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Velocity statistics in two-dimensional granular turbulence

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We studied the macroscopic statistical properties on the freely evolving quasielastic hard disk (granular) system by performing a large-scale (up to a few million particles) event-driven molecular dynamics systematically and found it to be remarkably analogous to an enstrophy cascade process in the decaying two-dimensional fluid turbulence. There are four typical stages in the freely evolving inelastic hard disk system, which are homogeneous, shearing (vortex), clustering, and final state. In the shearing stage, the self-organized macroscopic coherent vortices become dominant. In the clustering stage, the energy spectra are close to the expectation of Kraichnan-Batchelor theory and the squared two-particle separation strictly obeys Richardson law.

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The dynamics of granular materials becomes one of the most important topics in the studies of nonlinear, dissipative, and nonequilibrium statistical physics [1]. Granular media are collections of macroscopic particles with rough surfaces, and dissipative and frictional interactions. The granular systems require an energy source in order to be in a steady state and the external gravitational force much affects their dynamics.

To focus on the dissipative features, a smooth inelastic hard sphere (IHS) model is often used as an ideal model. The freely cooling granular fluid has been studied as an ideal dissipative particle system in the absence of external force. Since the system is only composed of an inelastic hard sphere and no relevant energy scale exists, the restitution coefficient between collision particles is the only parameter in terms of nonequilibrium. The assumption of an inelastic hard sphere potential is also employed in kinetic theory, which facilitates comparisons between theory and simulation. In order to construct the theory of the macroscopic phenomenology in nonequilibrium dissipative particle system, an IHS model is the most promising as a microscopic model. A linear stability analysis of hydrodynamic equations for IHS model has revealed that the initial spatially homogeneous cooling state is unstable in the formation of vortices and clusters. The shearing (vortex) and cluster instabilities were theoretically predicted and were tested by molecular dynamics (MD) simulations [2].

Since the total energy is monotonically decreasing in the freely evolving process, a steady state in terms of energy fluctuation can be realized by scaling the velocity of the entire particle to the total energy remaining constant [3]. Here, we introduce the new-scaled time t_s which is described by

$$t_s = \int_0^t \frac{dt}{\beta(t)}, \quad \beta(t) = \sqrt{T(0)/T(t)}, \tag{1}$$

where $T(t)$ is the average kinetic energy per particle as a function of usual time t. The great advantage of this scaling operation is that the trajectories of particles do not change compared with the nonscaled case in case of hard sphere system. Therefore, one simply replaces the usual time t with the new-scaled time t_s. This operation is the same as the well-known velocity scaling method [4] in the usual MD simulation, in which the velocities of all particles, $\mathbf{v}_i(t)$, are scaled following each collision by the factor $\beta(t)$ [i.e., $\beta(t)\mathbf{v}_i(t)$] and the total energy is kept fixed strictly all over the time.

The two-dimensional (2D) turbulence in nature is a large-scale fluid motion in the atmosphere or ocean dynamics on earth. The most remarkable feature of 2D turbulence is described by the enstrophy cascade dynamics, which is completely different from that of 3D turbulence represented by the K41 theory. The existence of enstrophy cascade process was originally proposed by Kraichnan [5] and Batchelor [6]. The theory expected that the enstrophy injected at a prescribed scale is dissipated at smaller scales, undergoing a cascading process at a constant enstrophy transfer rate; this led to predicting a k^{-3} spectrum for the energy, in a range of scales extending from the injection to the dissipative scale. The granular kinetic energy spectrum in connection with the fluid turbulence was first pointed out by Taguchi [7] in 2D granular vibrated beds. He obtained the results of the $k^{-5/3}$ spectrum in his simulation with a few hundred particles. Another important nature of 2D turbulence is the self-organized coherent vortices, which develop into larger ones through the merging process of vortices with the same sign of circulation.

In this paper, we especially focus on the velocity statistics and statistical laws of the fluid turbulence. To specify what is the universal character in a dissipative system, both macroscopic equation and microscopic dissipative particle, we performed extensive event-driven molecular dynamics simulation systematically on a freely cooling process in 2D IHS model with velocity scaling thermostat. We found a strong similarity between the 2D IHS model and 2D Navier-Stokes (NS) fluid turbulence.

