Anomalous diffusion, localization, aging, and subaging effects in trap models at very low temperature

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We study in detail the dynamics of the one-dimensional symmetric trap model via a real-space renormalization procedure which becomes exact in the limit of zero temperature. In this limit, the diffusion front in each sample consists of two δ peaks, which are completely out of equilibrium with each other. The statistics of the positions and weights of these δ peaks over the samples allows to obtain explicit results for all observables in the limit $T \to 0$. We first compute disorder averages of one-time observables, such as the diffusion front, the thermal width, the localization parameters, the two-particle correlation function, and the generating function of thermal cumulants of the position. We then study aging and subaging effects: our approach reproduces very simply the two different aging exponents and yields explicit forms for scaling functions of the various two-time correlations. We also extend the real-space renormalization group method to include systematic corrections to the previous zero temperature procedure via a series expansion in T. We then consider the generalized trap model with parameter $\alpha \in [0,1]$ and obtain that the large scale effective model at low temperature does not depend on α in any dimension, so that the only observables sensitive to α are those that measure the "local persistence," such as the probability to remain exactly in the same trap during a time interval. Finally, we extend our approach at a scaling level for the trap model in $d=2$ and obtain the two relevant time scales for aging properties.

I. INTRODUCTION

A. Trap models as toy models to study aging phenomena

Trap models provide a simple mechanism for aging [1,2]. The basic phenomenological idea is that the slow dynamics of glassy systems is governed by metastable states defined as "traps" in the coarse-grained configurational complicated landscape. The distribution of the energy of the traps is usually taken to be exponential,

$$\rho(E) = \theta(E) \frac{1}{T_g} e^{-E/T_g}. \tag{1}$$

On one hand, this exponential form describes the lowest energies in the random energy model [3] and the distribution of free energy of states in the replica theory of spin glasses [4]. On the other hand, it appears for the largest barriers in the biased one-dimensional Sinai diffusion [1,5] as well as in more complex disordered systems such as fractals and percolation clusters [6], elastic manifolds [7], bubble dynamics in DNA [8], and sequence alignment algorithms [9]. The ubiquity of this exponential form actually comes [10] from the exponential tail of the Gumbel distribution which represents one universality class of extreme-value statistics [11].

The exponential density of energy (1) corresponds for the Arrhenius trapping time $\tau = e^{\beta E}$ to the algebraic law

$$q(\tau) = \theta(\tau > 1) \frac{\mu}{\tau^{1+\mu}}, \tag{2}$$

with the temperature-dependent exponent

$$\mu = \frac{T}{T_g}. \tag{3}$$

At low temperatures $T < T_g$, the mean trapping time $\int d\tau \, \tau q(\tau)$ is infinite and this directly leads to aging effects. This mechanism shows that the presence of broad distribution of trapping times (2) is rather generic at low temperatures, since it simply emerges from the exponential tail of extreme-value statistics for the energy barriers.

B. Previous results on aging properties in trap models

The dynamics of the trap model has been studied in detail in its the mean field version [12–15] as well as in the one-dimensional *directed* version, in relation with the biased Sinai diffusion [1,16–20]. In both the cases, aging properties are characterized by scaling functions of the ratio (t/t_w) of the two times involved.

More recently, it has been proposed in Ref. [21] to study trap models on a hypercubic lattice in an arbitrary dimension d with the following generalized dynamics: the particle can jump from site i to any of the $2d$ nearest-neighbor sites j with a hopping rate per unit time given by

$$w_{i \to j}(\alpha) = \frac{1}{2d} e^{+\beta \alpha E_j - \beta(1-\alpha)E_i} \tag{4}$$

in terms of the parameter $\alpha \in [0,1]$. The case $\alpha = 0$ represents the usual trap model where the rate

$$w_{i \to j}(\alpha=0) = \frac{1}{2d \tau_i} \tag{5}$$

depends only on the initial site via the trapping time $\tau_i = e^{\beta E_i}$ distributed with Eq. (2): the particle spends at site i a time t_i distributed with the exponential distribution of mean τ_i,

$$f_{\tau_i}(t_i) = \frac{1}{\tau_i} e^{-t_i/\tau_i}, \tag{6}$$

and then jumps with equal probability $1/(2d)$ to one of its $(2d)$ nearest-neighbor sites. Another interesting case is $\alpha = \frac{1}{2}$, where the rate $w_{i \to j}$ depends on the energy difference $(E_j - E_i)$.

Monte Carlo simulations and scaling arguments [21] have shown the possibility of a so-called "subaging" behavior for the probability $\Pi(t + t_w, t_w)$ of no jump during the interval $[t_w, t_w + t]$,

$$\Pi(t + t_w, t_w) \simeq \Pi\left(\frac{t}{t_w^{\nu(\alpha)}}\right), \tag{7}$$

with an exponent $\nu < 1$ given in one dimension by [21]

$$\nu(\alpha) = \frac{1 - \alpha}{1 + \mu}. \tag{8}$$

This exponent was proven to be exact by mathematicians, first for the usual trap model $\alpha = 0$ [22], and then for arbitrary α [24]. On the other hand, the correlation function $C(t + t_w, t_w)$, defined as the probability to be at $(t + t_w)$ in the same trap as it was at time t_w, was shown to present a "full aging" behavior,

$$C(t + t_w, t_w) \simeq \mathcal{C}_\mu\left(\frac{t}{t_w}\right), \tag{9}$$

for the usual trap model $\alpha = 0$ [22], and then for arbitrary α [24]. So there are two different time scales t_w^ν and t_w which play a role in the aging of this model. Asymptotic forms have also been heuristically proposed and numerically tested in Ref. [23] for $\Pi(t + t_w, t_w)$ and $C(t + t_w, t_w)$. Finally, let us mention a recent interesting application: these properties of aging and subaging for the trap model are relevant to explain the numerical simulations on the dynamics of denaturation bubbles in random DNA sequences [8].

C. Previous results on anomalous diffusion and localization properties

Apart from aging properties discussed above, trap models are interesting for their anomalous diffusion and localization properties. In particular, in dimension $d = 1$, the averaged diffusion front is expected to take the following scaling form at large times [23,25,27]:

$$\overline{P(n, t|0,0)} \underset{t \to \infty}{\simeq} \frac{1}{\xi(t)} g_\mu\left(\frac{n}{\xi(t)}\right), \tag{10}$$

where the characteristic length scale $\xi(t)$ follows the subdiffusive behavior

$$\xi(t) \sim t^{\mu/(1+\mu)}. \tag{11}$$

This exponent can be found via a simple scaling argument on Lévy sums [25,26] or by a real-space block-RG (renormalized group) analysis [27]. However, the scaling function itself $g(X)$ is not known, but has recently been studied numerically in Ref. [23] together with asymptotic behaviors proposed in the limit $\mu \to 1$.

Another important issue concerns the localization properties. The localization parameters

$$Y_k(t) = \sum_{n=-\infty}^{+\infty} \overline{P^k(n, t|0,0)} \tag{12}$$

represent the disorder averaged probabilities that k independent particles starting at site 0 at time 0 in the same random environment are at the same site at time t. It has been proven in Ref. [22] that the limit $Y_2(\infty)$ is strictly positive in the full domain $0 \leq \mu < 1$. The values of the limits $Y_k(\infty)$ have been numerically studied in Ref. [23] with various approximations.

D. Summary of main results

The aim of this paper is to provide a probabilistic description, sample by sample, of the symmetric trap model in the limit of very low temperature $\mu \to 0$. We have previously developed a similar analysis for the *directed* version of the trap model [20]. Here in the undirected version, each site may be visited many times and this leads to essential changes. In particular, at time t, the important traps are the traps having a trapping time $\tau > R(t)$, where the scale $R(t)$ is not linear in t as in the *directed* version [20], but is sublinear in time

$$R(t) = \left(\frac{t}{\tilde{T}_0(\mu)}\right)^{1/(1+\mu)}, \tag{13}$$

where $\tilde{T}_0(\mu)$ may be expanded in μ as

$$\tilde{T}_0(\mu) \underset{\mu \to 0}{\simeq} 2e^{-1-\gamma_E}[1 + O(\mu)]. \tag{14}$$

The corresponding mean distance between these important traps is then given by

$$\xi(t) = \xi_0(\mu) t^{\mu/(1+\mu)}. \tag{15}$$

where the exponent agrees with previous studies (11) described above, and where the prefactor reads

$$\xi_0(\mu) = 1 + O(\mu). \tag{16}$$

In terms of these scales, we obtain the following explicit results in the limit $\mu \to 0$.

(a) Scaling function of the disorder averaged diffusion front (10)

$$g_\mu(X) = e^{-|X|} \int_0^{+\infty} du \, e^{-u} \frac{u}{|X| + u} + O(\mu). \tag{17}$$

(b) Localization parameters (12)

$$Y_k(\mu) = \frac{2}{(k+1)} + O(\mu). \qquad (18)$$

(c) Generating function of thermal cumulants

$$Z_\mu(s) \equiv \overline{\ln\langle e^{-[s\,n/\xi(t)]}\rangle} = \int_0^{+\infty} d\lambda \ e^{-\lambda}\lambda\left(\frac{s\lambda}{2}\coth\frac{s\lambda}{2} - 1\right)$$
$$+ O(\mu). \qquad (19)$$

The series expansion in s yields the disorder averages of rescaled thermal cumulants: the first ones are the thermal width

$$c_2(\mu) \equiv \lim_{t\to\infty} \frac{\overline{\langle n^2\rangle - \langle n\rangle^2}}{\xi^2(t)} = 1 + O(\mu), \qquad (20)$$

and the fourth cumulant

$$c_4(\mu) \equiv \lim_{t\to\infty} \frac{\overline{\langle n^4\rangle - 4\langle n^3\rangle\langle n\rangle - 3\langle n^2\rangle^2 + 12\langle n^2\rangle\langle n\rangle^2 - 6\langle n\rangle^4}}{\xi^4(t)}$$
$$= -4 + O(\mu), \qquad (21)$$

and more generally

$$c_k(\mu) = (2k+1)!B_{2k} + O(\mu) \qquad (22)$$

in terms of the Bernoulli numbers B_n.

(d) Two-particle correlation function

$$C(l,t) \equiv \overline{\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} P(n,t|0,0)P(m,t|0,0)\,\delta_{l,|n-m|}}$$
$$\underset{t\to\infty}{\simeq} Y_2(\mu)\,\delta_{l,0} + \frac{1}{\xi(t)}\,\mathcal{C}_\mu\left(\frac{l}{\xi(t)}\right), \qquad (23)$$

where the weight of the δ peak at the origin corresponds as it should to the localization parameter $Y_2 = 2/3 + O(\mu)$ (18), whereas the second part presents a scaling form of the variable $\lambda = l/\xi(t)$. The scaling function $\mathcal{C}_\mu$ reads

$$\mathcal{C}_\mu(\lambda) = e^{-\lambda}\frac{\lambda}{3} + O(\mu). \qquad (24)$$

(e) The probability $\Pi(t+t_w,t_w)$ of no jump during the interval $[t_w, t_w+t]$ takes the scaling form (7)

$$\Pi(t+t_w,t_w) = \widetilde{\Pi}_\mu\left(g = \frac{t}{R(t_w)}\right)$$
$$= \widetilde{\Pi}_\mu\left(g = [\widetilde{T}_0(\mu)]^{1/(1+\mu)}\frac{t}{t_w^{1/(1+\mu)}}\right), \qquad (25)$$

with the scaling function

$$\widetilde{\Pi}_\mu^{(0)}(g) = \int_0^1 dz \ \mu z^{\mu-1}e^{-zg}. \qquad (26)$$

In particular, we obtain the asymptotic behavior

$$\Pi(t+t_w,t_w) \underset{\frac{t}{t_w^{1/(1+\mu)}}\to+\infty}{\simeq} \left(\frac{t}{t_w^{1/(1+\mu)}}\right)^{-\mu}[\mu + O(\mu^2)]. \qquad (27)$$

(f) The probability $C(t+t_w,t_w)$ to be at time $(t+t_w)$ in the same trap as it was at time t_w takes the scaling form (9)

$$C(t+t_w,t_w) = \widetilde{C}_\mu\left(h = \frac{t}{R^{1+\mu}(t_w)}\right) = \widetilde{C}_\mu\left(h = \widetilde{T}_0(\mu)\frac{t}{t_w}\right), \qquad (28)$$

with the scaling function which reads at lowest order in μ,

$$\widetilde{C}_\mu^{(0)}(h) = \widetilde{C}_\mu(h) = \frac{2\mu}{(2h)^\mu}\int_0^{\sqrt{2h}} dz \ z^{1+2\mu}K_1^2(z). \qquad (29)$$

In particular, we obtain the asymptotic behavior

$$C(t+t_w,t_w) \underset{t/t_w\to\infty}{\simeq} \left(\frac{t}{t_w}\right)^{-\mu}[\mu + O(\mu^2)]. \qquad (30)$$

We also extend the real-space renormalized group (RSRG) method to include systematic corrections to the zero temperature procedure via a series expansion in μ; the corrections of the order of μ of the observables described above are given in Appendix C. We also extend our analysis to the generalized trap models (4) and obtain that the only observable that depends on α is the two-time correlation $\Pi(t,t')$, which takes the scaling form

$$\Pi_\mu^{(\alpha)}(t+t_w,t) = \widetilde{\Pi}_{\mu/(1-\alpha)}^{(0)}\left(v = \frac{t}{[R(t_w)]^{1-\alpha}}\right) \qquad (31)$$

in terms of result (26) for the $\alpha = 0$ case, but that otherwise all other observables described above are exactly the same as in the case $\alpha = 0$. The reason is that the influence of α is purely local around a renormalized trap and does not change the renormalized effective model at large scales. Finally, we also extend our RSRG approach to the trap model in dimension $d = 2$ at a scaling level.

E. Organization of the paper

The paper is organized as follows. In Sec. II, we defined the renormalized landscape for the usual trap model $\alpha = 0$ and study its properties: in particular, we obtain the relevant length scale and the two relevant time scales. In Sec. III, we describe the effective dynamics in the limit $\mu\to 0$ and compute one-time and two-times observables in this limit. In Sec. IV, we study the corrections to the effective dynamics at first order in μ and we describe the hierarchical structure of the important traps that play a role at order μ^n. In Sec. V, we

extend our approach to the generalized trap with parameter $\alpha \in [0,1]$. In Sec. VI, we extend our RSRG approach to the trap model in $d=2$ at a scaling level. The conclusions are given in Sec. VII, and the Appendixes contain more technical details.

