Criticality and market efficiency in a simple realistic model of the stock market

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We discuss a simple model based on the minority game which reproduces the main *stylized facts* of anomalous fluctuations in finance. We present the analytic solution of the model in the thermodynamic limit. Stylized facts arise only close to a line of critical points with nontrivial properties, marking the transition to an unpredictable market. We show that the emergence of critical fluctuations close to the phase transition is governed by the interplay between the signal to noise ratio and the system size. These results provide a clear and consistent picture of financial markets, where stylized facts and verge of unpredictability are intimately related aspects of the same critical systems.

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Understanding the origin of the anomalous collective fluctuations arising in stock markets poses novel and fascinating challenges in statistical physics. Stock market prices are characterized by anomalous collective fluctuations—known as *stylized facts* [1]—which are strongly reminiscent of critical phenomena: Prices do not follow a simple random walk process, but rather price increments are fat tailed distributed and their absolute value exhibits long range autocorrelations, called volatility clustering.

The connection with critical phenomena is natural, because financial markets are indeed complex systems of many interacting degrees of freedom—the traders. However, the nature of the two phases is still unclear. By means of agent based modeling, it has been realized [2–6] that stylized facts are due to the way in which the trading activity of agents interacting in a market "dresses" the fluctuations arising from economic activity—the so-called *fundamentals*. Reference [6] has shown that very simple models based on the minority game [7] can reproduce a quite realistic and rich behavior. Their simplicity makes an analytical approach to these models possible, using tools of statistical physics. Although minority game models do not capture the full complexity of financial markets [8–10], the emergence of anomalous fluctuations in such simple models, besides providing a picture for the behavior of real markets, also poses novel questions in statistical physics which deserve interest in their own.

In this paper, we first introduce the simplest possible grand canonical minority game (GCMG) which reproduces the main stylized facts, i.e., fat tails and volatility clustering. Then we present the analytic solution of this model in the relevant thermodynamic limit. It shows that the behavior of GCMG, in this limit, exhibits Gaussian fluctuations for all parameter values, but on a line of critical points which marks a phase transition at which the market becomes informationally efficient (i.e., unpredictable). For finite size systems, numerical simulations reveal that stylized facts emerge close to the transition line, but they abruptly disappear as the system size increases. Remarkably, the vanishing of stylized facts when the system's size increases also occurs in a variety of models of financial markets [11]; note that the models of Refs. [8,9] are not affected by finite size effects. We present a theory of finite size effects which is fully confirmed by numerical simulations. This allows us to conclude that i) anomalous fluctuations are properties of the critical point in GCMG and ii) their occurrence is a consequence of markets being close to efficiency. Put differently, the standard model of mathematical finance where markets are efficient and price fluctuations are Gaussian [2] is never realized. It is exactly in the limit where markets become efficient that anomalous fluctuations arise.

The phase transition is quite unique as it mixes features which are typical of first order phase transitions—as discontinuities and phase coexistence—and of second order phase transitions—such as the divergence of correlation volumes and finite size effects.

In the market described by the minority game [7], agents $i=1,\ldots,N$ submit a bid $b_i(t)$ to the market in every period $t=1,2,\ldots$. Agents whose bid has the opposite sign of the total bid $A(t)=\Sigma_i b_i(t)$, win whereas the others lose. Agents bid according to a *trading strategy* which prescribes a bid $a_i^{\mu(t)}=\pm 1$ for each possible value of the public information variable $\mu(t)$, which is drawn uniformly from the integers $1,\ldots,P$ at each time. Each agent is assigned one trading strategy a_i^μ, randomly chosen from the set of 2^P possible strategies of this type. Agents are adaptive and may decide to refrain from playing if their strategy is not good enough [3,4]. More precisely, the bids of agents take the form $b_i(t)=\phi_i(t)a_i^{\mu(t)}$ where $\phi_i(t)=1$ or 0 according to whether agent i trades or not. In order to assess the performance of their strategy, agents assign scores $U_i(t)$ which they update by

