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# Slide-rule-like property of Wigner's little groups and cyclic *S* matrices for multilayer optics

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It is noted that 2×2 "*S*" matrices in multilayer optics can be represented by the Sp(2) group whose algebraic property is the same as the group of Lorentz transformations applicable to two spacelike and one timelike dimensions. It is also noted that Wigner's little groups have a slide-rule-like property that allows us to perform multiplications by additions. It is shown that these two mathematical properties lead to a cyclic representation of the *S* matrix for multilayer optics, as in the case of *ABCD* matrices for laser cavities. It is therefore possible to write the *N*-layer *S* matrix as a multiplication of the *N* single-layer *S* matrices resulting in the same mathematical expression with one of the parameters multiplied by *N*. In addition, it is noted, as in the case of lens optics, that multilayer optics can serve as an analog computer for the contraction of Wigner's little groups for internal space-time symmetries of relativistic particles.

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## I. INTRODUCTION

In our previous paper on multilayer optics [1], it was shown that the complex 2×2 *S* matrix formalism is equivalent to a 2×2 real matrix representation of the Sp(2) group, which shares the same algebraic property as the Lorentz group applicable to two spacelike and one timelike dimensions. This group has three independent parameters. It was shown, furthermore that, under certain conditions, one of the off-diagonal elements vanishes and the three remaining elements can be computed analytically. We called this the Iwasawa effect [1]. In this paper, we remove those "certain conditions" and achieve the same kind of simplification for all possible multilayer cases.

Indeed, the group Sp(2) plays the central role in both quantum and classical optics, including multilayer optics [2]. It consists of 2×2 real matrices whose determinant is 1. Each matrix contains at most three independent parameters. It is thus a simple matter to multiply two or three matrices. However, multiplication of a large number of matrices presents a new problem. The product of that many matrices will also be one 2×2 matrix with a unit determinant, but how can we calculate their elements?

For example, let us look at laser cavities. It consists of a chain of *N* identical two-lens systems, where *N* is the number of cycles the light beam performs. The resulting *ABCD* matrix can be written as a multiplication of *N* identical matrices, but the resulting matrix has the same mathematical form as that for the single cycle [3].

Can we then expect a similar cyclic property in multilayer optics? We have shown in Ref. [1] that the *N* dependence can be made quite transparent if the multilayer *S* matrix [4] is reduced to the Iwasawa form. In this paper, we present the cyclic property for the most general form of multilayers, without the restriction we imposed in our previous paper [1]. We shall show that the core of the *S* matrix takes the form

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}. \tag{1}$$

These matrices form the core of Wigner's little groups applicable to the internal space-time symmetries of relativistic particles [5,6]. We note here that these matrices have the following interesting property.

We cannot write $(\cos\alpha_1 \cos\alpha_2) = \cos(\alpha_1+\alpha_2)$ because it is wrong. However, in the 2×2 matrix form

$$\begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{pmatrix}\begin{pmatrix} \cos\alpha_2 & -\sin\alpha_2 \\ \sin\alpha_2 & \cos\alpha_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) \end{pmatrix}, \tag{2}$$

and we have similar expressions for the remaining matrices in Eq. (1). We call this the slide-rule-like property of Wigner's little groups.

If they are cycled *N* times, they take the forms

$$\begin{pmatrix} \cos(N\alpha) & -\sin(N\alpha) \\ \sin(N\alpha) & \cos(N\alpha) \end{pmatrix}, \quad \begin{pmatrix} \cosh(N\beta) & \sinh(N\beta) \\ \sinh(\beta) & (\cosh\beta) \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ N\gamma & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & N\gamma \\ 0 & 1 \end{pmatrix}, \tag{3}$$

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respectively. This mathematical instrumentation works for laser cavity optics [3]. The question is whether this is applicable to multilayer optics.

The purpose of this paper is to show that the answer to the above question is yes. We note first that the $S$ matrix consists of $N$ cycles. Each cycle consists of two phase-shift matrices, one boundary matrix, and its inverse, and this cycle does not take any of the forms given in Eq. (1) if we start the cycle from the boundary. In this paper, we show that it is possible to obtain the core in the form of Eq. (1) if we start the cycle from somewhere within one of the media between the two boundaries.