II. DEFINITION AND PROPERTIES OF THE RENORMALIZED LANDSCAPE

A. Notion of renormalized landscape at scale R

We wish to adapt the real-space renormalization procedure already defined for the Sinai model [5] and for the directed trap model [20] to the undirected trap model. The basic idea is that the dynamics at large time is dominated by the statistical properties of the large trapping times. The renormalized landscape at scale R is defined as follows: all traps with trapping time $\tau_i < R$ are decimated and replaced by a "flat landscape," whereas all traps with waiting time $\tau_i > R$ remain unchanged. The distribution of the distance l between two traps of the renormalized landscape at scale R reads

$$P_R(l) = \left[1 - \int_R^{+\infty} d\tau \, q(\tau) \right]^{l-1} \int_R^{+\infty} d\tau \, q(\tau), \quad (32)$$

where the first part $[\cdots]^{l-1}$ represents the probability that $(l-1)$ traps have a trapping time $\tau_i < R$, and where the last part represents the probability that the lth trap has a trapping time $\tau_i > R$. So the appropriate rescaled length variable at large scale R is

$$\lambda = \frac{l}{R^\mu}, \quad (33)$$

and the scaling distribution is simply exponential,

$$\mathcal{P}(\lambda) = e^{-\lambda}. \quad (34)$$

The distribution of the trapping times of the traps in the renormalized landscape at scale R is simply

$$q_R(\tau) = \theta(\tau > R) \frac{q(\tau)}{\int_R^{+\infty} d\tau' \, q(\tau')} = \theta(\tau > R) \frac{\mu}{\tau} \left(\frac{R}{\tau} \right)^\mu. \quad (35)$$

In the directed version of the model, the particle visits each site only once, and the RSRG analysis directly deals with the trapping times τ_i. However, here in the undirected version of the model, each site may be visited many times, and thus, it is necessary to introduce the notion of "escape time" as we now explain.

B. Notion of escape time from a renormalized trap to another renormalized trap

We now introduce the notion of the escape time T from a trap τ_0 existing in the renormalized landscape at scale R. This trap is surrounded by two renormalized traps that are at
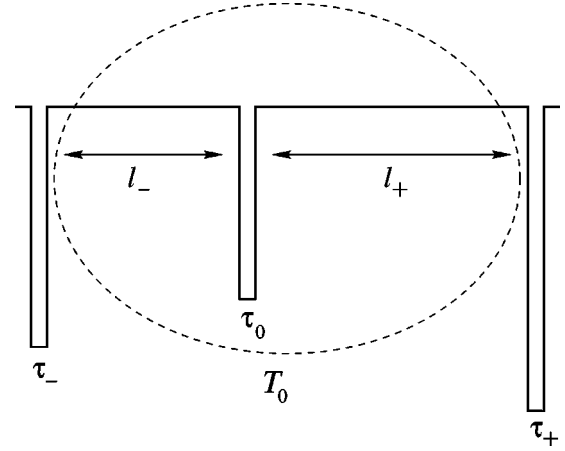

FIG. 1. Definition of the escape time from a trap in the renormalized landscape: the trap of escape time τ_0 existing in the renormalized landscape at scale R is surrounded by two renormalized traps that are at distances l_+ and l_- on each side. The escape time T_0 is the mean time needed to reach either τ_+ or τ_- when starting at τ_0.

distances l_+ and l_- on each side (see Fig. 1). Whenever the particle escapes from the trap τ_0, it can escape on either side with probability 1/2. If it escapes on the left, it will succeed to reach the trap τ_- with probability $1/l_-$ and if it escapes on the right, it will succeed to reach the trap τ_+ with probability $1/l_+$. Otherwise, it will be reabsorbed again by the trap τ_0.

1. Number of sojourns in a renormalized trap before escape to a neighbor renormalized trap

As a consequence, the probability $E_R(n)$ to need $(1+n)$ successive sojourns in the trap τ_0 before the particle succeeds to escape either to the trap τ_- or to the trap τ_+ reads

$$E_R(n) = \left[1 - \frac{1}{2} \left(\frac{1}{l_+} + \frac{1}{l_-} \right) \right]^n \frac{1}{2} \left(\frac{1}{l_+} + \frac{1}{l_-} \right). \quad (36)$$

For large R, since we have $l_\pm = R^\mu \lambda_\pm$ (33), the number n follows the same scaling: the rescaled variable

$$w = \frac{n}{R^\mu} \quad (37)$$

is distributed exponentially,

$$\mathcal{E}(w) = a(\lambda_+, \lambda_-) e^{-wa(\lambda_+, \lambda_-)}, \quad (38)$$

where the coefficient

$$a(\lambda_+, \lambda_-) = \frac{1}{2} \left(\frac{1}{\lambda_+} + \frac{1}{\lambda_-} \right) \quad (39)$$

depends on the two rescaled distances to the next traps. Its distribution $k(a)$ over the samples has the following for the Laplace transform in terms of the Bessel function K_1 (E1):

$$\hat{k}(s) \equiv \int_0^{+\infty} da \; e^{-sa} k(a)$$

$$= \left[\int_0^{+\infty} d\lambda \; \mathcal{P}(\lambda) e^{-s/2\lambda} \right]^2 = 2sK_1^2(\sqrt{2s}). \quad (40)$$

In particular, the nonanalytic behavior at small s (E3),

$$\hat{k}(s) \underset{s \to 0}{\simeq} 1 - s \ln\frac{1}{s} + O(s), \quad (41)$$

corresponds to the following algebraic decay at large a:

$$k(a) \underset{a \to \infty}{\simeq} \frac{1}{a^2}. \quad (42)$$

The decay at large s (E2),

$$\hat{k}(s) \underset{s \to \infty}{\simeq} \pi \sqrt{\frac{s}{2}} e^{-2\sqrt{2s}} \quad (43)$$

corresponds to the following essential singularity at small a:

$$k(a) \underset{a \to 0}{\simeq} \frac{\sqrt{2\pi}}{a^{5/2}} e^{-2/a}. \quad (44)$$

2. Total time spent inside a renormalized trap before escape to a neighbor renormalized trap

Let us now consider the probability distribution $P_{in}(t_{in})$ of the total time t_{in} spent inside the trap τ_0 before its escape. It can be decomposed into the number n of sojourns, where n is distributed with Eq. (36),

$$t_{in} = \sum_{i=1}^{1+n} t_i, \quad (45)$$

where t_i is the time spent during the sojourn i in the trap τ_0, so it is distributed with the exponential distribution (6) with $\tau_i = \tau_0$. Actually, since n is large in the large R limit, we have the central-limit theorem

$$t_{in} \underset{n \to \infty}{\simeq} n\langle t_i \rangle = n\tau_0. \quad (46)$$

This explains the numerical observation [23] that the results are unchanged if the particle spends a time exactly equal to τ_i at each visit to site i, instead of a random time t_i distributed with Eq. (6).

Since the number n is distributed with Eqs. (37) and (38), we finally obtain that t_{in} is also exponentially distributed,

$$\hat{P}_{in}(t_{in}) \underset{R \to \infty}{\simeq} \frac{1}{T_0} e^{-t_{in}/T_0}, \quad (47)$$

with the characteristic time

$$T_0 = \frac{1}{a} R^\mu \tau_0. \quad (48)$$

Since the smallest trapping time existing in the renormalized landscape at scale R is $\tau_0 = R$, the time spent inside the trap τ_0 before it succeeds to escape scales as

$$t_{in} \underset{R \to \infty}{\sim} R^{1+\mu}. \quad (49)$$

3. Total time spent during the unsuccessful excursions before the escape

Among the n unsuccessful excursions, there are m excursions on the left and $(n-m)$ excursions on the right, where m is distributed with the binomial distribution $2^{-n} C_n^m$. Since n and m are large, we again have a central-limit theorem

$$t_{out} = \sum_{i=1}^{m} t_i^- + \sum_{j=1}^{n-m} t_j^+ \simeq m\langle t_- \rangle + (n-m)\langle t_+ \rangle, \quad (50)$$

where $\langle t_\pm \rangle$ represents the mean time needed to return to 0 when starting at 1 without touching the point $l_\pm$ in a flat landscape. The asymptotic behavior (see Appendix A for more details)

$$\langle t_\pm \rangle \underset{l_\pm \to \infty}{\simeq} \frac{l_\pm}{3} \quad (51)$$

gives that the scale of t_{out} at large R reads

$$t_{out} \sim nl \sim R^{2\mu}, \quad (52)$$

which is negligible with respect to t_{in} (49) for $\mu < 1$.

4. Time spent during the successful excursion to escape

We finally consider the diffusion time t_{diff} of the successful escape to the neighbor renormalized landscape. The free diffusion over a length $l \sim R^\mu$ takes a time of the order of l^2 (see Appendix A for more details) and thus the scale of t_{diff} at large R,

$$t_{diff} \sim R^{2\mu} \quad (53)$$

is the same as t_{out} (52) but is negligible with respect to t_{in} (49) for $\mu < 1$.

5. Conclusion

So we obtain that the total time

$$t_{esc} = t_{in} + t_{out} + t_{diff} \quad (54)$$

needed to escape is actually simply given by the time t_{in} spent inside the trap τ_0. So the distribution of t_{esc} is given by exponential (47) with the escape time T_0.

In conclusion, a trap of the renormalized landscape at scale R has a trapping time τ distributed with Eq. (35), but has an escape time proportional to τ,

$$T = \frac{1}{a} R^\mu \tau, \quad (55)$$

with a factor R^μ that explains the occurrence of two different time scales in this model, and with a prefactor a distributed with Eq. (40).

C. Distribution of escape times in the renormalized landscape

The distribution of the escape time T in the renormalized landscape at scale R reads

$$Q_R(T) = \int_R^{+\infty} d\tau \, q_R(\tau) \int_0^{+\infty} da \, k(a) \delta\left(T - \frac{1}{a} R^\mu \tau \right). \tag{56}$$

It thus presents the scaling form

$$Q_R(T) = \frac{1}{R^{1+\mu}} Q_\mu\left(\tilde{T} = \frac{T}{R^{1+\mu}} \right), \tag{57}$$

where the scaling function reads

$$Q_\mu(\tilde{T}) = \frac{\mu}{\tilde{T}^{1+\mu}} \int_0^{+\infty} da \, \frac{k(a)}{a^\mu} \theta\left(a > \frac{1}{\tilde{T}} \right). \tag{58}$$

In particular, for large $\tilde{T}$, there is the same algebraic decay with index $(1+\mu)$ as for the distribution of trapping times

$$Q_\mu(\tilde{T}) \underset{\tilde{T} \to \infty}{\simeq} \frac{\mu}{\tilde{T}^{1+\mu}} c(\mu), \tag{59}$$

where the constant $c(\mu)$ may be computed from the Laplace transform (40) using Eq. (E4)

$$c(\mu) \equiv \int_0^{+\infty} da \, \frac{k(a)}{a^\mu}$$

$$= \frac{1}{\Gamma(\mu)} \int_0^{+\infty} ds \, s^{\mu-1} \tilde{k}(s)$$

$$= 2^\mu \frac{(1+\mu)\Gamma^3(1+\mu)}{\Gamma(2+2\mu)}$$

$$= 1 - \mu(1 + \gamma_E - \ln 2) + O(\mu^2). \tag{60}$$

For small $\tilde{T}$, we use the asymptotic behavior of $k(a)$ at large a (42),

$$k(a) \underset{a \to 0}{\simeq} \frac{1}{a^2} + \frac{\ln a + (\ln 2 - 1 - \gamma_E)}{a^3} + O\left(\frac{\ln a}{a^4} \right) \tag{61}$$

to obtain

$$Q_\mu(\tilde{T}) \underset{\tilde{T} \to 0}{\simeq} \frac{\mu}{1+\mu} + \frac{\mu}{2+\mu} \tilde{T}\left(\ln\frac{1}{\tilde{T}} + \ln 2 - \gamma_E - 1 + \frac{1}{2+\mu} \right)$$

$$+ O\left(\tilde{T}^2 \ln\frac{1}{\tilde{T}} \right). \tag{62}$$

Using Eq. (40), the most convenient way to characterize the scaling function in closed form is by the following transform in terms of a variable ω:

$$\int_0^{+\infty} d\tilde{T} \, e^{-\omega^2/2\tilde{T}} Q_\mu(\tilde{T})$$

$$= \int_0^1 \mu \, dv \, v^{\mu-1} \int_0^{+\infty} da \, k(a) e^{-(\omega^2/2)va}$$

$$= \frac{2\mu}{\omega^{2\mu}} \int_0^\omega dz \, z^{1+2\mu} K_1^2(z). \tag{63}$$

D. Choice of the renormalization scale R as a function of time

For small μ, the probability distribution $Q_\mu(\tilde{T})$ is dominated by its long tail (59), and we may approximate it by

$$Q_\mu(\tilde{T}) \simeq \theta(\tilde{T} > \tilde{T}_0(\mu)) \frac{\mu}{\tilde{T}} \left(\frac{\tilde{T}_0(\mu)}{\tilde{T}} \right)^\mu, \tag{64}$$

where the cutoff $\tilde{T}_0$ chosen to preserve the normalization is determined by the coefficient for the long tail part (59) and (60),

$$\tilde{T}_0(\mu) = [c(\mu)]^{1/\mu} \underset{\mu \to 0}{\simeq} 2e^{-1-\gamma_E}\left[1 + \mu \frac{18 - \pi^2}{12} + O(\mu^2) \right]. \tag{65}$$

For the unrescaled probability distribution (57), this corresponds to the cutoff

$$T_0(\mu) = R^{1+\mu} \tilde{T}_0(\mu). \tag{66}$$

It is thus convenient to associate at time t the renormalization scale $R(t)$ such that

$$T_0(\mu) = t, \tag{67}$$

meaning that at time t, only traps with escape times $T > t$ have been kept, whereas all traps with escape times $T < t$ have been removed and replaced by a flat landscape. This leads to the explicit choice

$$R(t) = \left(\frac{t}{\tilde{T}_0(\mu)} \right)^{1/(1+\mu)}. \tag{68}$$

The corresponding mean distance between traps reads at this renormalization scale (33)

$$\xi(t) \equiv [R(t)]^\mu = \xi_0(\mu) t^{\mu/(1+\mu)}, \tag{69}$$

with the prefactor

$$\xi_0(\mu) = [c(\mu)]^{-1/(1+\mu)} = 1 + \mu(1 + \gamma_E - \ln 2) + O(\mu^2). \tag{70}$$

III. EFFECTIVE MODEL AT LARGE TIME IN THE LIMIT $\mu \to 0$

A. Effective rules for the dynamics

The prescription for the dynamics is as follows.