$$U_i(t+1)=U_i(t)-a_i^{\mu(t)}A(t)-\epsilon_i, \tag{1}$$

where $A(t)=\Sigma_i\phi_i(t)a_i^{\mu(t)}$. Agents trade ($\phi_i=1$) only if their score $U_i(t)$ is large enough. Here we suppose that [12]

$$\text{Prob}\{\phi_i(t)=1\}=\frac{1}{1+e^{\Gamma U_i(t)}}, \tag{2}$$

where $\Gamma>0$ is a constant. The connection with markets goes along the lines of Refs. [4–6,10], which show that $A(t)$ is proportional to the difference of price logarithms; here, we take $\ln p(t+1)=\ln p(t)+A(t)$.

In words, an agent reward his strategy if it prescribes bids a_i^μ which tend to coincide with those $b(t)=-\text{sign}\,A(t)$ of the minority of agents. If $-a_i^{\mu(t)}A(t)$ is larger than ϵ_i, the score U_i increases. The threshold ϵ_i in Eq. (1) models the incentives of agents for trading in the market. Investors who need to trade in the market for exchanging goods or assets will have $\epsilon_i<0$. On the contrary, speculators who only trade for profiting of price fluctuations typically have $\epsilon_i>0$. Of course there may be a whole range of types of traders, from prudent investors ($\epsilon_i>0$) to risk-lover speculators ($\epsilon_i<0$). Here we focus, for simplicity, on the case $\epsilon_i=\epsilon$ for $i\leqslant N_s$ and $\epsilon_i=-\infty$ for $N_s<i\leqslant N$. The $N_p=N-N_s$ agents who have $\epsilon_i=-\infty$, called *producers* after Refs. [13,14], trade no matter what, whereas the remaining N_s, the *speculators*, trade only if their strategy puts them on the minority side often enough.

If the conditional time average $\langle A|\mu\rangle$ of $A(t)$ given $\mu(t)=\mu$ is nonzero, then the knowledge of $\mu(t)$ allows a statistical prediction of the sign of $A(t)$. A measure of predictability is hence given by

$$H_0=\overline{\langle A\rangle^2}=\frac{1}{P}\sum_{\mu=1}^{P}\langle A|\mu\rangle^2,$$

where we introduced the notation $\overline{(\cdots)}$ for averages over μ whereas $\langle\cdots\rangle$ denotes averages on the stationary state. When $H_0=0$ the market is unpredictable or *informationally efficient*. Volatility is instead defined as $\sigma^2=\overline{\langle A^2\rangle}$ and it measures market's fluctuations. A further quantity of interest is the number of active speculators, $N_{\rm act}(t)=\Sigma_i\langle\phi_i(t)\rangle$ in the market.

Exact results can be obtained in the thermodynamic limit, which is defined as the limit $N_s,N_p,P\to\infty$, keeping constant the reduced number of speculators and producers $n_s=N_s/P$ and $n_p=N_p/P$. In this limit, both σ^2 and H_0 diverge with the system size, since $A(t)\sim\sqrt{N}$. Hence we shall consider the rescaled quantities H_0/P or σ^2/P. A detailed account of the calculation will be given elsewhere [15]. Here we just discuss the main step and the results. Following Ref. [16], we derive an Ito stochastic differential equations for the strategy scores $y_i(\tau)=U_i(t)$ in the rescaled continuous time $\tau=t/N$:

$$\frac{\mathrm{d}y_i}{\mathrm{d}\tau}=-\overline{a_i\langle A\rangle_y}-\epsilon+\eta_i. \tag{3}$$

Here η_i is a zero average Gaussian noise term with

$$\langle\eta_i(\tau)\eta_j(\tau')\rangle=\frac{1}{N}\overline{a_ia_j\langle A^2\rangle_y}\delta(\tau-\tau'). \tag{4}$$

In Eqs. (3, 4) averages $\langle\cdots\rangle_y$ are taken on the distribution of $\phi_i(t)$ in Eq. (2), which depends on $y_i(\tau)$ in a nonlinear way: $\text{Prob}\{\phi_i(t)=1\}=1/[1+e^{\Gamma y_i(\tau)}]$. Hence Eq. (3) is a quite complex system of nonlinear equations with a noise strength proportional to the time dependent volatility $\overline{\langle A^2\rangle_y}$. This feedback will be responsible for the emergence of volatility buildups.