The freely 2D IHS (granular) model is so simple that the system is completely characterized by only three dimensionless parameters: the restitution coefficient r, the total number of disks N, and the packing fraction ν. The system size in the unit of disk diameter d is $L/d = \sqrt{\pi N/\nu}/2$. All the disks are identical, i.e., the system is monodispersed. The collision is

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instantaneous and only binary collisions occur. When two disks, i and j, with respective velocities $\mathbf{v}_i$ and $\mathbf{v}_j$ collide, the velocities after the collision, $\mathbf{v}_i'$ and $\mathbf{v}_j'$, are given by

$$\mathbf{v}_i' = \mathbf{v}_i - \tfrac{1}{2}(1+r)[\mathbf{n}\cdot(\mathbf{v}_i - \mathbf{v}_j)]\mathbf{n}, \tag{2}$$

$$\mathbf{v}_j' = \mathbf{v}_j + \tfrac{1}{2}(1+r)[\mathbf{n}\cdot(\mathbf{v}_i - \mathbf{v}_j)]\mathbf{n}, \tag{3}$$

where $\mathbf{n}$ is the unit vector parallel to the relative position of the two colliding disks in contact. Our system consists of more than 250 thousand hard disks (up to a few million) placed in a square box with periodic boundaries without any external force. To perform such a large-scale simulation, we implemented the simple and efficient event-driven algorithm, which can actually simulate more than a few million particles even in the personal computer [8]. Initially, the system is prepared as the equilibrium state by the long enough preliminary run with the restitution coefficient $r=1$, in which the density is uniform and the disk velocities are Maxwell-Boltzmann distribution. The packing fraction and the restitution coefficient (ν,r) were varied from dilute to dense and within shearing regime, which is estimated by the criterion of McNamara and Young [9], respectively. In the case of $(N,\nu)=(1024,0.25)$, McNamara and Young have found that the final states have three typical states, which are kinetic, shearing, and collapse regime. However, the spatiotemporal structure of the shearing regime is not known yet especially at the macroscopic level.

The criterion of kinetic-shearing boundary in Ref. [9] is based on the results of Jenkins and Richman [10], in which the high wave number cutoff for the unstable shear modes was derived. On the contrary, the shearing-collapse boundary is estimated by 1D theoretical analyses for inelastic collapse, which are known as the phenomenon on the divergence of the collision number during a finite time. By using the regime criterion described by McNamara and Young [9], one can find that both regime boundaries become close to the unity in the thermodynamic limit. These theoretical expectations indicate that the system is always unstable even in the quasielastic limit. This fact implies the important conjecture discussed later when we consider a large-scale IHS model as the macroscopic fluid model.

In the large-scale simulations, the restitution coefficients within the shearing regime become quasielastic $(r\sim 1)$. In our simulations by changing various parameters within the shearing regime, the system evolves to the final steady state after several stages. As described by McNamara and Young [9], we can calculate the packing fraction $\nu(\mathbf{x})$, velocity $\mathbf{u}(\mathbf{x})=(u_x(\mathbf{x}),u_y(\mathbf{x}))$, and temperature $T(\mathbf{x})$ at any point $\mathbf{x}$. Figure 1 presents a typical evolving process for four normalized properties as a function of new-scaled time t_s in the 2D IHS model. During the relaxation process, four stages can be distinguished. After the homogeneous cooling state (HCS) continues for a certain time from initial thermal equilibrium state (first stage), the short-range velocity correlation for each time (i.e., both precollisional and postcollisional velocity correlation), within the distance $\Sigma_{\mathbf{x}}\mathbf{u}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x}')$, sharply deviates from zero (solid line in Fig. 1), where $\mathbf{x}'$ is the location around $\mathbf{x}$ with a distance of disk diameter d (second stage). Note that there are several works on the existence of the short-range velocity correlations even in HCS [11,12]. Our simulation also seems to show that velocity correlation ''gradually'' increases from the beginning of simulation in the first stage. Therefore, it might be difficult to determine the exact time between first and second stages. In this stage, coherent vortices self-organize and the coherent vortices develop into larger ones through the mutual confluence and the merging process among vortices with the same sign of circulation [Fig. 2(a)]. These self-organized vortices were found first by McWilliams [13] in the direct numerical simulation (DNS) of 2D NS fluid turbulence. In the third stage, the density fluctuation ν_{rms} (dotted line in Fig. 1), which is calculated by the square root of the space average $[\nu(\mathbf{x})-\nu]^2$, exhibits instability compressive flow [Fig. 2(b)]. Finally, steady state is realized (fourth stage). In the final steady states, the spatial correlation of inelastic hard disks, which gradually increases from the beginning of simulation, might reach beyond the system size and begin to interfere with each other through the periodic boundary condition. In our simulations, in the shearing regime, there is no sign of the inelastic hard disks assembling to one cluster during the simulation time. This is because the shear mode expanded to the whole system might be stable through the periodic boundary condition. Therefore, we call them ''final steady states.'' We found there are four characteristic final steady states, which are shear (laminar, oscillating, and turbulent) and vortex (one pair of vortices with opposite sign of circulation) flows, by changing both packing fraction and the restitution coefficient within the shearing regime, systematically. Vortex flows of final pattern are also observed in the DNS simulation for 2D incompressible turbulence with the periodic boundary condition.