At time t, the particle starting at the origin O will be at time t either in the first trap M_+ of the renormalized landscape at scale $R(t)$ on its right or in the first trap M_- of the renormalized landscape on its left. The weight p_{M_+} of the trap M_+ is given by the probability to reach M_+ before M_- for a particle performing a pure random walk, so it is simply given by the ratio of the distances from its starting point

$$p_{[M_-M_+]}(M_+|0) = \frac{M_-O}{M_-M_+} = \frac{l_-}{l_+ + l_-}, \quad (71)$$

$$p_{[M_-M_+]}(M_-|0) = \frac{OM_+}{M_-M_+} = \frac{l_+}{l_+ + l_-}. \quad (72)$$

This rule for the effective dynamics is consistent upon iteration. Suppose there are three consecutive traps: the trap M_- is at a distance l_- from the origin on the left, the trap M_+ is at a distance l_+ from the origin on the right, and the trap M_{++} is at a distance l from the trap M_+ on the right. Suppose that the trap M_+ is decimated before the traps M_- and M_{++}. The new weights for the traps M_- and M_{++} become

$$p'_{M_-} = p_{[M_-M_+]}(M_-|0)$$

$$+ p_{[M_-M_{++}]}(M_-|M_+)p_{[M_-M_+]}(M_+|0)$$

$$= \frac{l_+ + l}{l_- + l_+ + l} = p_{[M_-M_{++}]}(M_-|0), \quad (73)$$

$$p'_{M_{++}} = p_{[M_-M_{++}]}(M_{++}|M_+)p_{[M_-M_+]}(M_+|0)$$

$$= \frac{l_-}{l_- + l_+ + l} = p_{[M_-M_{++}]}(M_{++}|0), \quad (74)$$

and thus rules (72) for the occupancies of renormalized traps are consistent upon decimation of traps in the renormalized landscape.

B. Diffusion front

In this effective model, the diffusion front in a given sample thus reads

$$P_t^{(0)}(n) = \frac{1}{\xi(t)} \mathcal{P}^{(0)}\left(X = \frac{n}{\xi(t)}\right), \quad (75)$$

where the scaling function reads

$$\mathcal{P}^{(0)}(X) = \frac{X_+}{X_+ + X_-} \delta(X + X_-) + \frac{X_-}{X_+ + X_-} \delta(X - X_+) \quad (76)$$

in terms of the two rescaled distances $X_\pm$ between the origin and the nearest renormalized traps. Since the joint distribution of the two rescaled distances is completely factorized,

$$\mathcal{D}(X_+, X_-) = \theta(X_+)\theta(X_-)e^{-X_+ - X_-}, \quad (77)$$

we obtain that the scaling function for the disorder averaged diffusion front (10) reads at lowest order in μ

$$g^{(0)}(X) = \int_0^{+\infty} dX_+ \int_0^{+\infty} dX_- \, \mathcal{D}(X_+, X_-)\mathcal{P}^{(0)}(X)$$

$$= e^{-|X|} \int_0^{+\infty} du \, e^{-u} \frac{u}{|X| + u}. \quad (78)$$

In particular, its asymptotic behaviors read

$$g^{(0)}(X) \underset{|X| \to \infty}{\simeq} \frac{1}{|X|} e^{-|X|}, \quad (79)$$

$$g^{(0)}(X) \underset{|X| \to 0}{\simeq} 1 - |X|\left[\ln\frac{1}{|X|} - \gamma_E\right] + O\left(|X|^2 \ln\frac{1}{|X|}\right). \quad (80)$$

It is interesting to compare with the simple exponential front e^{-X} obtained in the *directed* version of the same trap model. At infinity, the front is reduced by the power $1/|X|$ with respect to the exponential representing the distribution of the distance to a renormalized trap, because of the probability $1/l$ to escape to this trap instead of being absorbed by a nearer renormalized trap on the other side. On the other hand, near the origin, the front is enhanced by the log term because it is more probable to be absorbed by a trap which happens to be very near instead of being absorbed by the renormalized trap on the other side.

C. Localization parameters

In a given sample, the localization parameters read at this order

$$Y_k^{(0)} = p_{M_+}^k + p_{M_-}^k, \quad (81)$$

and thus averaging over the samples yields

$$\overline{Y_k^{(0)}} = \int_0^{+\infty} dX_+ \int_0^{+\infty} dX_- \, \mathcal{D}(X_+, X_-) \frac{X_+^k + X_-^k}{(X_+ + X_-)^k}$$

$$= \frac{2}{(k+1)}. \quad (82)$$

The agreement with the numerical simulations of Ref. [23] obtaining $Y_2 \to 2/3$ and $Y_3 \to 1/2$ is the clearest numerical evidence of the validity of the RSRG effective dynamics with only two relevant traps in each sample in the limit $\mu \to 0$.

D. Thermal width

In a given sample, the thermal width reads at this order

$$\langle \Delta n^2(t) \rangle^{(0)} = (n_{M_+} + n_{n_-})^2 p_{M_+} p_{M_-} = n_{M_+} n_{M_-}, \quad (83)$$

and thus we obtain after averaging over the samples

$$c_2(\mu \to 0) \equiv \lim_{t \to \infty} \frac{\overline{\langle \Delta n^2(t) \rangle}}{\xi^2(t)}$$

$$= \int_0^{+\infty} dX_+ \int_0^{+\infty} dX_- \ \mathcal{D}(X_+, X_-) X_+ X_- = 1. \tag{84}$$

E. Disorder averages of thermal cumulants

The generating function of disorder averages of rescaled thermal cumulants reads

$$Z(s) \equiv \overline{\ln \langle e^{-s[n/\xi(t)]} \rangle}, \tag{85}$$

thus reads at lowest order

$$Z^{(0)}(s) = \int_0^{+\infty} dX_+ \int_0^{+\infty} dX_- \ \mathcal{D}(X_+, X_-)$$

$$\times \ln\left(\frac{X_+}{X_+ + X_-} e^{sX_-} + \frac{X_-}{X_+ + X_-} e^{-sX_+} \right) \tag{86}$$

$$= \int_0^{+\infty} d\lambda \ e^{-\lambda} \lambda \left(\frac{s\lambda}{2} \coth \frac{s\lambda}{2} - 1 \right). \tag{87}$$

The series expansion in s gives the disorder averages of the rescaled thermal cumulants (22).

F. Two-particle correlation function

The two-particle correlation function reads

$$C(l,t) \equiv \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \overline{P(n)P(m)} \delta_{l,|n-m|}$$

$$\underset{t \to \infty}{\simeq} Y_2^{(0)} \delta_{l,0} + \frac{1}{\xi(t)} \mathcal{C}_\mu\left(\lambda = \frac{l}{\xi(t)} \right), \tag{88}$$

where the weight of the δ peak corresponds as it should to the localization parameter $Y_2^{(0)} = \frac{2}{3}$ discussed above, whereas the scaling function of the long-ranged part reads at lowest order

$$\mathcal{C}_\mu^{(0)}(\lambda) = \int_0^{+\infty} dX_+ \int_0^{+\infty} dX_- \ \mathcal{D}(X_+, X_-)$$

$$\times \frac{2X_+ X_-}{(X_+ + X_-)^2} \delta_{\lambda, X_+ + X_-} = e^{-\lambda} \frac{\lambda}{3}. \tag{89}$$

G. Aging and subaging properties

As explained in the Introduction, there are two different correlation functions which present different aging properties [22–24]. We now simply recover within our framework the expected subaging and aging exponents, and moreover, compute the scaling functions in the limit $T \to 0$.

1. Probability $\Pi(t+t_w, t_w)$ of no jump during the interval $[t_w, t_w+t]$

The probability $\Pi(t+t_w, t_w)$ of no jump during the interval $[t_w, t_w+t]$ is directly related to the probability $\psi_{t_w}(\tau)$ to be at time t_w in a trap of trapping time τ via the integral

$$\Pi(t+t_w, t_w) = \int_0^{+\infty} d\tau \ \psi_{t_w}(\tau) e^{-t/\tau}. \tag{90}$$

In our approach, we have assumed as a starting point that at lowest order in μ, the probability $\psi_t(\tau)$ was given by the trapping time distribution in the renormalized landscape at scale $R(t)$ (68),

$$\psi_t(\tau) = q_{R(t)}(\tau) = \theta(\tau > R(t)) \mu \frac{R^\mu(t)}{\tau^{1+\mu}}. \tag{91}$$

So we obtain the scaling form (7)

$$\Pi(t+t_w, t_w) = \tilde{\Pi}_\mu\left(g = \frac{t}{R(t_w)} \right)$$

$$= \tilde{\Pi}_\mu\left(g = [\tilde{T}_0(\mu)]^{1/(1+\mu)} \frac{t}{t_w^{1/(1+\mu)}} \right), \tag{92}$$

and the scaling function reads at lowest order in μ

$$\tilde{\Pi}_\mu^{(0)}(g) = \int_0^1 dz \ \mu z^{\mu-1} e^{-zg}. \tag{93}$$

In particular, its asymptotic behaviors read

$$\tilde{\Pi}_\mu^{(0)}(g) \underset{g \to 0}{\simeq} 1 - g \frac{\mu}{1+\mu} + O(g^2), \tag{94}$$

$$\tilde{\Pi}_\mu^{(0)}(g) \underset{g \to +\infty}{\simeq} \frac{\Gamma(1+\mu)}{g^\mu}. \tag{95}$$

2. Probability $C(t+t_w, t_w)$ to be at time $(t+t_w)$ in the same trap as it was at time t_w

According to the analysis of the escape time from a renormalized trap to a next nearest one, the total escape time (54) is actually dominated by the time t_{in} spent inside the renormalized trap. As a consequence, we have

$$C(t+t_w, t_w) = \int dT \ \phi_{t_w}(T) e^{-t/T}, \tag{96}$$

where $\phi_{t_w}(T)$ represents the probability to be in a trap of escape time T in the renormalized landscape at scale $R(t_w)$.

At lowest order in μ, $\phi_t(T)$ is simply given by the probability $Q_{R(t)}(T)$ given in Eq. (57). So we obtain the scaling form

$$C(t+t_w,t_w)=\widetilde{C}_\mu\left(h=\frac{t}{R^{1+\mu}(t_w)}\right)=\widetilde{C}_\mu\left(h=\widetilde{T}_0(\mu)\frac{t}{t_w}\right),$$
(97)

where the scaling function reads at lowest order in μ

$$\widetilde{C}_\mu(h)=\int_0^{+\infty}d\widetilde{T}\,\mathcal{Q}_\mu(\widetilde{T})e^{-h/\widetilde{T}}$$
(98)

in terms of the function $\mathcal{Q}_\mu(\widetilde{T})$ introduced in Eq. (58). Using Eq. (63), we thus obtain the scaling function in terms of the Bessel function K_1 as

$$\widetilde{C}_\mu(h)=\frac{2\mu}{(2h)^\mu}\int_0^{\sqrt{2h}}dz\,z^{1+2\mu}K_1^2(z).$$
(99)

In particular, the asymptotic behaviors read

$$\widetilde{C}_\mu(h)\underset{h\to 0}{\simeq}1-\frac{\mu}{1+\mu}h\ln\frac{1}{h}+O(h),$$
(100)

$$\widetilde{C}_\mu(h)\underset{h\to\infty}{\simeq}\frac{\Gamma(1+\mu)}{h^\mu}.$$
(101)

IV. CORRECTIONS TO THE ZERO TEMPERATURE EFFECTIVE DYNAMICS

In the preceding section, to compute all observables in the limit $\mu\to 0$, we have considered that the distribution of the escape time was infinitely broad in the following sense: all traps with $T_i<t$ were such that $T_i/t\sim 0$, whereas all traps with $T_i>t$ were such that $T_i/t\sim+\infty$. For finite μ, we have to take into account that these ratios are not really zero or infinite. This can be done in a systematic procedure order by order in μ as we now explain.

A. Corrections at order μ

At first order in μ, we need to consider the following effects (see Fig. 2).

(a) A trap M of the renormalized landscape has an escape time T_M which is not infinite. There is a small probability $(1-e^{-t/T_M})$ that the particle has already escaped from this trap at time t.

If it has escaped, it has been absorbed by one of the two renormalized neighbors, with probabilities given by the ratios of the distances. We will say that the particle is "in advance" with respect to the effective dynamics of the limit $\mu\to 0$.

(b) The biggest trap in an interval between two renormalized traps, which we will call S, has an escape time $T_S<t$ which is not zero and thus there is a small probability of the order of e^{-t/T_S} that the particle is still trapped in S at time t. We will say that the particle is "late" with respect to the effective dynamics of the limit $\mu\to 0$.

The first corrections of the order of μ for various observables are given in Appendix C. From a technical point of view, it turns out that the computation of averages over con-
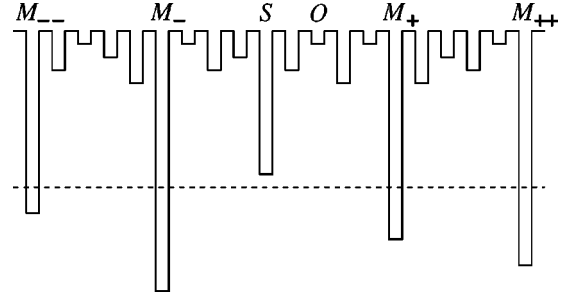

FIG. 2. Construction of the important traps in a given sample for a particle starting at the origin. The dashed line separates the "small" traps [that have a trapping time smaller than $R(t)$] from the "big" traps [that have a trapping time bigger than $R(t)$]. The first big traps on each side called M_+ and M_- are occupied with a weight of the order of $O(\mu^0)$. The next big traps M_{++}, M_{--} and the biggest small trap S in the interval $]M_-,M_+[$ are occupied with weights of the order of $O(\mu)$.

figurations becomes rapidly much more involved than the similar computations for the directed case [20]. This is due to the fact that the escape time depends on both the trapping time and the distances to the neighbor traps. So here, contrary to the directed case where we have computed corrections up to order μ^2 [20], we will not go further into the explicit corrections of higher orders, but simply describe the hierarchical structure of the important traps that appear at a given order in μ.

B. Hierarchical structure of the important traps

The procedure that we have described up to order μ can be generalized at an arbitrary order n as follows: all observables at order μ^n can be obtained by considering dispersion of the thermal packet over at most $(2+n)$ traps that have to be chosen among a certain number Ω_n of possible configurations of the traps. Our aim here is simply to get some insight into the set of important traps that play a role at a given order n.

