Hydrodynamic modes for granular gases

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The eigenfunctions and eigenvalues of the linearized Boltzmann equation for inelastic hard spheres ($d = 3$) or disks ($d = 2$) corresponding to $d + 2$ hydrodynamic modes are calculated in the long wavelength limit for a granular gas. The transport coefficients are identified and found to agree with those from the Chapman-Enskog solution. The dominance of hydrodynamic modes at long times and long wavelengths is studied via an exactly solvable kinetic model. A collisional continuum is bounded away from the hydrodynamic spectrum, assuring a hydrodynamic description at long times. The bound is closely related to the power law decay of the velocity distribution in the reference homogeneous cooling state.

The simplest model for a granular gas at low density is given by the Boltzmann equation for smooth, inelastic hard spheres or disks [1]. It provides an appropriate context in which we can address a number of fundamental issues. Primary among these is the existence of a macroscopic fluid dynamics analogous to the Navier-Stokes description for real gases. The derivation of hydrodynamic equations from the inelastic Boltzmann equation and identification of expressions for the transport coefficients has been a problem of interest for two decades [2–5].

Accurate prediction of the transport coefficients as a function of the restitution coefficient [6], and confirmation via Monte Carlo simulation [7], has been accomplished only within the last few years. However, the method used in these derivations (Chapman-Enskog) is formal and does not strictly establish either the existence or the context of a hydrodynamic description. Thus, there is a further point of principle that has not been addressed directly for granular gases. In the case of elastic collisions, this problem of existence and context has been solved by analysis of the spectrum of the Boltzmann operator using key theorems of Hilbert, Weyl, Carleman, McLennan, and Cergignani [8]. The primary results are (1) there exist $d + 2$ eigenfunctions (the summational invariants) with zero eigenvalue, corresponding to the local conservation laws, (2) these eigenfunctions become the hydrodynamic modes for finite wave vectors, and (3) the rest of the spectrum is bounded away from the hydrodynamic spectrum with a larger real part. The consequence of results (1) and (2) is the existence of hydrodynamics. The consequence of result (3) is the dominance of hydrodynamic modes at large times. These results are the necessary complementary studies to justify the more practical results obtained by the Chapman-Enskog procedure. The objective here is to establish the counterparts of results (1) and (2) for a granular gas described by the Boltzmann equation, and to prove result (3) for the special case of a model Boltzmann collision operator.

The states considered are obtained from small initial spatial perturbation of the "universal" homogeneous cooling state (HCS). The response is provided by the linearized in-elastic Boltzmann equation, with all possible excitations described by the spectrum of the linearized Boltzmann collision operator. We first show the existence and form of the hydrodynamic modes at asymptotically long wavelengths, corresponding to the summational invariants in the elastic case. Next, the eigenvalues at finite wave vector are constructed by perturbation theory showing the expected form for hydrodynamic modes. These are the primary results of this paper. In addition to the point of principle, existence of hydrodynamics, there are many practical applications made possible by identification of the hydrodynamic modes that go beyond the Boltzmann equation (e.g., hydrodynamic part of linear response and fluctuations, mode coupling phenomena). In particular, hydrodynamic mode coupling for the asymptotic decay of time correlation functions is expected to be more complex for granular systems due to the different wave vector dependence of the eigenvalues. Controlled analysis of this problem using the modes obtained here will be reported elsewhere.

Moreover, some support is provided for the isolation of the hydrodynamic spectrum by an exact analysis of a model Boltzmann equation [9,6]. It exhibits the dominant features of granular gases, including a nontrivial HCS with algebraic decay at large velocities, discussed extensively for related Maxwell models [10]. It is shown that the pure point spectrum for elastic collisions develops a continuum for any degree of inelasticity, extending toward the hydrodynamic spectrum. However, it remains bounded away from the latter by an amount that is controlled by the power law of the algebraic decay of the velocity distribution.

