is using $n!$ vectors indexed by σ, each vector having $n!$ components:

$$\vec{a}(\sigma)=\begin{pmatrix} a(\tau_1,\sigma)\\ a(\tau_2,\sigma)\\ \vdots\\ a(\tau_{n!},\sigma)\end{pmatrix}, \tag{B4}$$

where $\tau_1,\ldots,\tau_{n!}$ are the $n!$ permutations of $\{1,...,n\}$ sorted in an arbitrary way chosen once for all. (The order must, of course, be the same for all values of σ.) We now introduce the matrices M_k defined by

$$\begin{pmatrix} a(\tau_1\circ T_k,\sigma)\\ a(\tau_2\circ T_k,\sigma)\\ \vdots\\ a(\tau_{n!}\circ T_k,\sigma)\end{pmatrix}=M_k\vec{a}(\sigma). \tag{B5}$$

Those matrices M_k just shuffle the components of the vector $\vec{a}(\sigma)$; there is thus exactly one "1" per raw and per column and all the other components are "0." In a concise way, we can write M_k as

$$(M_k)_{i,j}=\delta_{\tau_i}^{\tau_j\circ T_k}. \tag{B6}$$

The matrices M_k are a representation of the permutations T_k. As such, they have the same standard commutation properties as the permutations:

$$M_k^2=I,\quad M_kM_{k+1}M_k=M_{k+1}M_kM_{k+1},$$

$$M_kM_{k'}=M_{k'}M_k\quad\text{if } |k-k'|>1 \tag{B7}$$

(I being the identity matrix). Equation (B3) is then simply written as

$$\vec{a}(\sigma\circ T_k)=Y_k^{\sigma(k),\sigma(k+1)}\vec{a}(\sigma), \tag{B8}$$

where $Y_k^{i,j}$ is the Yang-Baxter operator defined [29] by

$$Y_k^{i,j}=\frac{(\lambda_i-\lambda_j)M_k-\gamma I}{\lambda_i-\lambda_j+\gamma}. \tag{B9}$$

It is clear from (B8) that any vector $\vec{a}(\sigma)$ can be obtained from the knowledge of one of them. However, as in the symmetric case, one has to check that the result does not depend on the way the permutations are decomposed as a product of the elementary permutations T_k. There are no explicit formula [32] such as (A5) of $\vec{a}(\sigma)$, but one can check that the $(n-1)n!$ relations (B8) are indeed self compatible. This is implied by the following "Yang-Baxter" relations [29]

$$Y_k^{i,j}Y_k^{j,i}=I,$$

$$Y_k^{i,j}Y_{k'}^{i',j'}=Y_{k'}^{i',j'}Y_k^{i,j}\quad\text{if } |k-k'|>1, \tag{B10}$$

$$Y_k^{i,j}Y_{k+1}^{i,l}Y_k^{j,l}=Y_{k+1}^{j,l}Y_k^{i,l}Y_{k+1}^{i,j},$$

which can be easily checked using Eqs. (B7) and (B9). With the first of those three relations, using (B8) twice to compute $\vec{a}(\sigma\circ T_k\circ T_k)$ gives correctly $\vec{a}(\sigma)$. The second relation implies $\vec{a}(\sigma\circ T_k\circ T_{k'})=\vec{a}(\sigma\circ T_{k'}\circ T_k)$ if $|k-k'|>1$ and, finally, the third relation gives $\vec{a}(\sigma\circ T_k\circ T_{k+1}\circ T_k)=\vec{a}(\sigma\circ T_{k+1}\circ T_k\circ T_{k+1})$. It is a well known property of the symmetric group that those three necessary conditions are actually sufficient to ensure that the relations (B8) are self-consistent.

One still needs to write the boundary conditions (29) with the parameters $\{a(\tau,\sigma)\}$. From (B1), one gets

$$a(\tau,\sigma)=\exp(z_{\tau(1)}+\lambda_{\sigma(1)})a(\tau\circ\mathcal{C},\sigma\circ\mathcal{C}), \tag{B11}$$

where $\mathcal{C}$ is, as in Eq. (A6), the circular permutation.