At order μ^n, the important traps are the following: (a) the main traps M_- and M_+; (b) The next n large renormalized traps on each side, $M_{2+},\ldots,M_{(n+1)+}$ on the half line $]M_+,+\infty[$, $M_{(n+1)-},\ldots,M_{2-}$ on the half line $]-\infty,M_-[$; (c) the n biggest traps $S_1^{(0)},\ldots,S_n^{(0)}$ among the small traps in the interval $]M_-,M_+[$; (d) the $(n-1)$ biggest traps $S_2^{+(1)},\ldots,S_n^{+(1)}$ among the small traps in the interval $]M_+,M_{2+}[$ and the $(n-1)$ biggest traps $S_2^{-(1)},\ldots,S_n^{-(1)}$ among the small traps in the interval $]M_{2-},M_-[$; (e) the $(n-2)$ biggest traps $S_3^{+(2)},\ldots,S_n^{+(2)}$ among the small traps in the interval $]M_{2+},M_{3+}[$ and the $(n-2)$ biggest traps $S_3^{-(2)},\ldots,S_n^{-(2)}$ among the small traps in the interval $]M_{3-},M_{2-}[$; (f) so on; (g) the biggest trap $S_n^{+(n-1)}$ among the small traps in the interval $]M_{(n-1)+},M_{n+}[$ and the biggest trap $S_n^{-(n-1)}$ among the small traps in the interval $]M_{n-},M_{(n-1)-}[$.

The total number of traps is thus

$$T_n = 2 + n + 2 \sum_{i=1}^{n} i = 2 + n(n+2), \qquad (102)$$

which generalizes the number $T_0 = 2$ (M_-, M_+) at order $n = 0$ and the number $T_1 = 5$ $(M_-, M_+, M_{--}, M_{++}, S)$ at order $n = 1$.

With these T_n traps, one has to construct the possible Ω_n configurations of $(2+n)$ traps, that are ordered in positions, and that contribute up to order μ^n, as explained in more detail in Appendix D.

C. Discussion

In conclusion, the effective dynamics for the trap model valid in the limit $\mu \to 0$ is also a good starting point to study the full aging phase $0 < \mu < 1$, since one can build a systematic series expansion in μ for all observables. Moreover, the hierarchical structure of the traps that are important at order n give a clear insight into the structure of the dynamics for finite μ.

V. PROPERTIES OF THE RENORMALIZED LANDSCAPE FOR THE GENERALIZED MODEL

We now consider the generalized model defined by the hopping rates (4) with parameter $\alpha \in [0,1]$ and study the changes with respect to the usual trap model corresponding to the special case $\alpha = 0$ that we have studied in the previous sections.

A. Definition of the renormalized landscape

Here, the renormalized landscape is defined for the energy variable: at scale Γ, all traps with energy $E < \Gamma$ are decimated and replaced by a flat landscape, whereas all traps with energy $E > \Gamma$ remain unchanged. So the distribution of the energy in the renormalized landscape simply reads (1)

$$\rho_\Gamma(E) = \theta(E > \Gamma) \frac{1}{T_g} e^{-(E-\Gamma)/T_g}. \qquad (103)$$

B. Trapping time of a renormalized trap

Let us consider a trap called A with energy $E > \Gamma$ existing in the renormalized landscape: it is surrounded by traps (B_1, B_2) on the right and (B_{-1}, B_{-2}) on the left which have $E \sim 0$ (see Fig. 3). As a consequence, the hopping rates to escape from A are

$$w_{A \to B_{\pm 1}} = \tfrac{1}{2} e^{-\beta(1-\alpha)E}, \qquad (104)$$

so we may associate to the trap A the trapping time

$$\tau_E = e^{\beta(1-\alpha)E}. \qquad (105)$$

The distribution of τ_E in the renormalized landscape at scale Γ reads

$$q_\Gamma^{(\alpha)}(\tau) = \theta(\tau > \tau_\Gamma) \frac{\mu(\alpha)}{\tau} \left(\frac{\tau_\Gamma}{\tau} \right)^{\mu(\alpha)}, \qquad (106)$$
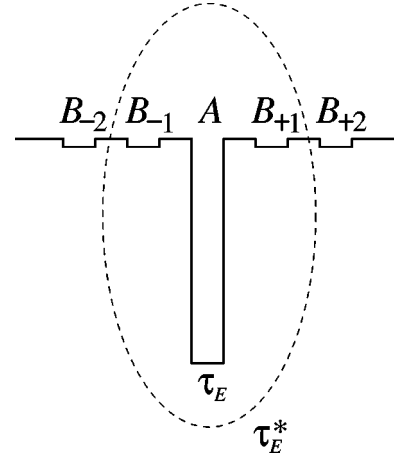

FIG. 3. Definition of the two escape times: a deep trap A of the renormalized landscape has for nearest neighbors $B_{\pm 1}$ and for next nearest neighbors $B_{\pm 2}$. In the case $\alpha > 0$, the trapping time τ_E of A alone depends on α, whereas the trapping time τ_E^* of the interval made with the three consecutive traps $B_{-1} A B_{+1}$ does not depend on α.

with the cutoff

$$\tau_\Gamma = e^{\beta(1-\alpha)\Gamma}, \qquad (107)$$

and with the new exponent

$$\mu(\alpha) = \frac{\mu}{1-\alpha}, \qquad (108)$$

instead of Eq. (3).

C. Trapping-time of the vicinity of a renormalized trap

The hopping rates from the site B_1 (or B_{-1}) nearest neighbor of the renormalized trap A of energy E read

$$w_{B_1 \to A} = \tfrac{1}{2} e^{\beta \alpha E}, \qquad (109)$$

$$w_{B_1 \to B_2} = \tfrac{1}{2}, \qquad (110)$$

so there is a very high probability to return immediately to the deep trap A.

We are interested into the probability distribution $P_*(t^*)$ of the total time t^* spent at A before the first passage at B_2 or B_{-2} when starting at A. We may decompose t^* into the number k of sojourns at A needed

$$t^* = \sum_{i=1}^{k} t_i, \qquad (111)$$

where t_i is the time spent in A in one sojourn, so it is distributed with $f_{\tau_E}(t_i)$ (6). Denoting p_2 the probability to hop to B_2 when starting at B_1,

$$p_2 = \frac{w_{B_1 \to B_2}}{w_{B_1 \to B_2} + w_{B_1 \to A}} = \frac{1}{1 + e^{+\beta \alpha E}} \sim e^{-\beta \alpha E}, \qquad (112)$$

we may express the probability distribution $W(k)$ of $k = 1, 2, \ldots$, as

$$W(k) = (1 - p_2)^{k-1} p_2. \tag{113}$$

The Laplace transform of t^* (111) thus reads

$$\int_0^{+\infty} dt^* \, P_*(t^*) e^{-st^*} = \sum_{k=1}^{+\infty} W(k) \left(\frac{1}{1 + s\tau_E} \right)^k = \frac{1}{1 + s\frac{\tau_E}{p_2}}, \tag{114}$$

so t^* is exponentially distributed

$$P_*(t^*) = \frac{1}{\tau_E^*} e^{-t^*/\tau_E^*}, \tag{115}$$

with the mean time

$$\tau_E^* = \frac{\tau_E}{p_2} = e^{\beta E}(1 + e^{-\beta \alpha E}) \simeq e^{\beta E}, \tag{116}$$

which is independent of the parameter α and exactly coincides with the trapping time of the usual trap model ($\alpha = 0$) distributed with Eq. (2). In conclusion, as α grows from 0 to 1, the mean sojourn time τ_E (105) in the trap A decays, but this is exactly compensated by the large probability to return immediately back to A when on nearest-neighbor sites $B_{\pm 1}$.

D. Escape time from a renormalized trap

Since we have found that the trapping time τ_E^* (116) of the region $B_{-1}AB_{+1}$ containing a renormalized trap A with its two neighbors was exactly the same as the trapping time τ of the usual trap model ($\alpha = 0$) studied in the previous sections, we obtain that the escape properties from a renormalized trap to another renormalized trap are exactly the same as for $\alpha = 0$. So the distribution of the escape time T is given by Eq. (57), where the RG scale $R(t)$ for the trapping times τ_E^* is related to the energy RG scale $\Gamma(t)$ via (116)

$$R(t) = e^{\beta \Gamma(t)}. \tag{117}$$

E. Conclusions in $d=1$

All results based on the escape times T in the renormalized landscape do not depend on α: in particular, the diffusion front, the localization parameters, the disorder averages of thermal cumulants, and the correlation $C(t', t)$ do not depend on α. Among the observables we have considered, the only change will be for the probability $\Pi(t_w + t, t_w)$ of no jump between t_w and $t_w + t$, since it directly involves the distribution of trapping time τ_E (90). At lowest order in μ, we have that the probability distribution $\psi_t(\tau)$ to be in a trap of trapping time τ at time t is now given by (106)

$$\psi_t(\tau) = q_\Gamma^{(\alpha)}(\tau) = \theta(\tau > \tau_{\Gamma(t)}) \frac{\mu(\alpha)}{\tau} \left(\frac{\tau_{\Gamma(t)}}{\tau} \right)^{\mu(\alpha)}, \tag{118}$$

where the cutoff is related to the RG scale $R(t)$ (68) via (107) and (116)

$$\tau_{\Gamma(t)} = e^{\beta(1-\alpha)\Gamma(t)} = [R(t)]^{1-\alpha}. \tag{119}$$

So we obtain the scaling form

$$\Pi_\mu^{(\alpha)}(t + t_w, t) = \widetilde{\Pi}_{\mu(\alpha)}^{(0)} \left(v = \frac{t}{[R(t_w)]^{1-\alpha}} \right) \tag{120}$$

in terms of the scaling function (93) with the modified exponent (108).

F. Effect of α in arbitrary dimension d

It is clear that the analysis presented above in $d=1$ can easily be generalized for the trap model on a hypercubic lattice in arbitrary dimension d: a very deep trap A of energy E has a trapping time τ_E (105) distributed with an algebraic law of exponent $\mu(\alpha) = \mu/(1 - \alpha)$ (108), whereas the cluster made out of A and its ($2d$) nearest-neighbor sites has for trapping time $\tau_E^* \simeq e^{\beta E}$ (116) distributed with an algebraic law of exponent $\mu(0) = \mu$. This explains why the quenched model and the annealed model remain different even in arbitrary dimension d when $\alpha > 0$, as was found in the numerical simulations [21]: in the quenched model, the exponent μ in the distribution of τ_E^* comes from the very high probability to return immediately to a deep trap when leaving it, and thus in the annealed model where these returns are absent, the only relevant exponent is $\mu(\alpha) = \mu/(1 - \alpha)$. As noted in Ref. [21], this effect is particularly clear in the region $0 < \mu < 1 < \mu/(1 - \alpha)$: the annealed model is above its glass transition, whereas the quenched model is below its glass transition.

In conclusion, in the glassy phase $\mu < 1$ in arbitrary dimension d, the diffusion front, the localization parameters, the disorder averages of thermal cumulants, and the correlation $C(t', t)$ do not depend on α, and the only observables sensitive to α will be again those that measure the "local persistence," such as the probability $\Pi(t_w + t, t_w)$ to remain exactly on the same site between t_w and $t_w + t$. This conclusion is in agreement with the qualitative argument given in Appendix B 3 of Ref. [21].

G. Generalization to arbitrary jump rates satisfying detailed balance

It has been recently argued in Ref. [28] how the decay time for the function $\Pi(t_w + t, t_w)$ should depend on the form of the jump rates satisfying detailed balance

$$e^{\beta E_i} w(E_i \text{ to } E_j) = e^{\beta E_j} w(E_i \text{ to } E_j), \tag{121}$$

when they are not of the special form (4). It is thus interesting to consider this question from the RSRG perspective.

The arguments presented in detail above for the special form (4) of the rates may be straightforwardly generalized as follows. The trapping time associated to a trap A with energy $E > \Gamma$ existing in the renormalized landscape (105) is more generally given by

$$\tau_E = \frac{1}{w(E \text{ to } 0)}, \tag{122}$$

whereas the trapping time of the vicinity of this renormalized trap (116) reads more generally

$$\tau_E^* = \frac{\tau_E}{p_2} = \frac{w(0 \text{ to } 0) + w(0 \text{ to } E)}{w(E \text{ to } 0)w(0 \text{ to } 0)}. \tag{123}$$

The detailed balance condition (121) for the special case $E_i = E$ and $E_j = 0$ thus gives

$$\tau_E^* = \frac{e^{\beta E}}{w(0 \text{ to } 0)} + \frac{1}{w(E \text{ to } 0)} = \frac{e^{\beta E}}{w(0 \text{ to } 0)} + \tau_E. \tag{124}$$

So the condition to have exactly the same properties on large scales as in the usual trap model (4) with $\alpha = 0$, is simply

$$\lim_{E \to \infty} (e^{-\beta E} \tau_E) < +\infty, \tag{125}$$

since this condition is sufficient to get $\tau_E^* \sim e^{\beta E}$. In this case, the only observable sensitive to the precise form of the rates will be again the probability $\Pi(t_w + t, t_w)$ to remain exactly on the same site between t_w and $t_w + t$. Indeed, this probability will be a scaling function of the ratio $[t/\tau_w(t_w)]$, where $\tau_w(t_w)$ is the cutoff $\tau_{\Gamma(t_w)}$ of the renormalized landscape at time t_w, i.e., at scale $R(t_w)$ (68). In terms of the energy cutoff $\Gamma(t_w)$ (117),

$$\Gamma(t_w) = \frac{\ln R(t_w)}{\beta} \sim \frac{\ln t_w}{\beta(1 + \mu)}, \tag{126}$$

we thus obtain (122) the general prediction

$$\tau_w(t_w) \equiv \tau_{\Gamma(t_w)} = \frac{1}{w(\Gamma(t_w) \text{ to } 0)} = \frac{1}{w\left(\frac{\ln t_w}{\beta(1 + \mu)} \text{ to } 0\right)}. \tag{127}$$

Vice versa, if one is interested in constructing a model with a given decay time [28] $\tau_w(t_w)$ for the function $\Pi(t_w + t, t_w)$, it is sufficient to choose a form for the rates that satisfies the following asymptotic behavior:

$$w(E \text{ to } 0) \underset{E \to \infty}{\sim} \frac{1}{\tau_w(e^{\beta E(1 + \mu)})}. \tag{128}$$

In particular, to obtain the ultraslow logarithmic aging dynamics $\tau_w(t_w) = (\ln t_w)^{\chi}$, the jump rates have to behave as the power law $w(E \text{ to } 0) \sim E^{-\chi}$ for $E \to \infty$, in agreement with Ref. [28].