The inelastic nonlinear Boltzmann equation for the density $f(\mathbf{r}, \mathbf{v}, t)$ of particles of mass m at position $\mathbf{r}$ with velocity $\mathbf{v}$ at time t has the form

$$(\partial_t + \mathbf{v} \cdot \boldsymbol{\nabla}) f = J[f, f], \qquad (1)$$

where $J[f, f]$ is a bilinear functional of f [1]. There is no stationary solution to this equation for an isolated system. However, a homogeneous cooling solution (HCS) is assumed to exist with the scaling property

$$f_{HCS}(\mathbf{v},t) = n_H v_0^{-d}(t)\chi_0(\mathbf{c}), \quad \mathbf{c} = \frac{\mathbf{v}}{v_0(t)}. \quad (2)$$

Here n_H is the homogeneous density and $v_0(t) = \sqrt{2T(t)/m}$ is the thermal velocity defined in terms of the temperature $T(t)$. The latter obeys the equation $T^{-1}\partial_t T = -\zeta_0$, where the cooling rate $\zeta_0 = \zeta[\chi_0,\chi_0]$ also is a specified bilinear functional. Both χ_0 and ζ_0 must be determined self-consistently from the Boltzmann equation,

$$\frac{1}{2}\zeta_0 \frac{\partial}{\partial \mathbf{c}} \cdot (\mathbf{c} f_{HCS}) = J[f_{HCS}, f_{HCS}]. \quad (3)$$

The existence of f_H is supported by approximate polynomial expansions [11] and by Monte Carlo simulation [12].

To study the relaxation of small spatial perturbations of the HCS, Δ is defined by $f = f_{HCS}[1+\Delta]$, and terms up through linear order in Δ are retained in the Boltzmann equation. The resulting equation is then written in dimensionless form with the velocities scaled relative to $v_0(t)$ and the time expressed in terms of $s = \int^t dt' \nu_0(t')$, where $\nu_0(t)$ is the average collision frequency, proportional to $\sqrt{T(t)}$. It is sufficient to consider a single Fourier component $\tilde{\Delta}(\mathbf{k})$ that obeys

$$(\partial_s - i\mathbf{k}\cdot\mathbf{c} + \mathcal{L})\tilde{\Delta} = 0, \quad (4)$$

$$\mathcal{L}\tilde{\Delta} = -\chi_0^{-1}\left\{ J[\chi_0, \chi_0\tilde{\Delta}] + J[\chi_0\tilde{\Delta}, \chi_0] - \frac{\zeta_0}{2}\frac{\partial}{\partial \mathbf{c}}\cdot(\mathbf{c}\chi_0\tilde{\Delta}) \right\}. \quad (5)$$

The solution is sought in a Hilbert space defined by the scalar product

$$(a,b) = \int d\mathbf{c}\,\chi_0(\mathbf{c})a^*(\mathbf{c})b(\mathbf{c}). \quad (6)$$

The relevant eigenvalue problem is

$$(-i\mathbf{k}\cdot\mathbf{c} + \mathcal{L})\phi_i = \omega_i(k)\phi_i. \quad (7)$$

It is understood that the index i may be discrete or continuous. The hydrodynamic spectrum can be defined precisely as follows. The linearized Boltzmann equation provides exact balance equations for the moments of Δ corresponding to the density, flow velocity, and temperature. The latter are the $d+2$ hydrodynamic fields. The spectrum of these balance equations can be calculated in the limit $\mathbf{k}=0$. For elastic collisions, there is a $(d+2)$-fold degenerate point at zero eigenvalue, corresponding to the conservation laws for binary collisions. Their perturbation at finite $\mathbf{k}$ defines the hydrodynamic modes more generally. For inelastic collisions, the spectrum is again $d+2$ points, now at $0, \zeta_0/2, -\zeta_0/2$, with the latter being d-fold degenerate. Again, the hydrodynamic modes more generally are defined as those solutions to Eq. (7) that are continuously connected as functions of k to these special $k=0$ solutions. For elastic collisions, it is possible to prove analyticity about $k=0$ [8], and it is assumed that this is the case here as well.