As $\mathcal{C}=T_1\circ T_2\circ\cdots\circ T_{n-1}$, the matrix that shuffles the lines of the vectors $\vec{a}(\sigma)$ according to the permutation $\mathcal{C}$ is simply the product of the matrices M_k. Thus, we have

$$\vec{a}(\sigma)=\exp(\lambda_{\sigma(1)})ZM_1M_2\cdots M_{n-1}\vec{a}(\sigma\circ\mathcal{C}), \tag{B12}$$

where Z is the diagonal matrix defined by

$$(Z)_{i,j}=\delta_i^j\exp(z_{\tau_i(1)}). \tag{B13}$$

Moreover, using several times (B8), we get, from the definition of $\mathcal{C}$,

$$\vec{a}(\sigma\circ\mathcal{C})=Y_{n-1}^{\sigma(1),\sigma(n)}Y_{n-2}^{\sigma(1),\sigma(n-1)}\cdots Y_1^{\sigma(1),\sigma(2)}\vec{a}(\sigma). \tag{B14}$$

Putting together (B12) and (B14), we see that $\vec{a}(\sigma)$ must be, for each σ, the eigenvector of some operator,

$$\exp(\lambda_{\sigma(1)})ZM_1\cdots M_{n-1}Y_{n-1}^{\sigma(1),\sigma(n)}\cdots Y_1^{\sigma(1),\sigma(2)}\vec{a}(\sigma)=\vec{a}(\sigma). \tag{B15}$$

There exists a nonzero $\vec{a}(\sigma)$ such as (B15) holds only for certain values of the $\{\lambda_\alpha\}$. However, as the $n!$ vectors $\{\vec{a}(\sigma)\}$ are not independent variables, we need to check that the $n!$ relations (B15) are compatible: they must hold simultaneously for the same values of the $\{\lambda_\alpha\}$. As explained in Appendix C, this is the case. To obtain the values of the $\{\lambda_\alpha\}$ we are looking for, we use (B15) when σ is the identical permutation. A nonzero $\vec{a}(\sigma)$ exists if and only if

$$\det[I-\exp(\lambda_1)ZM_1\cdots M_{n-1}Y_{n-1}^{1,n}Y_{n-2}^{1,n-1}\cdots Y_1^{1,2}]=0, \tag{B16}$$

or, using the properties (B10) of the operators Y,

$$\det(Y_1^{2,1}Y_2^{3,1}\cdots Y_{n-1}^{n,1}-\exp(\lambda_1)ZM_1\cdots M_{n-1})=0, \tag{B17}$$

or, using the definition (B9) of the operators Y,

$$\det\left(\prod_{\alpha=1}^{n-1}\frac{(\lambda_1-\lambda_{\alpha+1})M_\alpha+\gamma I}{\lambda_1+\lambda_{\alpha+1}-\gamma}-\exp(\lambda_1)Z\prod_{\alpha=1}^{n-1}M_\alpha\right)=0. \tag{B18}$$

That last equation relates $\exp(\lambda_1)$ to the $\{\lambda_\alpha\}$. There are $n-1$ other equations giving all the $\exp(\lambda_k)$ which we can obtain either by using (B15) with different permutations σ either, as they play symmetric roles, by shuffling the $\{\lambda_\alpha\}$ in Eq. (B18).

Finally, the wave function (B1) introduced is indeed an eigenvector of the Hamiltonian (8) with the boundary conditions (29), provided that the $\{\lambda_\alpha\}$ are such that (B18) and the $n-1$ other relations obtained by symmetry hold.

Note that the new Bethe ansatz equation (B18) can be regarded as a polynomial of degree $n!$ in $\exp(\lambda_1)$, so that we have not one value of $\exp(\lambda_1)$ as a function of the $\{\lambda_\alpha\}$, but $n!$. This could be expected, as the method we have used is known to generate, when $z_i=0$, not only the bosonic solution, but all the eigenvalues of Eq. (8) for arbitrary symmetry relations between the particles. So, if we write from (B18) the $n!$ possible expressions of $\exp(\lambda_1)$ and make the $\{z_i\}$ go to zero, we will recover the usual bosonic Bethe ansatz solution (10), but also the fermionic solution $\exp(\lambda_\alpha)=1$ and all the intermediate cases. In our problem, we are looking for the ground state energy of discernible particles and it is known, in this situation, that the ground state is given by the bosonic solution.