VI. SCALING ANALYSIS FOR THE TRAP MODEL IN DIMENSION $d = 2$

Since the random walk on a square lattice in dimension $d = 2$ is recurrent, we expect that here also, the scale of escape times in the renormalized landscape will be different

from the scale of trapping times. On the contrary, for $d > 2$, the random walk is not recurrent, and there will be only one-time scale given by the trapping times.

A. Renormalized landscape

As before, we define the renormalized landscape at scale R as follows: all traps with trapping time $\tau_i < R$ are decimated and replaced by a "flat landscape," whereas all traps with waiting time $\tau_i > R$ remain unchanged.

We now consider the probability $P_R(l)$ that the renormalized trap closest to the origin is at a distance l from the origin. At large scale, this means that the disc of surface $S = \pi l^2$ is empty of renormalized traps, whereas the disc of radius $l + dl$ is not, yielding

$$P_R(l) \underset{R \to \infty}{\simeq} \frac{2\pi l}{R^{\mu}} e^{-\pi l^2/R^{\mu}}, \tag{129}$$

which generalizes Eqs. (33) and (34). Given a renormalized trap τ_0, $P_R(l)$ also describes the distribution of the distance l of the nearest-neighbor renormalized trap.

B. Escape time for a renormalized trap in $d = 2$

We thus define the escape from a renormalized trap τ_0 as the event where a particle starting at τ_0 reaches for the first time the circle of distance l of the nearest renormalized trap: of course, contrary to the one-dimensional case, this does not mean that the particle has been really absorbed, but we expect that the particle has then a finite probability to be absorbed by a renormalized trap different from τ_0.

The probability to escape to the distance l without returning to τ_0 when starting nearby behaves as $c/(\ln l)$, where c is a constant (see Appendix B for more details), and thus the number n of sojourns at τ_0 before the particle succeeds to escape scales as

$$n \sim \ln l \sim \ln R^{\mu/2}. \tag{130}$$

So the time spent inside the trap τ_0 before the escape scales as

$$t_{in} \sim n\tau_0 \sim \tau_0 \ln R^{\mu/2} \gtrsim R \ln R^{\mu/2}. \tag{131}$$

The mean time $\langle t \rangle_l$ of one unsuccessful excursion scales as (see Appendix B for more details)

$$\langle t \rangle_l \underset{l \to \infty}{\simeq} c' \frac{l^2}{\ln^2 l}, \tag{132}$$

so that the total time t_{out} during the unsuccessful excursion scales before the escape scales as

$$t_{out} \sim n\langle t \rangle_l \sim \frac{l^2}{\ln l} \sim \frac{R^{\mu}}{\ln R^{\mu/2}}. \tag{133}$$

It is thus negligible with respect to t_{in} (131) at large scale.

Finally, the time t_{diff} of the successful excursion over the length l scales as

$$t_{diff} \sim l^2 \sim R^{\mu}, \tag{134}$$

and is thus also negligible with respect to t_{in} (131) at large scale.

In conclusion, the total time needed to escape from a trap τ_0 of the renormalized landscape at scale R,

$$t_{esc} = t_{in} + t_{out} + t_{diff}, \tag{135}$$

is dominated by t_{in} alone at large scale, and thus we may associate to the trap τ_0 an escape time

$$T_0 = a \tau_0 \ln R^{\mu/2}, \tag{136}$$

where a is a random variable of the order of 1 that characterizes the geometry of the neighboring renormalized traps around τ_0.

C. Choice of the renormalization scale R as a function of time

The distribution of the escape time T (136) in the renormalized landscape at scale R will thus present the scaling form

$$Q_R(T) = \frac{1}{R \ln R^{\mu/2}} \hat{Q}_\mu \left(\tilde{T} = \frac{T}{R \ln R^{\mu/2}} \right). \tag{137}$$

So the renormalization scale R has to be chosen as a function of time with the scaling

$$t \sim R(t) \ln R^{\mu/2}(t), \tag{138}$$

i.e., by inversion at leading order

$$R(t) \sim \frac{t}{\ln t^{\mu/2}}. \tag{139}$$

So the recurrence properties of the random walk in $d=2$ induces again a different scaling for trapping times and escape times in the renormalized landscape. As a consequence, the correlation $\Pi(t+t_w, t_w)$ will be a scaling function of $t/R(t_w)$ with Eq. (139), whereas the correlation $C(t + t_w, t_w)$ will be a scaling function of t/t_w, in agreement with the results obtained in Refs. [29,30].

D. Number of important traps

However, in contrast with the $d=1$ case, there are two important length scales in $d=2$ as we now explain.

On one hand, given choice (139) for the renormalization scale $R(t)$ as a function of time, the length scale for the distance between two renormalized traps is then given by (129)

$$\xi(t) \sim [R(t)]^{\mu/2} \sim \left[\frac{t}{\ln t^{\mu/2}} \right]^{\mu/2}. \tag{140}$$

On the other hand, let us now recall the scaling analysis using Lévy sums [21,25,26] in $d=2$: after N steps, the num-

ber of visited sites behaves as $S = N/(\ln N)$ and each site is typically visited $\ln N$ times, which yields the correspondence

$$t \sim \sum_{i=1}^{S} \tau_i (\ln N) \sim (\ln N) \left(\frac{N}{\ln N} \right)^{1/\mu}. \tag{141}$$

Since the typical distance reached after N steps scales as $r \sim \sqrt{N}$, this yields in terms of the time t after the inversion of Eq. (141) at leading order

$$r(t) \sim \frac{t^{\mu/2}}{(\ln t^{\mu})^{(\mu-1)/2}}. \tag{142}$$

So here, in contrast with the $d=1$ case, the typical distance $r(t)$ reached after time t does not coincide with the typical distance $\xi(t)$ (140) between two renormalized traps. As a consequence, the number $n(t)$ of important traps contained in the disk of radius $r(t)$ will not remain finite but will grow logarithmically in time,

$$n(t) \sim \frac{r^2(t)}{\xi^2(t)} \sim \ln t^{\mu}. \tag{143}$$

This is in agreement with the absence of localization in $d = 2$ as $t \to \infty$ [30].

However, in the regime where $t \to \infty$ and $\mu \to 0$ with t^{μ} fixed (143), we expect that the diffusion front in a given sample may be described as a sum of δ peaks that are out of equilibrium with respect to each other, i.e., that their weights depend on the geometry of the renormalized traps in the region around the origin but not on their energies. It would be, of course, very interesting to build some RSRG effective dynamics for $d=2$ case in this regime, but this goes beyond the scope of this paper.

VII. CONCLUSION

We have studied in detail the properties of the one-dimensional trap model via a disorder-dependent real-space renormalization procedure. The RSRG approach provides the correct exponents for the relevant length scale and the two relevant time scales in the full domain $0 \le \mu < 1$, and then allows to compute scaling functions in a systematic series expansion in μ. Since we have already summarized our main results for disorder averages of observables in Sec. I D, we will not describe them again but instead discuss the physical meaning of our construction.

We have seen that at lowest order in μ, in a given sample, the particle can be only on two sites, which are the two nearest renormalized traps M_+ and M_- existing at RG scale $R(t)$, and that the weights of these traps are simply given by the probabilities to reach one before the other one. This means that the dynamics remains out of equilibrium forever: the weights are not given by Boltzmann factors, they do not even depend on the energies of these two traps, but they only depend on the ratio of the distances to the origin. When including the first corrections in μ to this picture, we have taken into account the biggest trap S in the interval

]M_-,M_+[, and the next renormalized traps M_{--} and M_{++}, but the dynamics still remains out of equilibrium since the weights are still determined by the lengths between these traps and by escape rates of the form $e^{-t/T}$. This out-of-equilibrium character will thus persist in the whole localized phase $0 \leq \mu < 1$ where these deep traps keep finite weights. This explains why the numerical studies [21,23] have found that the ideas of "partial equilibrium" and "effective temperature" were not able to capture the long-time properties of the trap model. This is similar to what happens in the directed trap model or in the related biased Sinai diffusion, where the thermal packet is also broken into subpackets that remain out of equilibrium with each other even in the infinite time limit [20]; this is in contrast with the unbiased Sinai diffusion, where the thermal packet is asymptotically localized in one single infinite valley [31], in which particles are at equilibrium with each other [32].

From the point of view of numerical simulations on random walks in random media, this shows that to study the localization and the convergence or not to a quasiequilibrium regime at long times, it is interesting to study the dynamics in a single disordered sample, and not only disorder-averaged quantities: for instance, the pictures of a thermal packet in a given sample obtained in Ref. [33] for the Sinai diffusion and in Ref. [19] for the directed trap model, seems to give the clearer insight into the question of localization in one valley or in typically a few traps with finite weights. And whenever the structure of the thermal packet consists of a few subpackets, whose positions and weights are sample dependent, it is a very strong indication that some appropriate real-space RG description can be constructed to study the dynamics.

In a forthcoming paper [34], we will show how the present RSRG approach for the unbiased trap model can be generalized to obtain explicit results in the limit $\mu \to 0$ for the linear and nonlinear response to an external bias, when it is applied from the very beginning at $t = 0$ or after a waiting time t_w. Recently, this problem has been studied via scaling arguments and numerical simulations in Ref. [35], and was shown to satisfy a nonlinear fluctuation theorem [36].

APPENDIX A: STATISTICAL PROPERTIES OF EXCURSIONS IN $d=1$

As explained in the text, to study the excursions in the renormalized landscape (50) and (53), we have to study the following standard problem: what is the probability distribution $P_x(t)$ of the time t of the first passage at $x = 0$ without having touched the other boundary $x = l$ before, for a particle in a pure trap model?

For $x = 1, \ldots, l-1$ this probability distribution satisfies the equation

$$\partial_t P_x(t) = [P_{x+1}(t) + P_{x-1}(t) - 2P_x(t)], \quad \text{(A1)}$$

with the boundary conditions $P_0(t) = \delta(t)$ and $P_l(t) = 0$. So the Laplace transform with respect to t,

$$\hat{P}_x(s) \equiv \int_0^{+\infty} dt \, e^{-st} P_x(t), \quad \text{(A2)}$$

satisfies

$$\hat{P}_{x+1}(s) + \hat{P}_{x-1}(s) - (2+s)\hat{P}_x(s) = 0, \quad \text{(A3)}$$

for $x = 1, \ldots, l-1$ with the boundary conditions $\hat{P}_0(s) = 1$ and $\hat{P}_l(s) = 0$. The solution reads

$$\hat{P}_x(s) = \frac{\rho^x(s) - \rho^{2l-x}(s)}{1 - \rho^{2l}(s)}, \quad \text{(A4)}$$

$$\rho(s) = \frac{2 + s - \sqrt{s^2 + 4s}}{2}. \quad \text{(A5)}$$

In particular, the series expansion in s yields the first moments

$$\theta_k(x) \equiv \int_0^{+\infty} dt \, t^k P_x(t). \quad \text{(A6)}$$

For $n = 0$, the probability to reach 0 before l when starting at x reads as expected

$$\theta_0(x) = \frac{l-x}{l}. \quad \text{(A7)}$$

For $n = 1,2$

$$\theta_1(x) = \frac{x(l-x)(2l-x)}{6l}, \quad \text{(A8)}$$

$$\theta_2(x) = \frac{x(l-x)(2l-x)(4l^2 + 6xl - 3x^2 + 5)}{360l}. \quad \text{(A9)}$$

1. Unsuccessful excursions

For the unsuccessful excursions present in t_{out} (50), we need to consider the special case $x = 1$, where the first moments read

$$\theta_0(1) = 1 - \frac{1}{l}, \quad \text{(A10)}$$

$$\theta_1(1) = \frac{(l-1)(2l-1)}{6l} \underset{l \to \infty}{\simeq} \frac{l}{3}, \quad \text{(A11)}$$

$$\theta_2(1) = \frac{(l-1)(2l-1)(4l^2 + 6l - 3 + 5)}{360l} \underset{l \to \infty}{\simeq} \frac{l^3}{45}. \quad \text{(A12)}$$

The leading terms can be understood as follows: the time t is of the order of l^2 with a probability $1/l$, which is the probability to arrive very near the forbidden extremity $x=l$.

To get the leading behavior in l of all moments, we thus need to consider the rescaled Laplace transform

$$\hat{P}_1\left(\frac{p}{l^2}\right)\underset{l\to\infty}{\simeq}=1-\frac{1}{l}(\sqrt{p}\coth\sqrt{p}-1)+O\left(\frac{1}{l^2}\right),$$
(A13)

which corresponds to the following leading behavior at large l for the moments:

$$\theta_k(1)\underset{l\to\infty}{\simeq}l^{2k-1}\frac{2\Gamma(k+1)\zeta(2k)}{\pi^{2k}}.$$
(A14)

2. Successful excursions

For the successful excursion present in t_{diff} (53), we need to consider the special case $x=l-1$. Since in this case, the normalization is

$$\theta_0(l-1)=\frac{1}{l},$$
(A15)

we consider the normalized first moments

$$\frac{\theta_1(l-1)}{\theta_0(l-1)}\underset{l\to\infty}{\simeq}\frac{l^2}{6},$$
(A16)

$$\frac{\theta_2(l-1)}{\theta_0(l-1)}\underset{l\to\infty}{\simeq}\frac{7l^4}{360}.$$
(A17)

To get the leading behavior in l of all moments, we thus need to consider the rescaled Laplace transform

$$\hat{P}_{l-1}\left(\frac{p}{l^2}\right)\underset{l\to\infty}{\simeq}=\frac{1}{l}\frac{\sqrt{p}}{\sinh\sqrt{p}}+O\left(\frac{1}{l^2}\right).$$
(A18)

So the time t_{diff} for the successful diffusion scales as l^2 as expected.

APPENDIX B: STATISTICAL PROPERTIES OF EXCURSIONS IN $d=2$

To study the scaling properties of excursions in the renormalized landscape in dimension $d=2$, we have to study the following problem: what is the probability distribution $P_r(t)$ of the time t of the first passage on the circle $r=1$ without having touched the other circle $r=l$ before, for a particle starting at radius r and diffusing freely?

For $1<r<R$, this probability distribution satisfies the diffusion equation in radial coordinates

$$\partial_t P_r(t)=\Delta P_r(t)=\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)P_r(t),$$
(B1)

with the boundary conditions $P_1(t)=\delta(t)$ and $P_l(t)=0$.

So the Laplace transform with respect to t,

$$\hat{P}_r(s)\equiv\int_0^{+\infty}dt\,e^{-st}P_r(t),$$
(B2)

satisfies

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-s\right)P_r(s)=0,$$
(B3)

for $1<r<R$ with the boundary conditions $\hat{P}_0(s)=1$ and $\hat{P}_l(s)=0$.