The first step is to show that $\mathcal{L}$ includes the spectrum of the balance equations at $k=0$, and to determine the corresponding eigenfunctions. For elastic collisions, the stationary state is given by $J[f_M, f_M] = 0$, whose solution is the Maxwellian. Differentiating this stationary equation with respect to the hydrodynamic fields leads to the linearized Boltzmann operator acting on combinations of $1, \mathbf{c}, c^2$ independently being zero. These are the $d+2$ eigenfunctions with zero eigenvalue. A similar analysis works for the inelastic case, using the stationary condition (3). Straightforward calculations give

$$\mathcal{L}\phi_i^{(0)} = \omega_i^{(0)}\phi_i^{(0)}, \quad i = 1, \ldots, d+2, \quad (8)$$

$$\phi^{(0)} \rightarrow \left\{ d+1 + \mathbf{c}\cdot\frac{\partial \ln \chi_0}{\partial \mathbf{c}}, -d - \mathbf{c}\cdot\frac{\partial \ln \chi_0}{\partial \mathbf{c}}, -\frac{\partial \ln \chi_0}{\partial \mathbf{c}} \right\}, \quad (9)$$

$$\omega_i^{(0)} \rightarrow \{0, \zeta_0/2, -\zeta_0/2\}. \quad (10)$$

The evaluation of the hydrodynamic spectrum at finite k is now a well-defined technical problem. In general, the eigenvalues depend on both k and the restitution coefficient α. It is known from the Navier-Stokes equations that the behavior for small $k(\alpha)$ is not uniform with respect to $\alpha(k)$, so that a perturbation expansion in one or the other does not provide the entire mode structure. Here, for illustration, only the results for small k are reported. The expansion is defined by $\omega_i(k) = \omega_i^{(0)} + ik\omega_i^{(1)} + k^2\omega_i^{(2)} + \cdots$ and $\phi_i(k) = \phi_i^{(0)} + ik\phi_i^{(1)} + k^2\phi_i^{(2)} + \cdots$. The leading terms are those of Eqs. (9) and (10). Since $\mathcal{L}$ is not self-adjoint, it is necessary to introduce a set of functions $\{\psi_i\}$ that are biorthogonal to $\{\phi_i\}$. The leading terms of a similar expansion in k are found to be

$$\psi_i^{(0)} \rightarrow \left\{ 1, \frac{c^2}{d} + \frac{1}{2}, \hat{\mathbf{k}}\cdot\mathbf{c}, \hat{\mathbf{e}}^{(i)}\cdot\mathbf{c} \right\}, \quad (11)$$

where $\{\hat{\mathbf{k}}, \hat{\mathbf{e}}^{(i)}\}$ are d pairwise orthogonal unit vectors. The results to second order in k are $\omega_i^{(1)} = 0$,

$$\omega_i^{(2)} = (\psi_i^{(0)}, \hat{\mathbf{k}}\cdot\mathbf{c}\phi_i^{(1)}) + \delta_{i2}\zeta_0[\chi_0, \chi_0 \mathcal{Q}_1\phi_2^{(2)}], \quad (12)$$

$$\phi_i^{(1)} = \mathcal{Q}_i(\mathcal{L} - \omega_i^{(0)})^{-1}\hat{\mathbf{k}}\cdot\mathbf{c}\phi_i^{(0)}, \quad (13)$$

$$\phi_i^{(2)} = -\mathcal{Q}_i(\mathcal{L} - \omega_i^{(0)})^{-1}\hat{\mathbf{k}}\cdot\mathbf{c}\phi_i^{(1)}. \quad (14)$$