To sum up, what remains to be done is to single out from (B18) the expression of $\exp(\lambda_1)$ which goes to the standard bosonic equations (10) when the $\{z_i\}$ vanish, to write by symmetry the $n-1$ remaining equations giving all the $\{\exp(\lambda_\alpha)\}$ as functions of the $\{\lambda_\alpha\}$, to solve those nonalgebraic equations in order to write the ground state energy $E=-(1/2)\Sigma\lambda_\alpha^2$, and, finally, to take the limit $n\to 0$ and various derivatives with respect to the z_i to obtain the different quantities characterizing the winding number of the polymer.

As this seems to be a difficult task in the general case, we will go through this program only when all the $\{z_i\}$ have the same value z except for z_1.

2. Simplification when all the $\{z_i\}$ are equal except for z_1

When *all* the $\{z_i\}$ are set to zero, the matrix Z is the identity matrix and one of the solutions of Eq. (B18) must be the standard bosonic Bethe ansatz equation (10). One way to see it is to notice that, when (10) holds, the vector which cancels the matrix in Eq. (B18) is simply (1,...,1). Another way to see it is to notice that to derive (B18), we never actually used the matrix representation of M_k, $Y_k^{i,j}$, etc., but only the commutation properties of those matrices. If we were to choose other *representations* of those matrices having the same commutation properties, relation (B18) would still be valid with those representations. For example, when all the $\{z_i\}$ are zero, the bosonic solution is obtained from Eq. (B18) by choosing the trivial representation $M_k=1$. The fermionic solution $\exp(\lambda_\alpha)=1$ is obtained by choosing $M_k=-1$, etc.

In the situation where $z_i=z$ for $i\geqslant 2$, with only z_1 different from z, we can make a similar simplification. Indeed, in that case, the particles $x_2,\ldots,x_n$ play symmetric roles. Thus, the ground state solution must be symmetric in those variables. Back to the wave function (B1), this means that the parameters $a(\tau_1,\sigma)$ and $a(\tau_2,\sigma)$ must be equal if $\tau_1^{-1}(1)=\tau_2^{-1}(1)$. In other words, the parameters $a(\tau,\sigma)$ do not depend on the whole shuffling τ of the particles $\{x_i\}$, but only on the position of x_1 relatively to the other; for each σ, there is one value of $a(\tau,\sigma)$ corresponding to the first particle being the leftmost, another value when the first particle is the second leftmost, etc.

This suggests that Eq. (B18) can be written in that situation with a representation of the M_k as matrices of size $n\times n$ instead of $n!\times n!$. Indeed, we write the new vector $\vec{a}(\sigma)$ as

$$\vec{a}(\sigma)=\begin{pmatrix} a(\tau,\sigma)\ \text{ such that }\ \tau(1)=1\ \ (x_1\ \text{is the leftmost particle}) \\ a(\tau,\sigma)\ \text{ such that }\ \tau(2)=1\ \ (x_1\ \text{is the second leftmost particle}) \\ \vdots \\ a(\tau,\sigma)\ \text{ such that }\ \tau(n)=1\ \ (x_1\ \text{is the rightmost particle}) \end{pmatrix}. \tag{B19}$$

Then, the matrix M_k that switches the kth and $(k+1)$th particles is given by

$$(M_k)_{i,j}=\begin{cases}\delta_j^{k+1} & \text{if } i=k\\ \delta_j^{k} & \text{if } i=k+1\\ \delta_i^{j} & \text{otherwise,}\end{cases} \tag{B20}$$

that is,

$$M_1=\begin{pmatrix}\mathbf{0} & \mathbf{1} & 0 & 0 & \dots\\ \mathbf{1} & \mathbf{0} & 0 & 0 & \dots\\ 0 & 0 & 1 & 0 & \dots\\ 0 & 0 & 0 & 1 & \dots\\ \dots & \dots & \dots & \dots & \ddots\end{pmatrix},$$