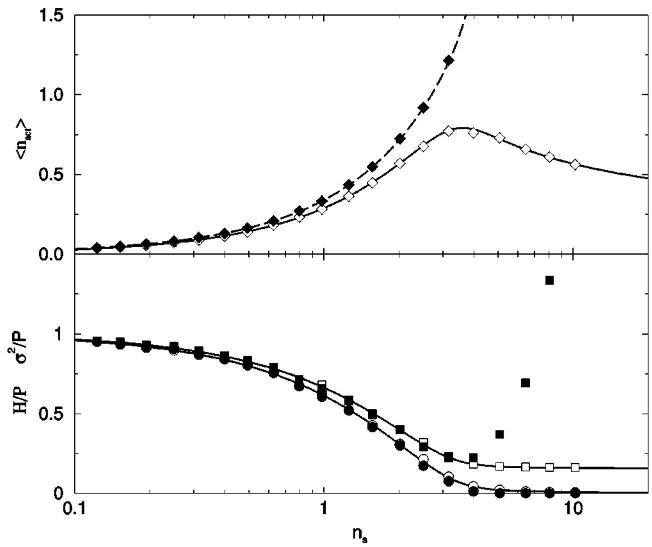

FIG. 1. Theory and numerical simulations: $n_{\rm act}$ (top) and σ^2/P and H/P (bottom) as a function of n_s for $\epsilon=0.1$ (solid line) and $\epsilon=-0.01$ (dashed line). Numerical results for $\epsilon=0.1$ (open symbols) and $\epsilon=-0.01$ (full symbols) are averages over 200 runs, with $N_sP=10\,000$ fixed and $\Gamma=\infty$.

Following Refs. [16,17] we find that the fraction $\langle\phi_i\rangle$ of times that agent i plays his active strategy in the stationary state is the solution of the minimization of the function

$$H_\epsilon=\frac{1}{P}\sum_{\mu=1}^{P}\left[\sum_{i=1}^{N}\langle\phi_i\rangle a_i^\mu+\sum_{i=N_s+1}^{N_s+N_p}a_i^\mu\right]^2+2\epsilon\sum_i\langle\phi_i\rangle, \tag{5}$$

with respect to $\langle\phi_i\rangle$. Note that for $\epsilon=0$ this function reduces to the predictability H_0. For $\epsilon\neq0$, the solution to this problem, and hence the stationary state, is unique. An exact statistical mechanics description of the solution $\{\langle\phi_i\rangle\}$ can be carried out with the replica method, because the replica symmetric ansatz is exact. Furthermore, the solution to the Fokker-Planck equation corresponding to Eq. (3) can be well approximated by a factorized ansatz for $\epsilon>0$. This means that the off-diagonal correlations vanish $[\langle(\phi_i-\langle\phi_i\rangle)(\phi_j-\langle\phi_j\rangle)\rangle=0$, for $i\neq j]$ and, as a consequence, the volatility turns out to be given by $\sigma^2=\overline{\langle A^2\rangle}=H_0+\Sigma_{i=1}^{N_s}\langle\phi_i\rangle(1-\langle\phi_i\rangle)$. The solution $\{\langle\phi_i\rangle\}$ of the minimization of H_ϵ provides a complete description of the model in the limit $N\to\infty$ for $\epsilon>0$. In particular the behavior of σ^2 is independent of Γ.