The operators $\mathcal{Q}_i = 1 - \mathcal{P}_i$ are projections orthogonal to $\phi_i^{(0)}$ with $\mathcal{P}_i a = \phi_i^{(0)}(\psi_i^{(0)}, a)$. The constants $\omega_i^{(2)}$ are transport coefficients which are identified from a similar calculation using the Navier-Stokes equations. For example, the shear viscosity η is

$$\eta = \frac{mv_0\omega_4^{(2)}}{\sigma^{d-1}} = -\frac{mv_0}{\sigma^{d-1}}\left[c_x c_y, \left(\mathcal{L} + \frac{1}{2}\zeta_0 \right)^{-1} c_x \frac{\partial \ln \chi_0}{\partial c_y} \right]. \quad (15)$$

This agrees with the result obtained from the Chapman-Enskog method [13]. A more complete connection with the

latter can be established as follows. Assuming that the hydrodynamic spectrum dominates for long times and small k, the solution to the Boltzmann equation for small perturbations becomes

$$f \to f_{hcs}(\mathbf{v}) \left[1 + \sum_{i=1}^{d+2} \phi_i(k) \, \delta y_i^h(\mathbf{k}, s) \right]. \qquad (16)$$

Here $\delta y_i^h(\mathbf{k}, s)$ are the hydrodynamic fields,

$$\delta y_i^h(\mathbf{k}, s) = (\psi_i, \tilde{\Delta}(\mathbf{k}, s)) = e^{s\omega_i(k)}(\psi_i, \tilde{\Delta}(\mathbf{k}, 0)). \qquad (17)$$

The connection between these fields and the macroscopic density, temperature, and flow velocity is obtained for small k by using $\psi_i \to \psi_i^{(0)}$ in Eq. (17). The fields $\delta y_i^h(\mathbf{k}, s)$ represent components of the initial distribution that obey the hydrodynamic equations for all times, whereas the density, temperature, and flow velocity become equal to these only on the long space and time scales [8]. Use of $\phi_i(k) \to \phi_i^{(0)} + ik\phi_i^{(1)}$ for small k in Eq. (16) leads to a result that is in agreement with the Navier-Stokes approximation of the Chapman-Enskog method [14].

The existence of a hydrodynamic spectrum is relevant only if the corresponding modes dominate for long times and long wavelengths. A sufficient condition is that the magnitude of the hydrodynamic eigenvalues be smaller than that of all other parts of the spectrum. This is the case for elastic collisions, but the proof does not extend directly to $-i\mathbf{k} \cdot \mathbf{c} + \mathcal{L}$ here, and the question remains open in general [15]. In fact, the mode at $\zeta_0/2$ gives reason for concern since it might become comparable to the microscopic excitations at large inelasticity. To explore this point, it is instructive to consider a model for the Boltzmann equation that preserves the essential physics of a nontrivial HCS, exact balance equations, and hydrodynamic modes, as described above. The Boltzmann collision operator can be written as

$$J[f, f] = -\nu(f - g), \qquad (18)$$

where νf is the "loss" contribution and νg is the "gain" contribution. Both the collision frequency ν and g are specified positive functionals of f.

The class of models to be considered here are characterized by the following two simplifications: (1) ν is independent of $\mathbf{v}$ and (2) ν and g are functionals of f only through its moments with respect to $1, \mathbf{v}, v^2$. The first condition is similar to that of "Maxwell models" that have been introduced recently [10]. The second condition means that ν and g depend on the state of the system only through the hydrodynamic fields. The forms of these functionals are subjected to preserve the exact properties of the collision operator required by the balance equations,

$$\int d\mathbf{v} \begin{pmatrix} 1 \\ \mathbf{v} \\ mv^2 \end{pmatrix} g(\mathbf{r}, \mathbf{v}, t) = \begin{pmatrix} n \\ n\mathbf{u} \\ nTd\left(\lambda^2 + \dfrac{mu^2}{Td}\right) \end{pmatrix}. \qquad (19)$$