$$M_2=\begin{pmatrix}1 & 0 & 0 & 0 & \dots\\ 0 & \mathbf{0} & \mathbf{1} & 0 & \dots\\ 0 & \mathbf{1} & \mathbf{0} & 0 & \dots\\ 0 & 0 & 0 & 1 & \dots\\ \dots & \dots & \dots & \dots & \ddots\end{pmatrix}, \tag{B21}$$

etc. With all the $\{z_i\}$ but z_1 equal to z, we can also write the new matrix Z in this representation. It is a diagonal matrix, on the first line we have $\exp(z_1)$ as x_1 is then the leftmost particle, and, on the other lines, we do not know which particle is the leftmost, but it is of no importance as we know it is not x_1 and as all the other particles have the same parameter z. Thus, we have

$$Z=\begin{pmatrix}e^{z_1} & 0 & 0 & \dots\\ 0 & e^{z} & 0 & \dots\\ 0 & 0 & e^{z} & \dots\\ \dots & \dots & \dots & \ddots\end{pmatrix}. \tag{B22}$$

Of course, the new matrices M_k have the correct commutation relations (B7) and the final result (B18) is still valid with the new matrices M_k and Z.

3. Explicit expression of the determinant

When $z_i=z$ for $i\geqslant 2$, using the new matrices M_k and Z, we can compute the determinant in (B18) by induction. First, we normalize $\exp(\lambda_1)$ by the standard Bethe ansatz expression and write

$$e^{\lambda_1+\zeta_1}=\prod_{\alpha=2}^{n}\frac{\lambda_1-\lambda_\alpha+\gamma}{\lambda_1-\lambda_\alpha-\gamma}, \tag{B23}$$

where ζ_1 is the quantity we are trying to determine.

Using this new variable, we write the determinant in (B18), up to a multiplicative prefactor, as

$$d_n=\det[A_n-\exp(-\zeta_1)B_n], \tag{B24}$$

where

$$A_n=\prod_{\alpha=1}^{n-1}\frac{(\lambda_1-\lambda_{\alpha+1})M_\alpha+\gamma}{\lambda_1-\lambda_{\alpha+1}+\gamma} \tag{B25}$$

and, as can easily be seen,

$$B_n=Z\prod_{\alpha=1}^{n-1}M_\alpha=\begin{pmatrix}0 & 0 & \dots & 0 & e^{z_1}\\ e^{z} & 0 & \dots & 0 & 0\\ 0 & e^{z} & \dots & 0 & 0\\ \dots & \dots & \ddots & \dots & \dots\\ 0 & 0 & \dots & e^{z} & 0\end{pmatrix}. \tag{B26}$$

As e^{z_1} appears only once in the matrix, the determinant d_n can be written as

$$d_n=\alpha_n-e^{z_1}\beta_n, \tag{B27}$$

where α_n and β_n do not depend on z_1.

Going from A_n to A_{n+1} is easy enough; singling out the last term in the product (B25), one has

$$A_{n+1}=\begin{pmatrix} & & & \vdots\\ & A_n & & 0\\ & & & \vdots\\ \dots & 0 & \dots & 1\end{pmatrix}\times\begin{pmatrix}1 & & & & \\ & \ddots & & (0) & \\ & & 1 & & \\ & (0) & & \mu_{n+1} & 1-\mu_{n+1}\\ & & & 1-\mu_{n+1} & \mu_{n+1}\end{pmatrix}, \tag{B28}$$

with

$$\mu_{n+1}=\frac{\gamma}{\lambda_1-\lambda_{n+1}+\gamma}. \tag{B29}$$

Doing this last multiplication, we see that the first $n-1$ columns of A_{n+1} are the first columns of A_n padded with one final zero, and that the nth and $(n+1)$th columns of A_{n+1} are the nth column of A_n with different multiplicative factors (respectively, μ_{n+1} and $1-\mu_{n+1}$) and different paddings (respectively, $1-\mu_{n+1}$ and μ_{n+1}.) Thus, if we develop d_{n+1} over the last line, there are only two terms which look very much like d_n. The only differences are that the last column is multiplied by some factor and that the term e^{z_1} is either missing or not multiplied by the numerical factor that affects its column. Finally, using (B27), one can get