Figure 1 shows that all these conclusions are perfectly supported by numerical simulations: With a fixed number n_p of producers, as the number n_s of speculators increases, the market becomes more and more unpredictable, i.e., H_0 decreases. At the same time also the volatility σ^2 decreases. In a market with few speculators ($n_s<1$ in Fig. 1), most of the fluctuations in $A(t)$ are due to the random choice of $\mu(t)$ (i.e., $\sigma^2\simeq H_0$) and the number $n_{\rm act}$ of active speculators grows approximately linearly with n_s.

When n_s increases further, the market reaches a point where it is barely predictable. Then, for $\epsilon>0$ the number of

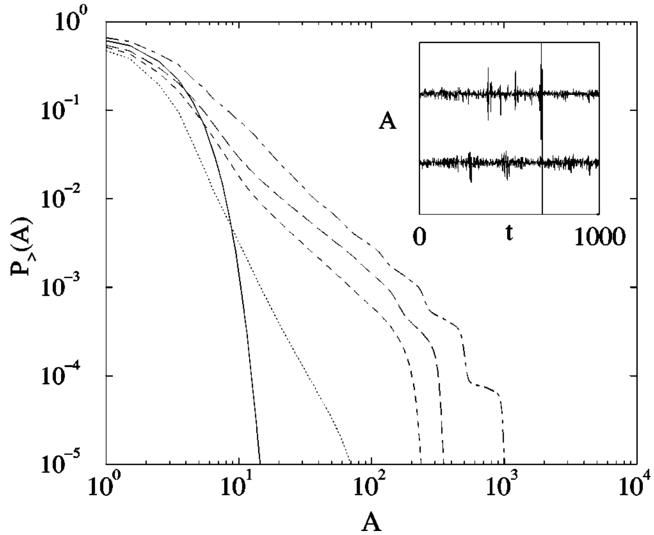

FIG. 2. Probability distribution of $A(t)>A$ for $n_s=10$ (continuous line), 20, 50, 100, and 200 (dash-dotted line) ($PN_s=16\,000$, $n_p=1$, $\epsilon=0.01$, $\Gamma=\infty$). Inset: time series of returns $A(t)$ showing volatility clustering for $n_s=20$ (lower curve), but not for $n_s=200$ (upper curve).

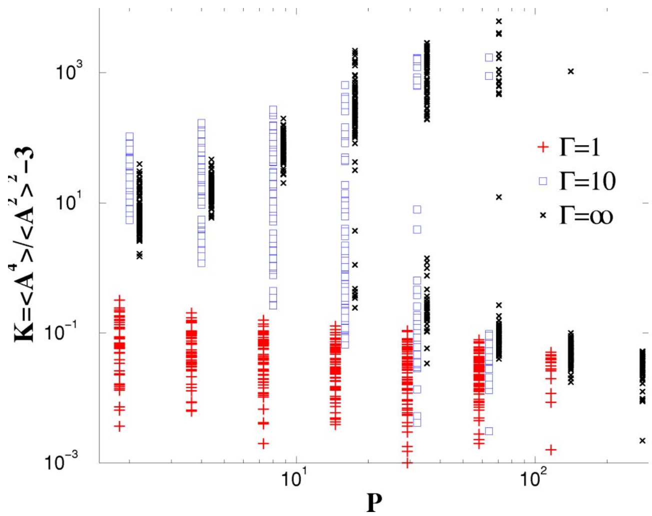

FIG. 3. (Color online) Excess kurtosis of $A(t)$ in simulations with $\epsilon=0.01$, $n_s=70$, $n_p=1$, and several different system sizes P for $\Gamma=1$, 10, and ∞.

active traders decreases and finally converges to a constant. This means that the market becomes highly selective: Only a negligible fraction of speculators trade ($\phi_i(t)=1$) whereas the majority is inactive ($\phi_i(t)=0$). The volatility σ^2 also remains constant in this limit.

For $\epsilon<0$ we see a markedly different behavior: The number of active speculators continues growing with n_s even if the market is unpredictable $H_0\approx 0$. The volatility σ^2/P has a minimum and then it increases with n_s in a way which depends on Γ. In other words, $\epsilon=0$ for $n_s\geqslant n_s^\star(n_p)$ ($=4.15\ldots$, for $n_p=1$) is the locus of a first order phase transition across which $N_{\rm act}$ and σ^2 exhibit a discontinuity. This same picture applies to a wider range of GCMG models such as that of Ref. [6].