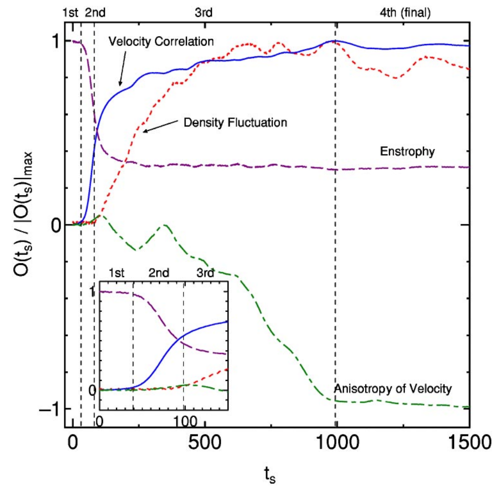

FIG. 1. The time evolution of various properties from an equilibrium, which are density fluctuation (dotted line), velocity correlation within a short distance (solid line), anisotropy of total velocity (dot-dashed line), and enstrophy (dashed line), respectively. The $O(t_s)$ indicated four normalized properties as a function of t_s. The parameters are fixed at $(r,N,\nu)=(0.991\,09,512^2,0.60)$ during the simulation. An inset in the below-left-hand corner shows the early stage of time evolution when shearing and clustering instability appears.

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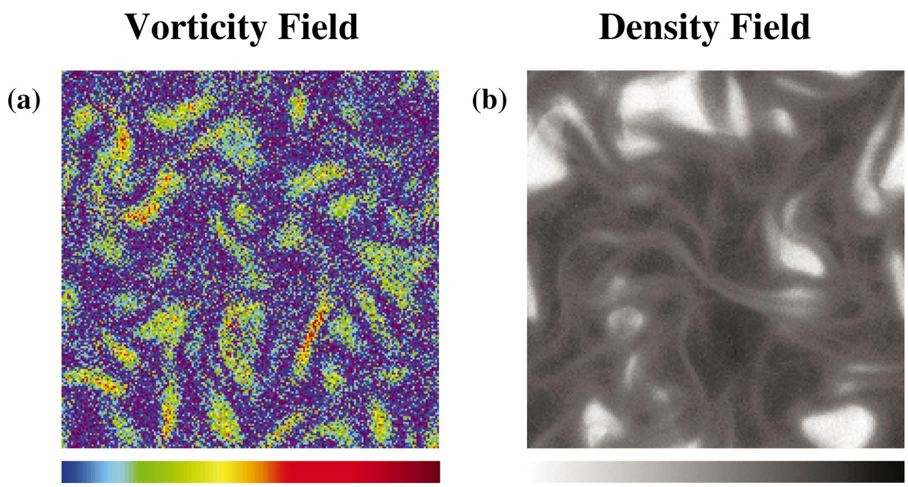

FIG. 2. (Color) The typical snapshots for coherent vortices and turbulent clustering patterns are shown. (a) The absolute vorticity field after 1200 collisions per particle with $(r,N,\nu)=(0.994\,52,640\,000,0.65)$. The self-organized coherent vortices in the vorticity field grow spontaneously from the initial equilibrium state. (b) The density field after 3590 collisions per particle in inelastic hard disk system with $(r,N,\nu)=(0.997\,25,2\,560\,000,0.65)$. The turbulent compressive flow appears in density field.

The velocity anisotropy $A=\Sigma[u_x(\mathbf{x})^2-u_y(\mathbf{x})^2]/[u_x(\mathbf{x})^2+u_y(\mathbf{x})^2]$, in which one can distinguish the final state as shear ($A=-1$ or 1) or vortex ($A=0$) flows, and the enstrophy $Z=\Sigma|\omega(\mathbf{x})|^2$, where $\omega(\mathbf{x})=rot\,\mathbf{u}(\mathbf{x})$, are also plotted by dot-dashed line and dashed line in Fig. 1, respectively. We confirmed that the total vorticity $\Sigma_{\mathbf{x}}\omega(\mathbf{x})$ is zero throughout the simulation.

The 2D fluid turbulence has different characters on the statistical law between forced and freely decaying cases. However, in the granular case, no systematic consideration seems to exist yet. In the previous studies related to energy spectrum in granular material, energy injections (thermostat) are driven by the vibration cycle [7] and the periodical-stochastic thermostat [14]. These energy injections resemble those of forced fluid turbulence. On the other hand, velocity scaling thermostat [3], in which the system is driven continuously, is thought as corresponding to a freely decaying case, because the statistics itself does not change by introducing the new-scaled time t_s. Actually, we found that the energy spectra (power spectra of velocity field) in the 2D IHS model with the velocity scaling thermostat are close to the expectation of Kraichnan-Batchelor theory $[E(k)\sim k^{-3}]$ after the third stage (Fig. 3). Therefore, our simulations show the enstrophy cascade, as is expected by the theory for freely decaying 2D fluid turbulence. We have confirmed this power law by changing several different system sizes. In Fig. 3, we can estimate the characteristic spatial scale ($k_d\sim 0.3$) for minimal dissipative domain (such as Kolmogorov scale in the fluid turbulence), which is composed of about a thousand disks.