$$d_{n+1}=\mu_{n+1}(\mu_{n+1}\alpha_n)-(1-\mu_{n+1}-e^{z-\zeta_1})[(1-\mu_{n+1})\alpha_n-\beta_n e^{z_1}], \tag{B30}$$

or

$$\alpha_{n+1}=[2\mu_{n+1}-1+(1-\mu_{n+1})e^{z-\zeta_1}]\alpha_n,$$

$$\beta_{n+1}=-(1-\mu_{n+1}-e^{z-\zeta_1})\beta_n \tag{B31}$$

with $\alpha_1=1$ and $\beta_1=\exp(-\zeta_1)$. It is now easy to compute α_n and β_n,

$$\alpha_n=\prod_{i=2}^{n}\frac{(\lambda_1-\lambda_i)(e^{z-\zeta_1}-1)+\gamma}{\lambda_1-\lambda_i+\gamma},$$

$$\beta_n=e^{-\zeta_1}\prod_{i=2}^{n}\frac{(\lambda_1-\lambda_i)(e^{z-\zeta_1}-1)+\gamma e^{z-\zeta_1}}{\lambda_1-\lambda_i+\gamma}, \tag{B32}$$

and, finally, the condition that the determinant (B18) is zero gives the following result:

$$e^{z}\prod_{i=1}^{n}\left[1+\frac{\lambda_1-\lambda_i}{\gamma}(e^{z-\zeta_1}-1)\right]=e^{z_1}\prod_{i=1}^{n}\left[e^{z-\zeta_1}+\frac{\lambda_1-\lambda_i}{\gamma}(e^{z-\zeta_1}-1)\right]. \tag{B33}$$

As all the $\{\lambda_\alpha\}$ play symmetric roles, this is exactly the result announced (45).

APPENDIX C: PROOF THAT EQ. (B15) CAN BE SATISFIED SIMULATANEOUSLY FOR ALL PERMUTATIONS σ

To prove Eq. (B15) are indeed compatible, we start by assuming that the $\{\vec{a}(\sigma)\}$ are such that (B8) holds for any σ and k. As a consequence, Eq. (B14) is true for any permutation and (B15) is equivalent to (B12).

Furthermore, we assume that Eq. (B15) [or (B12)] is true for a given permutation σ. To show that it is also true for any other permutation, it is sufficient, by induction, to prove that Eq. (B15) [or Eq. (B12)] holds for $\sigma\circ T_k$ with $1\leqslant k<n$.

It is necessary to distinguish the two cases $k\neq 1$ and $k=1$.

The case $1<k<n$. When $k\neq 1$, we have the following properties:

$$T_k\circ\mathcal{C}=\mathcal{C}\circ T_{k-1},\quad Y_k^{i,j}Z=ZY_k^{i,j},$$

$$Y_k^{i,j}M_1M_2\cdots M_{n-1}=M_1M_2\cdots M_{n-1}Y_{k-1}^{i,j}. \tag{C1}$$

The first relation is a basic property of permutations, the second relation comes from the fact that $Y_k^{i,j}$ does not change the value of $\tau(1)$, and the third one, considering the definition (B9) of Y_k, is a rewriting of the first relation in the matrix representation.

We can now show that (B12) and, therefore, (B15) holds for $\sigma\circ T_k$,

$$\begin{aligned}\vec{a}(\sigma\circ T_k)&=Y_k^{\sigma(k),\sigma(k+1)}\vec{a}(\sigma)\\&=Y_k^{\sigma(k),\sigma(k+1)}\exp(\lambda_{\sigma(1)})ZM_1M_2\cdots M_{n-1}\vec{a}(\sigma\circ\mathcal{C})\\&=\exp(\lambda_{\sigma(1)})ZM_1M_2\cdots M_{n-1}Y_{k-1}^{\sigma(k),\sigma(k+1)}\vec{a}(\sigma\circ\mathcal{C})\\&=\exp(\lambda_{\sigma(1)})ZM_1M_2\cdots M_{n-1}\vec{a}(\sigma\circ\mathcal{C}\circ T_{k-1})\\&=\exp(\lambda_{\sigma(1)})ZM_1M_2\cdots M_{n-1}\vec{a}(\sigma\circ T_k\circ\mathcal{C}).\end{aligned} \tag{C2}$$

As $\sigma(1)=\sigma\circ T_k(1)$ for $k\neq 1$, this is indeed (B12) applied to the permutation $\sigma\circ T_k$.