Numerical simulations reproduce anomalous fluctuations similar to those of real financial markets close to the phase transition line. As shown in Fig. 2, the distribution of $A(t)$ is Gaussian for small enough n_s, and has fatter and fatter tails as n_s increases; the same behavior is seen for decreasing ϵ. In particular the distribution of $A(t)$ shows a power law behavior $P(|A|>x)\sim x^{-\beta}$ with an exponent which we estimated as $\beta\simeq 2.8$ and 1.4 for $n_s=20$ and 200 respectively and $\epsilon=0.01$. Note that a realistic value $\beta\approx 3$ [19] is obtained for $n_s=20$.

This is inconsistent, at first sight, with the theoretical results discussed previously for $N\to\infty$. Indeed, if the distribution of ϕ_i factorizes, $A(t)$ is the sum of N_s independent contributions and it satisfies the Central Limit Theorem. This implies that for $\epsilon\neq 0$ the variable $A(t)/\sqrt{N}$ converges in distribution to a Gaussian variable with zero average and variance σ^2/N in the limit $N\to\infty$ at fixed α. There are no anomalous fluctuations and no stylized facts. Figure 3 indeed shows that the anomalous fluctuations of Fig. 2 are finite size effects which disappear abruptly as the system size increases (or if Γ is small).

In order to understand these finite size effects, we note that volatility clustering arises because the noise strength in Eqs. (3,4) is proportional to the time dependent volatility $\langle A^2\rangle_y$. The noise term is a source of correlated fluctuations because $\overline{a_i a_j\langle A^2\rangle_y}/N\sim 1/\sqrt{N}$ is small but nonzero, for $i\neq j$. It is reasonable to assume that the dynamics will sustain collective correlated fluctuations in the y_i only if the correlated noise is larger than the signal $-\overline{a_i\langle A\rangle_y}-\epsilon$, which agents receive form the deterministic part of Eq. (3). Time dependent volatility fluctuations would be dissipated by the deterministic dynamics otherwise. A quantitative translation of this insight goes as follows: The noise correlation term is of order $\overline{a_i a_j\langle A^2\rangle_y}/N\sim\sigma^2/P^{(3/2)}$, for $i\neq j$. This should be compared to the square of the deterministic term of Eq. (3) $[\overline{a_i\langle A\rangle_y}+\epsilon]^2\sim[\sqrt{H_0/P}+\epsilon]^2$. Rearranging terms, we find that volatility clustering sets in when

$$\frac{H_0}{\sigma^2}+2\epsilon\sqrt{\frac{H_0}{P}}\,\frac{P}{\sigma^2}+\epsilon^2\frac{P}{\sigma^2}\simeq\frac{K}{\sqrt{P}}, \qquad (6)$$

where K is a constant. This prediction is remarkably well confirmed by Fig. 4: In the lower panel we plot the two sides of Eq. (6) as a function of n_s for different system sizes. The upper panel shows that the volatility σ^2/N starts deviating from the analytic result exactly at the crossing point $n_s^c(P)$ where Eq. (6) holds true. Furthermore the inset shows that the region $n_s>n_s^c(P)$ is described by a different type of scaling limit. Indeed the curves of Fig. 4 collapse one on top of the other when plotted against $n_s/n_s^c(P)$.

The nonlinearity of the response of agents is crucial for the onset of volatility time dependence. If Γ is small the response becomes smooth and anomalous fluctuations disappear (see Fig. 3). This picture is not affected by the introduction of a finite memory in the learning process of agents, for example in Ref. [18]. In particular the exponents of Fig. 2 do not depend on the memory.