Here $\mathbf{u}$ is the flow velocity and $\lambda^2 \equiv 1 - \zeta/\nu$. In principle, ζ is the same functional of f as obtained from the Boltzmann equation, although it can be eventually approximated similarly to ν and g. In any case, Eq. (19) must be preserved. In the following, we will assume that both ζ and ν scale as $T^{1/2}$ in order to mimic hard sphere behavior. With the above class of models, it is obtained from Eq. (3),

$$\chi_0(\mathbf{c}) = \int_1^\infty dx P(x)(\lambda x)^{-d} g^*(\mathbf{c}/\lambda x), \qquad (20)$$

where $P(x) \equiv px^{-(1+p)}$ with $p = 2\nu/\zeta$. The dimensionless gain function $g(\mathbf{v}; y^h(\mathbf{r}, t)) = n_H(\lambda v_0)^3 g^*(\mathbf{c}/\lambda)$ has been introduced. Further investigation requires specification of $g^*(\mathbf{c}/\lambda)$. However, it is easily verified that moments of degree equal or greater than p are not finite if the corresponding moments of g^* exist. This is consistent with a similar result for the Maxwell models where algebraic decay of $\chi_0(\mathbf{c})$ is observed for large velocities, also due to a slower decay of the loss term than the gain term.

Linearization of the kinetic model around this HCS leads again to Eq. (4), but with a simpler form for $\mathcal{L}$. It has a decomposition into the subspace of $\phi_i^{(0)}$ and its complement, $\mathcal{L} = \mathcal{P}L\mathcal{P} + \mathcal{Q}L\mathcal{Q}$, where $\mathcal{P} = \Sigma \mathcal{P}_i$ and $\mathcal{Q} = 1 - \mathcal{P}$. In the subspace of $\phi_i^{(0)}$, the spectrum is just the $d+2$ points of Eq. (10),

$$\mathcal{P}L\mathcal{P} = \sum_{i=1}^{d+2} \omega_i^{(0)} \mathcal{P}_i. \qquad (21)$$

In the complementary subspace the operator is

$$\mathcal{Q}L\mathcal{Q} = \nu\mathcal{Q} + \chi_0^{-1} \frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{c}} \cdot (\mathbf{c}\chi_0 \mathcal{Q}). \qquad (22)$$

Interestingly, there is no explicit dependence on the gain function g^* except through χ_0. For elastic collisions, the second term is missing and the model becomes the linearized Bhatnager-Gross-Krook model (BGK) [16], with a single degenerate point in the spectrum representing all the microscopic excitations of the Boltzmann operator. It is somewhat easier to analyze the corresponding adjoint problem,

$$\nu\phi - \frac{1}{2}\zeta_0 \mathbf{c} \cdot \frac{\partial}{\partial \mathbf{c}} \phi = \omega\phi, \quad \phi = \mathcal{Q}^\dagger \phi, \qquad (23)$$

where $\mathcal{Q}^\dagger$ is the adjoint of $\mathcal{Q}$. Solutions to this equation exist from the class of homogeneous functions

$$\phi \propto \prod_{i=1}^d c_i^{q_i}, \quad \sum_{i=1}^d q_1 = q. \qquad (24)$$

The property $\mathcal{Q}^\dagger \phi = \phi$ can be satisfied with a suitable choice of the linear combinations. The corresponding eigenvalue is $\omega = \nu - q\zeta_0/2$. Thus the spectrum of $\mathcal{L}$ includes a continuum with $\text{Re}\,\omega \leq \nu$. Of critical interest for hydrodynamics is whether the lower bound for this continuum on the real axis intersects the hydrodynamic point spectrum. The bound is determined by the maximum degree of such homogeneous

eigenfunctions admitted in the Hilbert space. The condition is that $\|c^q\|^2 = (c^q, c^q)$ exists. Using Eq. (20), this becomes

$$\|c^q\|^2 = \lambda^{2\mathrm{Re}q} p M_q \int_1^\infty dx\, x^{2\mathrm{Re}q - (1+p)}, \qquad (25)$$