When $k=1$. Equation (B15) express that $\vec{a}(\sigma)$ is an eigenvector of

$$A=ZM_1\cdots M_{n-1}Y_{n-1}^{\sigma(1),\sigma(n)}\times Y_{n-2}^{\sigma(1),\sigma(n-1)}\cdots Y_2^{\sigma(1),\sigma(3)}Y_1^{\sigma(1),\sigma(2)}, \tag{C3}$$

and we want to prove that $\vec{a}(\sigma\circ T_1)$ is also an eigenvector of

$$ZM_1\cdots M_{n-1}Y_{n-1}^{\sigma(2),\sigma(n)}Y_{n-2}^{\sigma(2),\sigma(n-1)}\cdots Y_2^{\sigma(2),\sigma(3)}Y_1^{\sigma(2),\sigma(1)}. \tag{C4}$$

As $\vec{a}(\sigma\circ T_1)=Y_1^{\sigma(1),\sigma(2)}\vec{a}(\sigma)$, this is equivalent to prove that $\vec{a}(\sigma)$ is an eigenvector of

$$B=Y_1^{\sigma(2),\sigma(1)}ZM_1\cdots M_{n-1}\times Y_{n-1}^{\sigma(2),\sigma(n)}Y_{n-2}^{\sigma(2),\sigma(n-1)}\cdots Y_2^{\sigma(2),\sigma(3)}. \tag{C5}$$

(The relation $Y_k^{i,j}Y_k^{j,i}=I$ has been used twice.)

To conclude, we presently show that $AB=BA$, which implies that A and B have the same eigenvectors. First, we define another diagonal matrix Z_2 by

$$(Z_2)_{i,j}=\delta_i^j\exp(z_{\tau_i(2)}). \tag{C6}$$

[Compare with Eq. (B13).] Clearly, we have

$$M_1Z=Z_2M_1\quad\text{and}\quad M_1Z_2=ZM_1. \tag{C7}$$

Moreover, as a consequence, the product $ZZ_2=Z_2Z$ commutes with M_1 and Y_1.

When computing AB, two matrices Y_1 cancel out. The matrix Z commutes with all the Y_k and all the M_k except Y_1 and M_1, so that we can "move" the second Z to the left and obtain

$$\begin{aligned}AB&=Z\left(\prod M_i\right)Y_{n-1}^{\sigma(1),\sigma(n)}\cdots Y_2^{\sigma(1),\sigma(3)}Z\times\left(\prod M_i\right)Y_{n-1}^{\sigma(2),\sigma(n)}\cdots Y_2^{\sigma(2),\sigma(3)},\\&=ZZ_2\left(\prod M_i\right)Y_{n-1}^{\sigma(1),\sigma(n)}\cdots Y_2^{\sigma(1),\sigma(3)}\times\left(\prod M_i\right)Y_{n-1}^{\sigma(2),\sigma(n)}\cdots Y_2^{\sigma(2),\sigma(3)}.\end{aligned} \tag{C8}$$