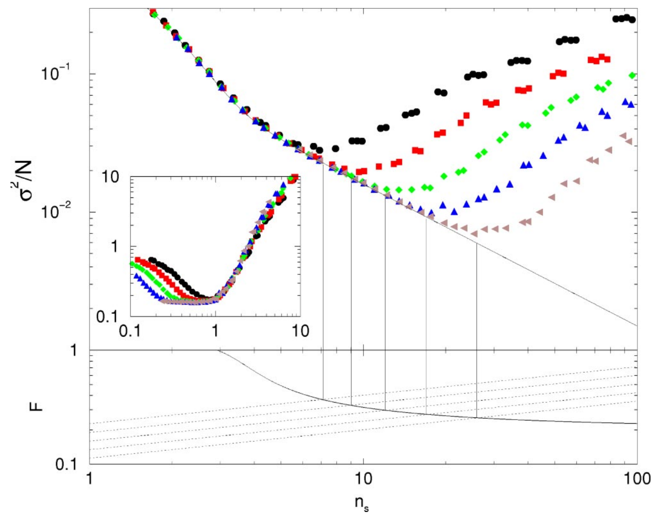

FIG. 4. (Color online) Onset of the anomalous dynamics for different system sizes. Top: σ^2/N for different series of simulations with $L\equiv PN_s$ constant: $PN_s=1000$ (circles), 2000 (squares), 4000 (diamonds), 8000 (up triangles) and 16 000 (left triangles). In all simulations $n_p=1$, $\epsilon=0.1$, and $\Gamma=\infty$. Bottom: Left-hand side of Eq. (6) (full line) from the exact solution and $K/\sqrt{P}=K(n_s/L)^{1/4}$ (parallel dashed lines) as a function of n_s ($K\simeq 1.1132$ in this plot). The intersection defines $n_s^c(P)$. Inset: Collapse plot of σ^2/N as a function of $n_s/n_s^c(P)$.

The fact that, in finite systems, stylized facts arise only close to the phase transition is reminiscent of finite size scaling in the theory of critical phenomena: In d-dimensional Ising model, for example, at temperature $T=T_c+\varepsilon$ critical fluctuations (e.g., in the magnetization) occur as long as the system size N is smaller than the correlation volume $\sim\varepsilon^{-d\nu}$. But for $N\gg\varepsilon^{-d\nu}$ the system shows the normal fluctuations of a paramagnet.

Equation (6) and $H_0/P\sim\epsilon^2$ imply that the same occurs in the GCMG with $d\nu=4$. In other words, the critical window shrinks as $N^{-1/4}$ when $N\to\infty$. However, because of the long range nature of the interaction, anomalous fluctuations either concern the whole system or do not affect it at all, as clearly shown in Fig. 3. In the critical region the Gaussian phase coexists probabilistically with a phase characterized by anomalous fluctuations. This and the discontinuous nature of the transition at $\epsilon=0$, are usually typical of first order phase transitions.

The picture of collective correlated fluctuations controlled by the signal to noise ratio appears to be universal for minority games. Finite size effects close to the phase transition of the standard minority game [7,15] are indeed explained by the same generic argument: When the signal to noise ratio H_0/σ^2 is of order $1/\sqrt{P}$ self-sustained collective fluctuations arise. In addition, finite size effects appear at a distance of order $P^{-1/4}$ from the critical point.

Volatility clustering also occur only close to the phase transition in the GCMG. The effect, in real markets is known to be due to wild fluctuations in the volume of trades [19]. Volume is the number of active traders $N_{\mathrm{act}}+N_p$ in the GCMG. Wild volume fluctuations can only occur because of correlated collective fluctuations which arise close to criticality. Numerical simulations suggest that exponents vary continuously on the line of critical points. This raises the question of why real markets self-organize close to the critical surface with $\beta\approx 3$.

We conclude that the GCMG exhibits a critical behavior which is very similar to that observed in real markets. This, with the observation that real markets are indeed close to being informationally efficient, strongly suggests that real markets operate close to criticality. The phase transition is quite peculiar, with properties of both continuous and discontinuous transitions. The extension of renormalization group approaches to this system promises to be a quite interesting challenge.

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