Throughout this paper, we avoid group theoretical languages and rely on explicit $2\times 2$ matrices with real elements. However, in doing so, we are exploiting an important group theoretical feature that became known to us only recently concerning contractions of Wigner's little groups. This aspect was discussed in detail in a recent paper on lens optics [7]. Thus, we shall borrow some of the mathematical identities from that paper.

In addition, in the present paper, we observe that Wigner's little group has slide-rule-like properties that allow us to convert multiplications into additions. This property was noted for one of the little groups in the paper by Han *et al.* In this paper, we shall show that all three of the little groups have the same slide-rule-like property, using Eq. (2).

In Sec. II, we formulate the problem in terms of the $S$ matrix method widely used in multilayer optics [4,8,9], and show that the complex $S$ matrices can be transformed to real matrices by a conjugate transformation, and thus to the algebra of the Sp(2) group which is by now a familiar mathematical language in optics. In Sec. III, we import from the literature mathematical identities useful for the purpose of the present paper. They are derivable from Wigner's little groups and their contractions. In Sec. IV, using the cyclic property of Eq. (3), it is possible to write the multilayer $S$ matrix as a multiplication of the $N$ single-layer $S$ matrices resulting in the same mathematical expression with one of the parameters multiplied by $N$. In Sec. V, it is pointed out that the mathematical identities presented in this paper can be tested experimentally. We discuss the condition under which the system can achieve the Iwasawa effect [1]. In Sec. VI, we explain what we do in this paper using group theoretical language, particularly in terms of Wigner's little groups which dictate internal space-time symmetries of relativistic particles.

## II. FORMULATION OF THE PROBLEM

It was noted in our previous paper that one cycle in $N$-layer optics starts with the boundary matrix of the form [10]

$$B(\eta)=\begin{pmatrix}\cosh(\eta/2) & \sinh(\eta/2)\\ \sinh(\eta/2) & \cosh(\eta/2)\end{pmatrix}, \tag{4}$$

which, as illustrated in Fig. 1, describes the transition from medium 2 to medium 1, taking into account both the transmission and the reflection of the beam. As the beam passes through medium 1, it undergoes the phase shift represented by the matrix

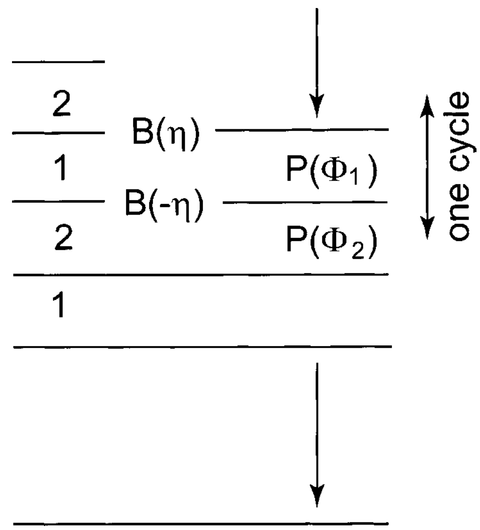


FIG. 1. Optical layers. There are phase-shift matrices for their respective layers. There is a boundary matrix for transition from the first to the second medium, and its inverse applies to the transition from the second to the first medium. The mathematics becomes much simpler if the cycle starts in the middle of the second layer.

$$P(\phi_1)=\begin{pmatrix}e^{-i\phi_1/2} & 0\\ 0 & e^{i\phi_1/2}\end{pmatrix}. \tag{5}$$

When the wave hits the surface of the second medium, the corresponding matrix is

$$B(-\eta)=\begin{pmatrix}\cosh(\eta/2) & -\sinh(\eta/2)\\ -\sinh(\eta/2) & \cosh(\eta/2)\end{pmatrix}, \tag{6}$$

which is the inverse of the matrix given in Eq. (4). Within the second medium, we write the phase-shift matrix as

$$P(\phi_2)=\begin{pmatrix}e^{-i\phi_2/2} & 0\\ 0 & e^{i\phi_2/2}\end{pmatrix}. \tag{7}$$