In particular, the moments appearing in the series expansion of $P_r(s)$ in s

$$\theta_k(r)\equiv\int_0^{+\infty}dt\,t^k P_r(t)$$
(B4)

may be computed by the recurrence

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\theta_k(r)=-\theta_{k-1}(r).$$
(B5)

For $k=0$, the probability $\theta_0(r)$ to reach first $r=1$ before $r=l$ when starting at r is the solution of

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\theta_0(r)=0,$$
(B6)

with the boundary conditions $\theta_0(1)=1$ and $\theta_0(l)=0$, and thus reads

$$\theta_0(r)=1-\frac{\ln r}{\ln l}.$$
(B7)

For $k=1$, $\theta_1(r)$ is the solution of

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\theta_1(r)=-\left(1-\frac{\ln r}{\ln l}\right),$$
(B8)

with the boundary conditions $\theta_1(1)=0$ and $\theta_1(l)=0$, and thus reads

$$\theta_1(r)=\frac{l^2-1}{4\ln^2 l}\ln r+\frac{(r^2-1)\left(\ln\frac{r}{l}-1\right)}{4\ln l}.$$
(B9)

In particular, to study the unsuccessful excursions, we may consider $r=2$ for starting point. The probability to escape is then

$$\theta_0(2)=1-\frac{\ln 2}{\ln l},$$
(B10)

and the mean time of an unsuccessful excursion has for leading behavior in l,

$$\theta_1(2) \underset{l\to\infty}{\simeq} \frac{l^2}{4\ln^2 l}\ln 2. \qquad (B11)$$

APPENDIX C: EFFECTIVE MODEL AT FIRST ORDER IN μ

In this Appendix, we compute the corrections at first order in μ by taking into account the two effects described in in Sec. IV A.

1. Statistical properties for particles in advance at order μ

There is a probability,

$$\pi_a = (1 - e^{-t/T_M}), \qquad (C1)$$

that a particle has already escaped at time t from a renormalized trap of escape time T_M.

We consider a particle starting at O that was in the trap M_+ in the effective dynamics of the preceding section. With probability π_a, the particle has already escaped from the trap M_+ at time t, and then it has been absorbed by M_- or by M_{++}, the two nearest renormalized traps, with probabilities given by the ratios of the distances.

We note as before $X_\pm$ the rescaled distances between the origin and $M_\pm$, and we note λ the rescaled distance between M_+ and M_{++}. The joint probability distribution of $(X_-, X_+, \lambda, \tau_M)$ is completely factorized

$$\mathcal{D}_{M_- M_+ M_{++}}(X_-, X_+, \lambda, \tau_M)$$

$$= \theta(X_+)\theta(X_-)\theta(\lambda)\theta(\tau_M > R(t))\mu$$

$$\times \frac{d\tau_M}{\tau_M}\left(\frac{R(t)}{\tau_M}\right)^\mu e^{-X_+ - X_- - \lambda}. \qquad (C2)$$

Since the escape time T_M is proportional to the trapping time,

$$T_M = \frac{1}{a_M}R^\mu \tau_M, \qquad (C3)$$

with a prefactor a_M that depends on the positions via the prefactor

$$a_M = \frac{1}{2}\left(\frac{1}{(X_+ + X_-)} + \frac{1}{\lambda}\right), \qquad (C4)$$

it is more convenient to use the variable

$$\lambda_0 = X_- + X_+ \qquad (C5)$$

instead of X_+. So measure (C2) becomes

$$\mathcal{D}_{M_- M_+ M_{++}}(X_-, \lambda_0, \lambda, \tau_M)$$

$$= \theta(0 < X_- < \lambda_0)\theta(\lambda)\theta(\tau_M > R(t))\mu\frac{d\tau_M}{\tau_M}$$

$$\times \left(\frac{R(t)}{\tau_M}\right)^\mu e^{-\lambda_0 - \lambda}. \qquad (C6)$$

To compute the average of an observable $A(X_-, \lambda_0, \lambda, \tau_M)$ with respect to this measure, it is more convenient to first integrate over $X_- \in (0, \lambda_0)$ and then to integrate over the remaining variables with the following notations:

$$\overline{A(X_-, \lambda_0, \lambda, \tau_M)} = \left\langle\!\!\left\langle\left\langle\int_0^{\lambda_0}dX_-\,A(X_-, \lambda_0, \lambda, \tau_M)\right\rangle\right\rangle\!\!\right\rangle_a, \qquad (C7)$$

where the notation $\langle\!\langle\cdots\rangle\!\rangle_a$ denotes

$$\langle\!\langle f(\lambda_0, \lambda, \tau_M)\rangle\!\rangle_a = \int_{R(t)}^{+\infty}d\tau_M\,\mu\frac{d\tau_M}{\tau_M}\left(\frac{R(t)}{\tau_M}\right)^\mu\int_0^{+\infty}d\lambda_0$$

$$\times\int_0^{+\infty}d\lambda\,e^{-\lambda_0 - \lambda}f(\lambda_0, \lambda, \tau_M). \qquad (C8)$$

For the simplest observables, we will need integrals of the following form, which can be computed in terms of Bessel functions via (E1)

$$W_{p,q}(\mu) \equiv \langle\!\langle(1 - e^{-(t/2R^\mu\tau_M)(1/\lambda + 1/\lambda_0)})\lambda_0^p\lambda^q\rangle\!\rangle_a$$

$$= 2\mu\int_0^{z_0(\mu)}\frac{dz}{z}\left(\frac{z}{z_0(\mu)}\right)^{2\mu}\left[\Gamma(1+p)\Gamma(1+q)\right.$$

$$\left. - \frac{z^{2+p+q}}{2^{p+q}}K_{1+p}(z)K_{1+q}(z)\right], \qquad (C9)$$

where the parameter z_0 depends only on μ via $\tilde{T}_0$ introduced in the explicit choice of the renormalization scale $R(t)$ (68),

$$z_0(\mu) \equiv \sqrt{\frac{2t}{R^{1+\mu}(t)}} = \sqrt{2\tilde{T}_0(\mu)}. \qquad (C10)$$

In particular, in the limit of vanishing μ, $z_0(\mu)$ has for limit (65)

$$z_0(0) = 2e^{-(1+\gamma_E)/2} = 0.908\,95, \qquad (C11)$$

so that the integrals read at lowest order in μ

$$W_{p,q}(\mu) \underset{\mu\to 0}{\simeq} 2\mu\int_0^{z_0(0)}\frac{dz}{z}\left[\Gamma(1+p)\Gamma(1+q)\right.$$

$$\left. - \frac{z^{2+p+q}}{2^{p+q}}K_{1+p}(z)K_{1+q}(z)\right] + O(\mu^2). \qquad (C12)$$

As the first example, let us consider the probability of having already escaped from a renormalized trap at time t:

$$\pi_a = 1 - e^{-t/T_M} = 1 - e^{-(t/2R^\mu \tau_M)(1/\lambda_0 + 1/\lambda)}. \quad (C13)$$

Its average over the configurations can be computed via

$$\overline{\pi_a} = \langle\langle \lambda_0 [1 - e^{-(t/2R^\mu \tau_M)(1/\lambda_0 + 1/\lambda)}]\rangle\rangle_a, \quad (C14)$$

which corresponds to form (C9) for the special case ($p = 1, q = 0$) and is thus of the order of μ (C12),

$$\overline{\pi_a} = W_{1,0}(\mu) \underset{\mu \to 0}{\simeq} 2\mu \int_0^{2e^{-(1+\gamma_E)/2}} \frac{dz}{z}\left[1 - \frac{z^3}{2}K_2(z)K_1(z)\right]$$

$$+ O(\mu^2) = \mu 0.678\,238 + O(\mu^2). \quad (C15)$$

Let us now write the diffusion front in a given sample: it is a linear combination of the two possibilities (72) and (74),

$$\mathcal{P}^{(0)+(1)}_{M_- M_+ M_{++}}(n) = e^{-t/T_M}[p_{[M_- M_+]}(M_-|0)\delta_{n,n_{M_-}}$$

$$+ p_{[M_- M_+]}(M_+|0)\delta_{n,n_{M_+}}] + (1 - e^{-t/T_M})$$

$$\times [p_{[M_- M_{++}]}(M_-|0)\delta_{n,n_{M_-}}$$

$$+ p_{[M_- M_{++}]}(M_{++}|0)\delta_{n,n_{M_{++}}}], \quad (C16)$$

and thus in rescaled distances, the correction with respect to the zeroth order (76) reads

$$\mathcal{P}^{(1)}_{M_- M_+ M_{++}}(X) \equiv \mathcal{P}^{(0)+(+1)}_{M_-,M_+,M_{++}}(X) - \mathcal{P}^{(0)}(X)$$

$$= (1 - e^{-t/T_M})\left[-\frac{X_-}{\lambda_0}\delta(X - (\lambda_0 - X_-))\right.$$

$$+ \frac{\lambda X_-}{\lambda_0(\lambda + \lambda_0)}\delta(X + X_-)$$

$$+ \left.\frac{X_-}{\lambda + \lambda_0}\delta(X - (\lambda + \lambda_0 - X_-))\right], \quad (C17)$$

where the parameters are distributed with measure (C6).

Similarly, if the particle has escaped from M_-, it has been absorbed by M_+ or by M_{--} and the properties are the same as above by the symmetry $X \to -X$.

2. Statistical properties of particles "in delay"

We are now interested in the trap S with the biggest trapping time τ_S in the interval $]M_-, M_+[$. Its position is uniformly distributed in this interval, so the joint distribution of (X_-, X_S, X_+, τ_S) reads

$$\mathcal{D}_{M_- S M_+}(X_-, X_S, X_+, \tau_S)$$

$$= \theta(X_+)\theta(X_-)\theta(-X_- < X_S < X_+)\theta(\tau_S < R(t))$$

$$\times \frac{\mu}{\tau_S}\left(\frac{R(t)}{\tau_S}\right)^\mu e^{-(X_+ + X_-)[R(t)/\tau_S]^\mu}. \quad (C18)$$

The measure for the positions alone reads

$$\int d\tau_s\, \mathcal{D}_{M_- S M_+}(X_-, X_S, X_+, \tau_S)$$

$$= \theta(X_+)\theta(X_-)\theta(-X_- < X_S < X_+)$$

$$\times \frac{1}{X_+ + X_-}e^{-(X_+ + X_-)}, \quad (C19)$$

whereas the measure for the trapping time alone reads

$$\int dX_+ \int dX_- \int dX_S\, \mathcal{D}_{M_- S M_+}(X_-, X_S, X_+, \tau_S)$$

$$= \theta(\tau_S < R(t))2\mu \frac{\tau_s^{2\mu-1}}{R^{2\mu}(t)}. \quad (C20)$$

Since the associated escape time

$$T_S = \frac{1}{a_S}R^\mu \tau_S \quad (C21)$$

depends on the positions via the prefactor

$$a_S = \frac{1}{2}\left(\frac{1}{(X_+ - X_S)} + \frac{1}{(X_- + X_S)}\right), \quad (C22)$$

it is more convenient to replace the rescaled distances (X_-, X_+) between $M_\pm$ and the origin by the rescaled distances between $M_\pm$ and the trap S,

$$\lambda_+ = X_+ - X_S, \quad (C23)$$

$$\lambda_- = X_- + X_S. \quad (C24)$$

With these new variables, measure (C18) becomes

$$\mathcal{D}_{M_- S M_+}(X_S, \tau_S; \lambda_+, \lambda_-)$$

$$= \theta(\lambda_+)\theta(\lambda_-)\theta(-\lambda_+ < X_S < \lambda_-)\theta(\tau_S < R(t))$$

$$\times \frac{\mu}{\tau_S}\left(\frac{R(t)}{\tau_S}\right)^\mu e^{-(\lambda_+ + \lambda_-)[R(t)/\tau_S]^\mu}. \quad (C25)$$

To compute the average of an observable $A(X_S, \lambda_+, \lambda_-, \tau_S)$ with respect to this measure, it is convenient to first integrate over $X_S \in [-\lambda_+, \lambda_-]$ and then average over the remaining variables

$$\overline{A(X_S, \lambda_+, \lambda_-, \tau_S)} = \left\langle\left\langle \int_{-\lambda_+}^{\lambda_-} dX_S\, A(X_S, \lambda_+, \lambda_-, \tau_S)\right\rangle\right\rangle_d, \quad (C26)$$

where the notation $\langle\langle\cdots\rangle\rangle_d$ is defined by

$$\langle\langle f(\lambda_+,\lambda_-,\tau_S)\rangle\rangle_d = \int_0^{R(t)} d\tau_S \frac{\mu}{\tau_S}\left(\frac{R(t)}{\tau_S}\right)^\mu \int_0^{+\infty} d\lambda_+$$

$$\times \int_0^{+\infty} d\lambda_- \, e^{-(\lambda_++\lambda_-)[R(t)/\tau_S]^\mu}$$

$$\times f(\lambda_+,\lambda_-,\tau_S). \quad (C27)$$

For the simplest observables, we will need integrals of the following form, which can be computed in terms of Bessel functions via (E1),

$$\Omega_{p,q}(\mu) \equiv \langle\langle e^{-t/T_S}\lambda_+^p\lambda_-^q\rangle\rangle_d$$

$$= \frac{2\mu}{1+\mu}\frac{z_0^{[2\mu/1+\mu](1+p+q)}(\mu)}{2^{p+q}}$$

$$\times \int_{z_0(\mu)}^{+\infty} dz\, z^{(1-\mu)/(1+\mu)(1+p+q)}$$

$$\times K_{1+p}(z)K_{1+q}(z), \quad (C28)$$

where the parameter z_0 has been introduced in Eq. (C10). At lowest order in μ, we thus obtain

$$\Omega_{p,q}(\mu) \underset{\mu\to 0}{\simeq} \frac{2\mu}{2^{p+q}} \int_{z_0(0)}^{+\infty} dz\, z^{(1+p+q)}K_{1+p}(z)K_{1+q}(z)$$

$$+ O(\mu^2). \quad (C29)$$

Let us first consider the probability

$$\pi_d = e^{-t/T_S} = e^{-(t/2R^\mu\tau_S)(1/\lambda_++1/\lambda_-)}, \quad (C30)$$

that a particle that has been trapped by S is still in the trap S at time t. Its average with respect to measure (C25) can be computed as

$$\overline{\pi_d} = \langle\langle(\lambda_++\lambda_-)e^{-(t/2R^\mu\tau_S)(1/\lambda_++1/\lambda_-)}\rangle\rangle_d. \quad (C31)$$