When computing BA, the second matrix Z can also travel to the left; we get

$$BA = Y_1^{\sigma(2),\sigma(1)} Z \left(\prod M_i \right) Y_{n-1}^{\sigma(2),\sigma(n)} \cdots Y_2^{\sigma(2),\sigma(3)} Z \times \left(\prod M_i \right) Y_{n-1}^{\sigma(1),\sigma(n)} \cdots Y_1^{\sigma(1),\sigma(2)}$$
$$= Y_1^{\sigma(2),\sigma(1)} Z Z_2 \left(\prod M_i \right) Y_{n-1}^{\sigma(2),\sigma(n)} \cdots Y_2^{\sigma(2),\sigma(3)} \times \left(\prod M_i \right) Y_{n-1}^{\sigma(1),\sigma(n)} \cdots Y_1^{\sigma(1),\sigma(2)}$$
$$= Z Z_2 Y_1^{\sigma(2),\sigma(1)} \left(\prod M_i \right) Y_{n-1}^{\sigma(2),\sigma(n)} \cdots Y_2^{\sigma(2),\sigma(3)} \times \left(\prod M_i \right) Y_{n-1}^{\sigma(1),\sigma(n)} \cdots Y_1^{\sigma(1),\sigma(2)}. \quad \text{(C9)}$$

Thus, the products AB and BA share the same prefactor ZZ_2, so that if $AB=BA$ is true when $z_i=0$ (the case studied by Yang [29]), then $AB=BA$ is true for arbitrary values of the $\{z_i\}$. As it is a well known fact that the operators commute in Yang's case, we could stop the proof here. However, for completeness, let us properly finish it.

We continue the simplification of AB; using (C1), we can have the whole first group of matrices Y_k in AB go to the right through the second product $M_1 \cdots M_{n-1}$. We get

$$AB = Z Z_2 \left(\prod M_i \right)^2 Y_{n-2}^{\sigma(1),\sigma(n)} \cdots Y_1^{\sigma(1),\sigma(3)} \times Y_{n-1}^{\sigma(2),\sigma(n)} \cdots Y_2^{\sigma(2),\sigma(3)}. \quad \text{(C10)}$$

We do the same for the product BA,

$$BA = Z Z_2 Y_1^{\sigma(2),\sigma(1)} \left(\prod M_i \right)^2 Y_{n-2}^{\sigma(2),\sigma(n)} \cdots Y_1^{\sigma(2),\sigma(3)} \times Y_{n-1}^{\sigma(1),\sigma(n)} \cdots Y_1^{\sigma(1),\sigma(2)}. \quad \text{(C11)}$$

Using

$$Y_1^{i,j} (M_1 \cdots M_{n-1})^2 = (M_1 \cdots M_{n-1})^2 Y_{n-1}^{i,j}, \quad \text{(C12)}$$

which can be deduced from the properties (B7) of the matrices M_k, we get

$$BA = Z Z_2 \left(\prod M_i \right)^2 Y_{n-1}^{\sigma(2),\sigma(1)} Y_{n-2}^{\sigma(2),\sigma(n)} \cdots \times Y_1^{\sigma(2),\sigma(3)} Y_{n-1}^{\sigma(1),\sigma(n)} \cdots Y_1^{\sigma(1),\sigma(2)}. \quad \text{(C13)}$$

AB and BA have the same prefactor $ZZ_2(\Pi M_i)^2$; we need to show that the two products of matrices Y_k are equal. We proceed by induction: It is clear for $n=1$ (or $n=2$) and we assume it is true for $n-1$. In both products, we "move" the matrices Y_{n-1} to the left. We get

$$AB = Z Z_2 \left(\prod M_i \right)^2 Y_{n-2}^{\sigma(1),\sigma(n)} Y_{n-1}^{\sigma(2),\sigma(n)} Y_{n-3}^{\sigma(1),\sigma(n-1)} \cdots \times Y_1^{\sigma(1),\sigma(3)} \times Y_{n-2}^{\sigma(2),\sigma(n-1)} \cdots Y_2^{\sigma(2),\sigma(3)}, \quad \text{(C14)}$$

and, using Eq. (B10),

$$BA = Z Z_2 \left(\prod M_i \right)^2 Y_{n-1}^{\sigma(2),\sigma(1)} Y_{n-2}^{\sigma(2),\sigma(n)} Y_{n-1}^{\sigma(1),\sigma(n)} \times Y_{n-3}^{\sigma(2),\sigma(n-1)} \cdots Y_1^{\sigma(2),\sigma(3)} \times Y_{n-2}^{\sigma(1),\sigma(n-1)} \cdots Y_1^{\sigma(1),\sigma(2)}$$
$$= Z Z_2 \left(\prod M_i \right)^2 Y_{n-2}^{\sigma(1),\sigma(n)} Y_{n-1}^{\sigma(2),\sigma(n)} Y_{n-2}^{\sigma(2),\sigma(1)} \times Y_{n-3}^{\sigma(2),\sigma(n-1)} \cdots Y_1^{\sigma(2),\sigma(3)} \times Y_{n-2}^{\sigma(1),\sigma(n-1)} \cdots Y_1^{\sigma(1),\sigma(2)}. \quad \text{(C15)}$$