Then, when the wave hits the first medium from the second, we have to go back to Eq. (4). Thus, one cycle consists of

$$M_1=\begin{pmatrix}\cosh(\eta/2) & \sinh(\eta/2)\\ \sinh(\eta/2) & \cosh(\eta/2)\end{pmatrix}\begin{pmatrix}e^{-i\phi_1/2} & 0\\ 0 & e^{i\phi_1/2}\end{pmatrix} \times\begin{pmatrix}\cosh(\eta/2) & -\sinh(\eta/2)\\ -\sinh(\eta/2) & \cosh(\eta/2)\end{pmatrix}\begin{pmatrix}e^{-i\phi_2/2} & 0\\ 0 & e^{i\phi_2/2}\end{pmatrix}. \tag{8}$$

This arrangement of matrices is illustrated in Fig. 1.

The $M_1$ matrix, Eq. (8), contains complex numbers, but we are interested in carrying out calculations with real matrices. This can be done if we make the following conjugate transformation [1].

Let us next consider the matrix

$$C=\frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\pi/4} & e^{i\pi/4}\\ -e^{-i\pi/4} & e^{-i\pi/4}\end{pmatrix}. \quad (9)$$

This matrix and its inverse can be written as

$$C=\frac{e^{i\pi/4}}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}, \quad C=\frac{e^{-i\pi/4}}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}. \quad (10)$$

We have shown in our previous paper that

$$M_2=CM_1C^{-1}, \quad (11)$$

with

$$M_2=\begin{pmatrix}e^{\eta/2} & 0\\ 0 & e^{-\eta/2}\end{pmatrix}\begin{pmatrix}\cos(\phi_1/2) & -\sin(\phi_1/2)\\ \sin(\phi_1/2) & \cos(\phi_1/2)\end{pmatrix}\times\begin{pmatrix}e^{-\eta/2} & 0\\ 0 & e^{\eta/2}\end{pmatrix}\begin{pmatrix}\cos(\phi_2/2) & -\sin(\phi_2/2)\\ \sin(\phi_2/2) & \cos(\phi_2/2)\end{pmatrix}. \quad (12)$$

The conjugate transformation of Eq. (11) changes the boundary matrix $B(\eta)$ of Eq. (4) to a squeeze matrix

$$S(\eta)=\begin{pmatrix}e^{\eta/2} & 0\\ 0 & e^{-\eta/2}\end{pmatrix}, \quad (13)$$

and the phase-shift matrices $P(\phi_1)$ of Eqs. (5) and (7) to rotation matrices

$$R(\phi_i)=\begin{pmatrix}\cos(\phi_i/2) & -\sin(\phi_i/2)\\ \sin(\phi_i/2) & \cos(\phi_i/2)\end{pmatrix}, \quad (14)$$

with $i=1,2$.

Indeed, the matrices $M_1$ and $M_2$ can be written as

$$M_1=B(\eta)P(\phi_1)B(-\eta)P(\phi_2),$$

$$M_2=S(\eta)R(\phi_1)S(-\eta)R(\phi_2). \quad (15)$$

The matrix $M_2$ can be obtained from $M_1$ by the conjugate transformation in Eq. (11). Conversely, $M_1$ can be obtained from $M_2$ through the inverse conjugate transformation:

$$M_1=C^{-1}M_2C. \quad (16)$$

In addition, the conjugate transformations have the following properties:

$$(M_2)^N=C(M_1)^NC^{-1}, \quad (M_1)^N=C^{-1}(M_2)^N\ C. \quad (17)$$

Thus, we can study $M_2$ in order to study $M_1$. The advantage of $M_2$ is that it consists of real matrices. The group of these matrices is called Sp(2) which is like (isomorphic) the Lorentz group applicable to three space and one time dimensions. This group contains very rich group-theoretical contents including those of Wigner's little groups. We intend to study $M_2$ in terms of those little groups.

The problem is that $M_2$ takes a simple form and $(M_2)^2$ is manageable, but we cannot predict what form $(M_2)^N$ takes. In this paper, we shall construct the core matrix of the form of Eq. (1) for multilayer optics. Then, as we can see in Eq. (3), the chain effect is straightforward. We shall calculate $(M_2)^N$ first and then $(M_1)^N$.