It is of form (C28) and is thus of the order of μ (C29),

$$\overline{\pi_d} = 2\Omega_{1,0}(\mu) \underset{\mu\to 0}{\simeq} 2\mu \int_{2e^{-(1+\gamma_E)/2}}^{+\infty} dz\, z^2 K_2(z)K_1(z)$$

$$\underset{\mu\to 0}{\simeq} \mu 1.202\,05\ldots + O(\mu^2). \quad (C32)$$

Let us now write the diffusion front in a given sample at this order:

$$\mathcal{P}_{M_-SM_+}^{(0)+(1)}(X) = e^{-t/T_S}\theta(X_S>0)[p_{[M_-S]}(M_-|0)\delta_{n,n_{M_-}}$$

$$+ p_{[M_-S]}(S|0)\delta_{n,n_S}] + e^{-t/T_S}\theta(X_S<0)$$

$$\times[p_{[SM_+]}(M_+|0)\delta_{n,n_{M_+}}$$

$$+ p_{[SM_+]}(S|0)\delta_{n,n_S}] + (1-e^{-t/T_S})$$

$$\times[p_{[M_-M_+]}(M_-|0)\delta_{n,n_{M_-}}$$

$$+ p_{[M_-M_+]}(M_+|0)\delta_{n,n_{M_+}}], \quad (C33)$$

and thus in rescaled distances, the correction with respect to the zeroth order (76) reads

$$\mathcal{P}_{M_-SM_+}^{(1)}(X) \equiv \mathcal{P}_{M_-SM_+,}^{(0)+(1)}(X) - \mathcal{P}^{(0)}(X)$$

$$= e^{-t/T_S}\theta(X_S>0)\left[\frac{X_S}{\lambda_-}\delta(X+(\lambda_--X_S))\right.$$

$$\left.+ \frac{(\lambda_--X_S)}{\lambda_-}\delta(X-X_S)\right] + e^{-t/T_S}\theta(X_S<0)$$

$$\times\left[\frac{(-X_S)}{\lambda_+}\delta(X-(\lambda_++X_S)) + \frac{\lambda_++X_S}{\lambda_+}\right.$$

$$\left.\times\delta(X-X_S)\right] - e^{-t/T_S}\left[\frac{\lambda_++X_S}{\lambda_++\lambda_-}\delta(X+\lambda_-\right.$$

$$\left.-X_S) + \frac{\lambda_--X_S}{\lambda_++\lambda_-}\delta(X-(\lambda_++X_S))\right], \quad (C34)$$

where the parameters are distributed with measure (C25).

3. Correction of the order of μ to the averaged diffusion front

a. Contribution of particles in advance

To compute contribution (C17) to the diffusion front of the configurations $M_-M_+M_{++}$, we first integrate over $X_- \in [0,\lambda_0]$,

$$\int_0^{\lambda_0} dX_- \mathcal{P}_{M_-M_+M_{++}}^{(1)}(X)$$

$$= (1-e^{-t/T_M})\left[-\frac{\lambda_0-|X|}{\lambda_0}\theta(0<X<\lambda_0)\right.$$

$$+ \frac{\lambda|X|}{\lambda_0(\lambda+\lambda_0)}\theta(0<-X<\lambda_0) + \frac{\lambda+\lambda_0-|X|}{\lambda+\lambda_0}$$

$$\left.\times\theta(\lambda<X<\lambda+\lambda_0)\right]. \quad (C35)$$

So taking into account the configurations $M_{--}M_-M_+$ via the symmetry $X\to-X$, we finally get that averaging with measure (C6) yields the following contribution of particles in advance to the scaling function at order μ:

$$g_a^{(1)}(X) = \mu \int_0^{+\infty} d\lambda_0 \int_0^{+\infty} d\lambda \int_0^1 \frac{dw}{w} e^{-(\lambda_0 + \lambda)}$$

$$\times (1 - e^{-(\tilde{T}_0(0)/2)w(1/\lambda_0 + 1/\lambda)})$$

$$\times \left[-2 \frac{\lambda_0 - |X|}{\lambda_0} \theta(0 < X < \lambda_0) \right.$$

$$\left. + \frac{\lambda + \lambda_0 - |X|}{\lambda + \lambda_0} \theta(0 < X < \lambda + \lambda_0) \right] + O(\mu^2).$$

$$(C36)$$

b. Contribution of particles in delay

To compute the average of the specific contribution (C34) of the configurations $S_- M S_+$ with respect to measure (C25), we first integrate over X_S,

$$\int dX_S \, \theta(-\lambda_+ < X_S < \lambda_-) \mathcal{P}_{M_- S M_+}^{(1)}(X)$$

$$= e^{-t/T_S} \left[\theta(|X| < \lambda_-) \frac{\lambda_- - |X|}{\lambda_-} \right.$$

$$+ \theta(|X| < \lambda_+) \frac{(\lambda_+ - |X|)}{\lambda_+}$$

$$\left. - \theta(|X| < \lambda_+ + \lambda_-) \frac{\lambda_- + \lambda_+ - |X|}{\lambda_+ + \lambda_-} \right], \quad (C37)$$

so that the correction to the averaged diffusion front reads

$$g_d^{(1)}(X) = \mu \int_0^{+\infty} d\lambda_+ \int_0^{+\infty} d\lambda_- \int_1^{+\infty} \frac{dw}{w} e^{-(\lambda_+ + \lambda_-)}$$

$$\times e^{-(\tilde{T}_0(0)/2)w(1/\lambda_+ + 1/\lambda_-)} \left[\theta(|X| < \lambda_-) \right.$$

$$\times \frac{\lambda_- - |X|}{\lambda_-} + \theta(|X| < \lambda_+) \frac{(\lambda_+ - |X|)}{\lambda_+}$$

$$\left. - \theta(|X| < \lambda_+ + \lambda_-) \frac{\lambda_- + \lambda_+ - |X|}{\lambda_+ + \lambda_-} \right] + O(\mu^2).$$

$$(C38)$$

c. Total correction at first order in μ

The two contributions (C36) and (C38) yield the following total contribution at first order in μ:

$$g^{(1)}(X) \equiv g_a^{(1)}(X) + g_d^{(1)}(X)$$

$$= \mu \int_0^{+\infty} d\lambda_0 \int_0^{+\infty} d\lambda \, e^{-(\lambda_0 + \lambda)} \left[\gamma_E + \ln \frac{\tilde{T}_0(0)}{2} \right.$$

$$\times \left(\frac{\lambda + \lambda_0}{\lambda_0 \lambda} \right) \left] \right[-2 \frac{\lambda_0 - |X|}{\lambda_0} \theta(0 < X < \lambda_0)$$

$$\left. + \frac{\lambda + \lambda_0 - |X|}{\lambda + \lambda_0} \theta(0 < X < \lambda + \lambda_0) \right] + O(\mu^2).$$

$$(C39)$$

4. Correction of the order of μ to the thermal width

a. Contribution of particles in advance

The specific contribution of the configurations $M_- M_+ M_{++}$ to the thermal width reads with measure (C6)

$$[c_2]_{M_- M_+ M_{++}}^{(1)} \equiv [c_2]_{M_- M_+ M_{++}}^{(0)+(1)} - [c_2]_{M_- M_+}^{(0)}$$

$$= \overline{(1 - e^{-t/T_M}) X_- \lambda}. \quad (C40)$$

After the integration over X_- (C7), we thus obtain an integral of type (C9) for the values $p = 2, q = 1$,

$$[c_2]_{M_- M_+ M_{++}}^{(1)} = \left\langle \left\langle (1 - e^{-t/T_M}) \frac{\lambda_0^2}{2} \lambda \right\rangle \right\rangle_a = \frac{1}{2} W_{2,1}(\mu).$$

$$(C41)$$

By symmetry, the configurations $M_{--} M_- M_+$ give exactly the same contribution, and thus the total contribution of particles in advance reads at lowest order in μ (C12)

$$[c_2]_a^{(1)} = W_{2,1}(\mu) \underset{\mu \to 0}{\simeq} \mu \int_0^{2e^{-(1+\gamma_E)/2}} \frac{dz}{z}$$

$$\times \left[4 - \frac{z^5}{4} K_3(z) K_2(z) \right] + O(\mu^2)$$

$$= \mu 0.5383. \quad (C42)$$

b. Contribution of particles in delay

The specific contribution of the configurations $M_- S M_+$ to the thermal width reads with measure (C25)

$$[c_2]_d^{(1)} \equiv [c_2]_{M_- S M_+}^{(0)+(1)} - [c_2]_{M_- M_+}^{(0)} = -\overline{e^{-t/T_S} [\theta(X_S > 0) \lambda_+ (\lambda_- - X_S) + \theta(X_S < 0) \lambda_- (\lambda_+ + X_S)]}. \quad (C43)$$

After integration over X_S, we thus obtain the integral of type (C28) for the special value $(p=2, q=1)$

$$
[c_2]_d^{(1)} = -\left\langle\left\langle e^{-t/T_S}\left[\lambda_+\frac{\lambda_-^2}{2} + \lambda_-\frac{\lambda_-^2}{2}\right]\right\rangle\right\rangle_d
$$

$$
= -\Omega_{2,1}(\mu) \underset{\mu\to 0}{\simeq} -\frac{\mu}{4}\int_{2e^{-(1+\gamma_E)/2}}^{+\infty} dz\, z^4 K_3(z) K_2(z)
$$

$$
+ O(\mu^2)
$$

$$
= -\mu 2.2172\ldots + O(\mu^2). \tag{C44}
$$

c. Total correction of the order of μ for the thermal width

Adding the contributions of particles in advance and in delay, we get the total correction at order μ

$$
[c_2]_{total}^{(1)} = [c_2]_a^{(1)} + [c_2]_d^{(1)}
$$

$$
= \mu\left[-\frac{5}{6} - 2 + 2\gamma_E\right] = -\mu 1.6789. \tag{C45}
$$

5. Correction of the order of μ to the localization parameters

a. Contribution of particles in advance

For a given configuration $M_- M_+ M_{++}$, the localization parameters read (C17)

$$
[Y_k]_{M_-M_+M_{++}}^{(0)+(1)} = \left[e^{-t/T_M}\frac{X_-}{\lambda_0}\right]^k + \left[(1-e^{-t/T_M})\frac{X_-}{\lambda_0+\lambda}\right]^k
$$

$$
+ \left[1 - e^{-t/T_M}\frac{X_-}{\lambda_0} - (1-e^{-t/T_M})\frac{X_-}{\lambda_0+\lambda}\right]^k. \tag{C46}
$$

The average of the specific contribution may be computed with Eq. (C7), so that taking into account the similar contribution of the configurations $M_{--}M_-M_+$, we finally obtain

$$
\overline{[Y_k]_a^{(1)}} = 2\overline{([Y_k]_{M_-M_+M_{++}}^{(0)+(1)} - [Y_k]_{M_-M_+}^{(0)})}
$$

$$
= 2\left\langle\left\langle\frac{\lambda_0}{k+1}\left[e^{-k(t/T_M)} - 1 + (1-e^{-t/T_M})^k\frac{\lambda_0^k}{(\lambda_0+\lambda)^k}\right.\right.\right.
$$

$$
\left.\left.\left. + \sum_{m=1}^k (1-e^{-t/T_M})^m\left(\frac{\lambda}{\lambda+\lambda_0}\right)^m\right]\right\rangle\right\rangle_a. \tag{C47}
$$

In particular, it reads for $k=2$

$$
\overline{[Y_2]_a^{(1)}} = \left\langle\left\langle\frac{2\lambda_0}{3}\left[(1-e^{-t/T_M})\left(\frac{\lambda}{(\lambda+\lambda_0)} - 2\right)\right.\right.\right.
$$

$$
\left.\left.\left. + (1-e^{-t/T_M})^2 2\left(1 - \frac{\lambda\lambda_0}{(\lambda+\lambda_0)^2}\right)\right]\right\rangle\right\rangle_a. \tag{C48}
$$

b. Contribution of particles in delay

For a given configuration M_-SM_+, the localization parameters read (C34) for the case $X_S > 0$

$$
[Y_k]_{M_-SM_+}^{(0)+(1)} = \left[e^{-t/T_S}\frac{\lambda_- - X_S}{\lambda_-}\right]^k + \left[(1-e^{-t/T_S})\frac{\lambda_- - X_S}{\lambda_+ + \lambda_-}\right]^k
$$

$$
+ \left[1 - e^{-t/T_S}\frac{\lambda_- - X_S}{\lambda_-}\right.
$$

$$
\left. - (1-e^{-t/T_S})\frac{\lambda_- - X_S}{\lambda_+ + \lambda_-}\right]^k, \tag{C49}
$$

and a similar expression for $X_S < 0$.

The integration over $X_S \in [0, \lambda_-]$ (C26) yields

$$
\int_0^{\lambda_-} dX_S [Y_k]_{M_-SM_+}^{(0)+(1)} = \frac{\lambda_-}{(k+1)}\left[e^{-k(t/T_S)} + (1-e^{-t/T_S})^k\right.
$$

$$
\times\frac{\lambda_-^k}{(\lambda_+ + \lambda_-)^k} + \sum_{m=0}^k (1-e^{-t/T_S})^m
$$

$$
\left.\times\left(\frac{\lambda_+}{\lambda_+ + \lambda_-}\right)^m\right]. \tag{C50}
$$

The case $X_S \in [-\lambda_+, 0]$ leads to the symmetric contribution via the exchange between (λ_+, λ_-), and thus the average of the specific contribution reads

$$
\overline{[Y_k]_d^{(1)}} \equiv \left\langle\left\langle\int_{-\lambda_+}^{\lambda_-} dX_S([Y_k]_{M_-SM_+}^{(0)+(1)} - [Y_k]_{M_-M_+}^{(0)})\right\rangle\right\rangle_d
$$

$$
= \left\langle\left\langle\frac{1}{(k+1)}\left[(e^{-k(t/T_S)} - 1)(\lambda_- + \lambda_+)\right.\right.\right.
$$

$$
+ (1-e^{-t/T_S})^k\frac{\lambda_-^{k+1} + \lambda_+^{k+1}}{(\lambda_+ + \lambda_-)^k} + \sum_{m=1}^k (1-e^{-t/T_S})^m
$$

$$
\left.\left.\left.\times\left(\frac{\lambda_-\lambda_+^m + \lambda_+\lambda_-^m}{(\lambda_+ + \lambda_-)^m}\right)\right]\right\rangle\right\rangle_d. \tag{C51}
$$

In particular, it reads for $k=2$

$$\overline{[Y_2]_d^{(1)}} = -\left\langle\!\left\langle\left[\frac{2}{3}e^{-t/T_S}(1-e^{-t/T_S})\right]\!\left[\lambda_- + \lambda_+\right.\right.\right.$$

$$\left.\left.\left.-\frac{\lambda_-\lambda_+}{\lambda_-+\lambda_+}\right]\right\rangle\!\right\rangle_d . \tag{C52}$$

6. Correction to aging and subaging properties

a. Probability $\Pi(t_w+t,t_w)$ of no jump during the interval $[t_w,t_w+t]$

Since the probability $\Pi(t_w+t,t_w)$ of no jump during the interval $[t_w,t_w+t]$ is directly related to the probability $\psi_{t_w}(\tau)$ to be at time t_w in a trap of trapping time τ via (90), the configurations $M_-M_+M_{++}$ do not give any correction, since the trapping time of M_{++} has the same statistical properties as the trapping times of M_- and M_+.