Leaving aside the common prefix $ZZ_2(\Pi M_i)^2$, the products AB and BA start with the same two Y matrices, and what remains are the products of Y matrices in the expressions of AB and BA at order $n-1$. This, by induction, proves that $AB=BA$.

Finally, putting everything together, we have shown that the $n!$ properties (B12) obtained from the boundary conditions are self-compatible.

[1] M. Kardar, Nucl. Phys. **B290**, 582 (1987).
[2] T. Halpin-Healy and Y.-C. Zhang, Phys. Rep. **254**, 215 (1995).
[3] É. Brunet and B. Derrida, Physica A **279**, 398 (2000).
[4] É. Brunet and B. Derrida, Phys. Rev. E **61**, 6789 (2000).
[5] T. Emig and M. Kardar, Nucl. Phys. **B604**, 479 (2001).
[6] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[7] J. Krug, Adv. Phys. **46**, 139 (1997).
[8] F. Spitzer, Trans. Am. Math. Soc. **87**, 187 (1958).
[9] B. Drossel and M. Kardar, Phys. Rev. Lett. **53**, 5861 (1996).
[10] B. Drossel and M. Kardar, in *Theoretical and Mathematical Models in Polymer Research*, edited by A. Grosberg (Academic Press, Orlando, FL, 1998), p. 187.
[11] D. R. Nelson and A. Stern, in *Complex Behavior of Glassy Systems, XIV Sitges Conference*, edited by Miguel Rudy and Conrado Perez-Vicente, Lecture Notes in Physics Vol. 492 (Springer-Verlag, Berlin, 1997).
[12] T. Nattermann, T. Giamarchi, and P. Le Doussal, Phys. Rev. Lett. **91**, 056603 (2003).
[13] T. Giamarchi and E. Orignac, in *Theoretical Methods for Strongly Correlated Electrons*, CRM Series in Mathematical Physics (Springer, Heidelberg, 2003).
[14] J.-P. Bouchaud and H. Orland, J. Stat. Phys. **61**, 877 (1990).
[15] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).
[16] C. N. Yang and C. P. Yang, J. Math. Phys. **10**, 1115 (1969).
[17] M. Gaudin, Phys. Rev. A **4**, 386 (1971).
[18] M. Jimbo, T. Miwa, Y. Môri, and M. Sato, Physica D **1**, 80 (1980).
[19] H. B. Thacker, Rev. Mod. Phys. **53**, 253 (1981).
[20] M. Gaudin, *La Fonction d'Onde de Bethe* (Masson, Paris, 1983).
[21] H. A. Bethe, Z. Phys. **71**, 205 (1931).

[22] S. F. Edwards and P. W. Anderson, J. Phys. F: Met. Phys. **5**, 965 (1975).
[23] J. B. Mc'Guire, J. Math. Phys. **5**, 622 (1964).
[24] M. Kardar, Phys. Rev. Lett. **55**, 2923 (1985).
[25] D. A. Huse and C. L. Henley, Phys. Rev. Lett. **54**, 2708 (1985).
[26] D. A. Huse, C. L. Henley, and D. S. Fisher, Phys. Rev. Lett. **55**, 2924 (1985).
[27] U. Schulz, J. Villain, É. Brézin, and H. Orland, J. Stat. Phys. **51**, 1 (1988).
[28] T. Hwa and D. S. Fisher, Phys. Rev. B **49**, 3136 (1994).
[29] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).
[30] B. Sutherland, Phys. Rev. Lett. **20**, 98 (1968).
[31] M. Takahashi, Prog. Theor. Phys. **44**, 899 (1970).
[32] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Lett. Math. Phys. **1**, 75 (1997).