$$M_q = \int_0^\infty d\mathbf{c}\, c^{2\mathrm{Re}q} g^*(c). \qquad (26)$$

For finite M_q, the value of q is bounded from above by $\mathrm{Re}q < q_m = p/2 = \nu/\zeta_0$. The continuous spectrum therefore is restricted by $\nu/2 < \mathrm{Re}\omega \leq \nu$. This means that the discrete spectrum is isolated from the continuum for all $\zeta_0/\nu < 1$. But ζ_0/ν must be smaller than unity because of the third condition required by Eq. (19). This condition is related with the fact that only a fraction of the energy of the particles can be lost in collisions. The continuum is singular in the sense that it is independent of the restitution coefficient but it is present only for finite inelasticity.

For this class of models, it is seen that the hydrodynamic spectrum at $k=0$ is isolated from the rest of the spectrum. The model also allows exact calculation of the hydrody-namic eigenvalues $\omega_i(k)$ at finite k, without the restrictions of the perturbation theory calculation above. For elastic collisions, such a calculation shows that the hydrodynamic modes exist only for k less than some maximum value. It is expected that a similar condition applies in the granular case as well.

In the case of elastic collisions, the kinetic model considered above reduces to the familiar BGK model that has been widely used for applications in gas dynamics [16]. The mathematical relevance for granular gases has been discussed recently [17]. Furthermore, comparisons of the model predictions for transport show good quantitative agreement with those of the Boltzmann equation in many cases. A qualification here is the relationship of the spectrum to the algebraic decay of the HCS distribution which is different from that of the Boltzmann equation. Calculations using a model without this algebraic decay are in progress.

[1] T.P. C van Noije and M.H. Ernst, in *Granular Gases*, edited by T. Pöschel and S. Luding, Lecture Notes in Physics 564 (Springer, Berlin, 2001); J.W. Dufty, Adv. Complex Syst. **4**, 397 (2001).

[2] J.T. Jenkins and M.W. Richman, Arch. Ration. Mech. Anal. **87**, 355 (1985).

[3] C.K.W. Lun, S.B. Savage, D.J. Jeffrey, and N. Chepurnuy, J. Fluid Mech. **140**, 223 (1984).

[4] A. Goldshtein and M. Shapiro, J. Fluid Mech. **282**, 41 (1995).

[5] N. Sela and I. Goldhirsch, J. Fluid Mech. **361**, 41 (1998).

[6] J.J. Brey, J.W. Dufty, C.S. Kim, and A. Santos, Phys. Rev. E **58**, 4638 (1998).

[7] J.J. Brey and D. Cubero, in *Granular Gases* (Ref. [1]).

[8] J.A. McLennan, *Introduction to Nonequilibrium Statistical Mechanics* (Prentice-Hall, Englewood Cliffs, NJ, 1989).

[9] J.J. Brey, F. Moreno, and J.W. Dufty, Phys. Rev. E **54**, 445 (1996).

[10] M.H. Ernst and R. Brito, Europhys. Lett. **58**, 182 (2002).

[11] T.P. C van Noije and M.H. Ernst, Granular Matter **1**, 57 (1998).

[12] J.J. Brey, M.J. Ruiz-Montero, and D. Cubero, Phys. Rev. E **54**, 3664 (1998).

[13] J.W. Dufty and J.J. Brey, J. Stat. Phys. **109**, 433 (2002).

[14] J.J. Brey, J.W. Dufty, and M.J. Ruiz-Montero, e-print cond-mat/0302180.

[15] I. Goldhirsh, in *Granular Gases* (Ref. [1]).

[16] C. Cercignani, *Theory and Application of the Boltzmann Equation* (Elsevier, New York, 1975).

[17] J.J. Brey, J.W. Dufty, and A. Santos, J. Stat. Phys. **97**, 281 (1999).