The first quantitative phenomenological observation in developed turbulence was shown by Richardson [15], in which the two-fluid particle separation $R=|\mathbf{r}_i-\mathbf{r}_j|$ obeys power law ($\langle R^2\rangle\sim t^3$). We also show that the time t_s dependence of the space-averaged squared two-disk separation in

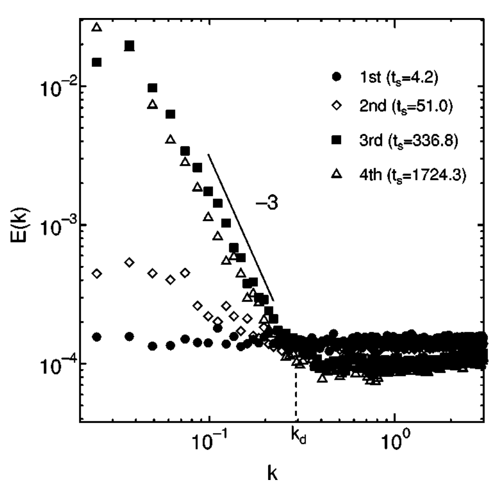

FIG. 3. Energy spectra of the velocity field are plotted for each stage. The parameters are at $(r,N,\nu)=(0.99109,512^2,0.60)$.

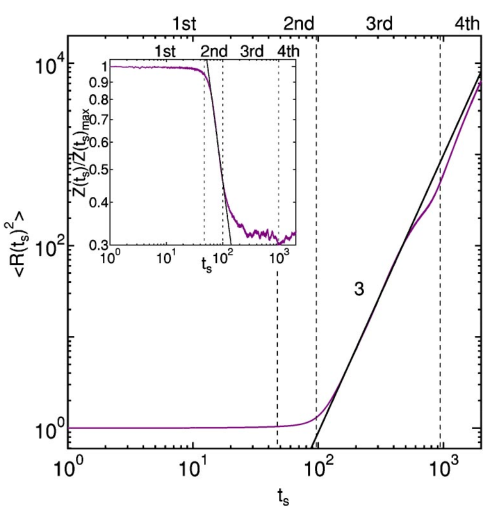

FIG. 4. The evolving squared two-particle separation in terms of new-scaled time t_s is plotted. An inset in the upper-left hand corner shows the time dependence of enstrophy decay. The parameters are at $(r,N,\nu)=(0.991\,09,512^2,0.60)$.

2D IHS model strictly obeys the Richardson dispersion law in the third stage (Fig. 4). In the inset of Fig. 4, the enstrophy evolution is also plotted in terms of the new-scaled time t_s. The enstrophy seems to decay as power-law behavior in the second stage, in which the coherent vortices self-organize. However, since the second stage itself is relatively short, this behavior needs further confirmation. As the intermittency of vorticity, McWilliams found that the probability distribution function of vorticity significantly deviates from the Gaussian [13]. By calculating flatness of vorticity ($f_\omega = \langle \omega^4 \rangle / \langle \omega^2 \rangle^2$) in the 2D IHS model, our simulations also show the deviation from the Gaussian after the third stage.

How should we understand the obtained results? The different points between fluid turbulence and granular turbulence are compressibility, the origin of dissipation (i.e., viscosity and inelasticity between particles), and the ratio of particle size and system size. Is the granular turbulence close to fluid turbulence in the thermodynamic limit? Let us assume the extreme condition, that is, dense, thermodynamic, and elastic limit. In this situation, a little amount of dissipation in the system always results in an unstable state. We also obtained the fact that the velocity correlation length (Kolmogorov scale) becomes larger in a dense (i.e., quasi-incompressible) system, but is less than system size when we consider the thermodynamic limit. Therefore, this extreme condition seems to really correspond to NS fluid turbulence.

In this paper, we showed the remarkably similar aspects on the statistical law of vorticity between 2D IHS (granular) turbulence and 2D NS fluid turbulence. These results were obtained by only solving a simple Newton's equation system for inelastic hard disks in terms of an event-driven scheme. From the microscopic dynamics of inelastic hard disk to the macroscopic fluid, it is important to study the origins of the statistical law for turbulence at the microscopic level, but there are very few studies so far from this point of view. The discussion for three limits (dense, thermodynamic, and elastic) might make a connection between 2D granular turbulence and 2D fluid turbulence.

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