On the contrary, the configurations M_-SM_+ will give a correction to $\psi_t(\tau)$. For the case $X_S>0$, we have

$$[\psi_t(\tau)]_{M_-SM_+}^{(0)+(1)} = e^{-t/T_S}\frac{\lambda_- - X_S}{\lambda_-}\delta(\tau-\tau_S)$$

$$+ (1-e^{-t/T_S})\frac{\lambda_- - X_S}{\lambda_+ + \lambda_-}\delta(\tau-\tau_{M_+})$$

$$+ \left[1 - e^{-t/T_S}\frac{\lambda_- - X_S}{\lambda_-} - (1-e^{-t/T_S})\right.$$

$$\left.\times\frac{\lambda_- - X_S}{\lambda_+ + \lambda_-}\right]\delta(\tau-\tau_{M_-}), \tag{C53}$$

so that the specific contribution yields after integration over $X_S\in[0,\lambda_-]$

$$\int_0^{\lambda_-} dX_S [\psi_t(\tau)]_{M_-SM_+}^{(1)}$$

$$= \int_0^{\lambda_-} dX_S \{[\psi_t(\tau)]_{M_-SM_+}^{(0)+(1)} - [\psi_t(\tau)]_{M_-M_+}^{(0)}\}$$

$$= e^{-t/T_S}\frac{\lambda_-}{2}[\delta(\tau-\tau_S) - \delta(\tau-\tau_{M_-})]$$

$$+ e^{-t/T_S}\frac{\lambda_-^2}{2(\lambda_+ + \lambda_-)}[\delta(\tau-\tau_{M_-}) - \delta(\tau-\tau_{M_+})]. \tag{C54}$$

After averaging over τ_{M_+} and τ_{M_-}, the second term will vanish. Since the case $X_S<0$ gives the symmetric contribution via the exchange (λ_+,λ_-), we finally obtain that the full correction to the probability distribution reads with measure (C27)

$$\overline{[\psi_t(\tau)]^{(1)}} = \int_{R(t)}^{+\infty}\frac{\mu}{\tau_M}\left(\frac{R(t)}{\tau_M}\right)^\mu\left\langle\!\left\langle e^{-t/T_S}\frac{\lambda_- + \lambda_-}{2}\right.\right.$$

$$\left.\left.\times[\delta(\tau-\tau_S) - \delta(\tau-\tau_{M_-})]\right\rangle\!\right\rangle_d . \tag{C55}$$

We thus finally obtain at first order in μ

$$\overline{[\psi_t(\tau)]^{(1)}} = \theta(\tau<R(t))\frac{\mu}{2\tau}z_0^3(0)\left(\frac{R(t)}{\tau}\right)^{3/2}$$

$$\times K_1\!\left(z_0(0)\left(\frac{R(t)}{\tau}\right)^{1/2}\right)K_2\!\left(z_0(0)\left(\frac{R(t)}{\tau}\right)^{1/2}\right)$$

$$- \Omega_{0,1}(\mu)\,\theta(R(t)<\tau)\frac{\mu}{\tau}\left(\frac{R(t)}{\tau}\right)^\mu, \tag{C56}$$

with Eq. (C11). The prefactor $\Omega_{0,1}(\mu)$ (C28) of the second term represents the probability to be in a trap of type S at time t and is of the order of μ (C32). So in the domain $\tau>R$, the total distribution $[\psi_t(\tau)]^{(0)+(1)}$ of τ keeps the same form that at zeroth order, but the amplitude is $[1-\Omega_{0,1}(\mu)]$. There is now a contribution of the domain $\tau<R$ which was absent at zeroth order. In particular, in the limit $\tau\to 0$, we obtain the essential singularity

$$[\psi_t(\tau)]^{(1)} \underset{\tau\to 0}{\simeq} \frac{\mu\pi z_0^2(0)R(t)}{4\tau^2}e^{-2z_0(0)(R(t)/\tau)^{1/2}}. \tag{C57}$$

We now compute the correction to the scaling function $\tilde\Pi(v)$ (92) at first order in μ,

$$\tilde\Pi^{(1)}(v) = \mu\int_{z_0(0)}^{+\infty}dz\,z^2 K_1(z)K_2(z)e^{-v[z/z_0(0)]^2}$$

$$- \Omega_{0,1}(\mu)\int_0^1 dy\,\mu y^{\mu-1}e^{-yv}. \tag{C58}$$

So for large v, the only correction to the scaling function $\tilde\Pi^{(0)}(v)$ (93) comes from the second term

$$\tilde\Pi^{(1)}(v) \underset{v\to\infty}{\simeq} -\Omega_{0,1}(\mu)\frac{\mu}{v^\mu}, \tag{C59}$$

which corresponds to a correction of the order of μ^2 for the amplitude of the algebraic decay $1/v^\mu$.

b. Probability $C(t+t_w,t_w)$ to be at time $(t+t_w)$ in the same trap as it was at time t_w

Since in our framework, $C(t+t_w,t_w)$ is determined by the probability $\phi_{t_w}(T)$ to be at time t_w in a trap of escape

time T (96), we obtain as for $\Pi(t_w+t,t_w)$ that the configurations $M_-M_+M_{++}$ do not give any correction because the escape time of M_{++} has the same statistical properties as the escape times of M_- and M_+.

The correction due to the configurations M_-SM_+ can be written for the case $X_S>0$ as

$$[\phi_t(T)]_{M_-SM_+}^{(0)+(1)}=e^{-t/T_S}\frac{\lambda_--X_S}{\lambda_-}\delta(T-T_S)$$

$$+(1-e^{-t/T_S})\frac{\lambda_--X_S}{\lambda_++\lambda_-}\delta(T-T_{M_+})$$

$$+\left[1-e^{-t/T_S}\frac{\lambda_--X_S}{\lambda_-}-(1-e^{-t/T_S})\right.$$

$$\left.\times\frac{\lambda_--X_S}{\lambda_++\lambda_-}\right]\delta(T-T_{M_-}),\qquad(C60)$$

so that the specific contribution yields after integration over $X_S\in[0,\lambda_-]$

$$\int_0^{\lambda_-}dX_-[\phi_t(\tau)]_{M_-SM_+}^{(1)}$$

$$=\int_0^{\lambda_-}dX_-\{[\phi_t(T)]_{M_-SM_+}^{(0)+(1)}-[\phi_t(\tau)]_{M_-M_+}^{(0)}\}$$

$$=e^{-t/T_S}\frac{\lambda_-}{2}[\delta(T-T_S)-\delta(T-T_{M_-})]$$

$$+e^{-t/T_S}\frac{\lambda_-^2}{2(\lambda_++\lambda_-)}[\delta(T-T_{M_-})-\delta(T-T_{M_+})].$$

$$(C61)$$

After averaging over T_{M_+} and T_{M_-}, the second term will vanish. Since the case $X_S<0$ gives the symmetric contribution via the exchange (λ_+,λ_-), we finally obtain that the full correction to the probability distribution reads with measure (C27)

$$[\phi_t(T)]^{(1)}=\left\langle\left\langle\int_R^{+\infty}\mu\frac{d\tau_M}{\tau_M}\left(\frac{R}{\tau_M}\right)^\mu e^{-t/T_S}\frac{\lambda_-+\lambda_+}{2}[\delta(T-T_S)-\delta(T-T_{M_-})]\right\rangle\right\rangle_d$$

$$=\mu\int_0^R\frac{d\tau_S}{\tau_S}\left(\frac{R}{\tau_S}\right)^\mu\int_0^{+\infty}d\lambda_+\int_0^{+\infty}d\lambda_-\,e^{-(\lambda_++\lambda_-)[R(t)/\tau_S]^\mu}\lambda_-e^{-t/T}\delta\left(T-\frac{2R^\mu\tau_S}{\left(\frac{1}{\lambda_+}+\frac{1}{\lambda_-}\right)}\right)$$

$$-\mu\int_0^R\frac{d\tau_S}{\tau_S}\left(\frac{R}{\tau_S}\right)^\mu\int_0^{+\infty}d\lambda_+\int_0^{+\infty}d\lambda_-\,e^{-(\lambda_++\lambda_-)[R(t)/\tau_S]^\mu}\lambda_-e^{-(t/2R^\mu\tau_S)(1/\lambda_++1/\lambda_-)}$$

$$\times\int_0^{+\infty}d\lambda\,e^{-\lambda}\int_R^{+\infty}\mu\frac{d\tau_M}{\tau_M}\left(\frac{R}{\tau_M}\right)^\mu\delta\left(T-\frac{2R^\mu\tau_M}{\left(\frac{1}{\lambda}+\frac{1}{\lambda_++\lambda_-}\right)}\right),\qquad(C62)$$

so the correction to the scaling function $\widetilde{C}_\mu(h)$ (97) reads

$$\widetilde{C}_\mu^{(1)}(h)=\int_0^{+\infty}dT[\phi_t(T)]^{(1)}e^{-ht/T}$$

$$=\mu\int_1^{+\infty}\frac{dv}{v}v^\mu\int_0^{+\infty}d\lambda_+\int_0^{+\infty}d\lambda_-\,e^{-(\lambda_++\lambda_-)v^\mu}$$

$$\times\lambda_-e^{-(1+h)(\overline{T}_0(0)/2)v(1/\lambda_++1/\lambda_-)}$$

$$-\mu\int_1^{+\infty}\frac{dv}{v}v^\mu\int_0^{+\infty}d\lambda_+\int_0^{+\infty}d\lambda_-\,e^{-(\lambda_++\lambda_-)v^\mu}$$

$$\times\lambda_-e^{-(\overline{T}_0(0)/2)v(1/\lambda_++1/\lambda_-)}\int_0^{+\infty}d\lambda\,e^{-\lambda}$$

$$\times\int_0^1\mu\frac{dw}{w}w^\mu e^{-h(\overline{T}_0(0)/2)w[1/\lambda+1/(\lambda_++\lambda_-)]}.\qquad(C63)$$

APPENDIX D: SET OF THE IMPORTANT CONFIGURATIONS AT ORDER n

With the T_n traps described in text (102), we have to construct the possible Ω_n configurations of $(2+n)$ traps, that are ordered in positions, and that contribute up to order μ^n. We have

$$\Omega_n=\Omega_{n-1}+\omega_n=\sum_{i=0}^n\omega_i,\qquad(D1)$$

where ω_n represents the number of configurations that begin to contribute at order n. We may decompose

$$\omega_n=\sum_{k\geq1,l\geq1,k+l\leq2+n}a_n^{(M_{k-},M_{l+})},\qquad(D2)$$

where $a_n^{(M_{k-},M_{l+})}$ is the number of configurations that contain M_{k-} as leftmost trap and M_{l+} as rightmost trap. For $(k=1,l=1+n)$, there is only $a_n^{(M_-,M_{(1+n)+})}=1$ configuration $\{M_-,M_+,M_{2+},\ldots,M_{(n+1)+}\}$, whereas for $(k=1,l=1)$, there are

$$a_n^{(M_-,M_+)}=n! \tag{D3}$$

configurations, since we have to order in space the n traps $S_1^{(0)},\ldots,S_n^{(0)}$ in the interval $]M_-,M_+[$. More generally, at order (k,l), to construct the configurations of $(n+2)$ traps containing $M_{k-}M_{(k-1)-},\ldots,M_-,M_+,M_{2+},\ldots,M_{l+}$, which represent $(k+l)$ fixed traps, we have to choose $(n+2-k-l)$ traps among the $(k+l-1)$ available intervals and to count the possible positional orders in each interval

$$a_n^{(M_{k-},M_{l+})}=\sum_{p_1=0}^{+\infty}\cdots\sum_{p_{k+l-1}=0}^{+\infty}\delta\left(\sum_{i=1}^{k+l-1}p_i=n+2-k-l\right)$$
$$\times p_1!\ldots p_{k+l-1}!. \tag{D4}$$

The final result is thus that the number of new configurations that appear at order n reads

$$\omega_n=\sum_{k\geqslant1,l\geqslant1,k+l\leqslant2+n}\left[\sum_{p_1=0}^{+\infty}\cdots\sum_{p_{k+l-1}=0}^{+\infty}\right.$$
$$\left.\times\delta\left(\sum_{i=1}^{k+l-1}p_i=n+2-k-l\right)p_1!\ldots p_{k+l-1}!\right], \tag{D5}$$

which generalize what we have found before for the lowest orders with $\omega_0=1$ corresponding to (M_-,M_+), and $\omega_1=3$ corresponding to (M_{--},M_-,M_+), (M_-,M_+,M_{++}), and (M_-,S,M_+).

APPENDIX E: USEFUL PROPERTIES OF BESSEL FUNCTIONS

The following integrals yield Bessel function of type K:

$$\int_0^{+\infty}dx\;x^\nu e^{-ax-b/x}=2\left(\frac{b}{a}\right)^{(1+\nu)/2}K_{1+\nu}(2\sqrt{ab}). \tag{E1}$$

The asymptotic behavior at infinity is independent of ν and reads

$$K_\nu(z)\underset{z\to\infty}{\simeq}\sqrt{\frac{\pi}{2z}}e^{-z}. \tag{E2}$$

Near the origin, the behavior depends on ν. We will need the behavior for $\nu=1$,

$$K_1(z)\underset{z\to0}{\simeq}\frac{1}{z}+\frac{z}{2}\left[\ln\frac{z}{2}+\gamma_{Euler}-1/2\right]+O(z^3\ln z). \tag{E3}$$

Another useful integral is

$$\int_0^{+\infty}dz\;z^{2k-1}K_\nu^2(z)=\frac{\sqrt{\pi}\Gamma(k)\Gamma(k+\nu)\Gamma(k-\nu)}{4\Gamma(k+\frac{1}{2})}. \tag{E4}$$

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