## III. MATHEMATICAL IDENTITIES FROM THE LORENTZ GROUP

Wigner's little groups were formulated for internal spacetime symmetries of relativistic particles [5,6]. However, they produced many mathematical identities useful in other branches of physics, including classical layer optics, which depends heavily on 2×2 matrices. The correspondence between the 2×2 and 4×4 representations of the Lorentz group has been repeatedly discussed in the literature [1,3,7]. In the 2×2 representation, we write the rotation matrix around the $y$ axis as

$$\begin{pmatrix}\cos(\phi/2) & -\sin(\phi/2)\\ \sin(\phi/2) & \cos(\phi/2)\end{pmatrix}, \quad (18)$$

and the boost matrices along the $z$ and $x$ axes as

$$\begin{pmatrix}e^{\eta/2} & 0\\ 0 & e^{-\eta/2}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\cosh\lambda & \sinh\lambda\\ \sinh\lambda & \cosh\lambda\end{pmatrix}, \quad (19)$$

respectively. We shall use only these three matrices in this paper.

We use the following identity that Baskal and Kim introduced recently in their paper on lens optics and group contractions [7,11]:

$$\begin{pmatrix}e^{\eta/2} & 0\\ 0 & e^{-\eta/2}\end{pmatrix}\begin{pmatrix}\cos(\phi/2) & -\sin(\phi/2)\\ \sin(\phi/2) & \cos(\phi/2)\end{pmatrix}\begin{pmatrix}e^{-\eta/2} & 0\\ 0 & e^{\eta/2}\end{pmatrix}=\begin{pmatrix}\cos(\theta/2) & -\sin(\theta/2)\\ \sin(\theta/2) & \cos(\theta/2)\end{pmatrix}\begin{pmatrix}\cosh\lambda & \sinh\lambda\\ \sinh\lambda & \cosh\lambda\end{pmatrix}\times\begin{pmatrix}\cos(\theta/2) & -\sin(\theta/2)\\ \sin(\theta/2) & \cos(\theta/2)\end{pmatrix}, \quad (20)$$

with

$$\cos(\phi/2)=\cosh\lambda\cos\theta,$$

$$e^{2\eta}=\frac{\cosh\lambda\sin\theta+\sinh\lambda}{\cosh\lambda\sin\theta-\sinh\lambda}. \quad (21)$$

The left-hand side of the above expression is one rotation matrix sandwiched between one boost matrix and its inverse, while the right-hand side consists of one boost matrix sandwiched between two identical rotation matrices.

The left-hand side of Eq. (20) is the same as the first three matrices of the core matrix $M_2$ given in Eq. (12). However, the fourth matrix is a rotation matrix. Since one rotation matrix multiplied by another rotation matrix is still a rotation matrix, the core matrix $M_2$ is one boost matrix sandwiched between two different rotation matrices. Thus, the problem is to find a transformation that will make those two rotation matrices the same, and go back to the form of the left-hand side of Eq. (20). We shall come back to this problem in Sec. IV.

If we complete the matrix multiplications of both sides, the result is

$$\begin{pmatrix} \cos(\phi/2) & -e^{\eta}\sin(\phi/2) \\ e^{-\eta}\sin(\phi/2) & \cos(\phi/2) \end{pmatrix} = \begin{pmatrix} \cosh\lambda\cos\theta & -(\cosh\lambda\sin\theta+\sinh\lambda) \\ \cosh\lambda\sin\theta-\sinh\lambda & \cosh\lambda\cos\theta \end{pmatrix}. \tag{22}$$

Then, we can write $\phi$ and $\eta$ in terms of $\lambda$ and $\theta$ as given in Eq. (21). The parameters $\lambda$ and $\theta$ can be written in terms of $\phi$ and $\eta$ as

$$\cosh\lambda=(\cosh\eta)\sqrt{1-\cos^2(\phi/2)\tanh^2\eta},$$

$$\cos\theta=\frac{\cos(\phi/2)}{(\cosh\eta)\sqrt{1-\cos^2(\phi/2)\tanh^2\eta}}. \tag{23}$$

The above relation is valid only if $(\cosh\lambda\sin\theta/2-\sinh\lambda)$ is positive. If it is negative, the left-hand side of the above expression should be

$$\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}\begin{pmatrix} \cosh(\chi/2) & -\sinh(\chi/2) \\ -\sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}\begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix} = \begin{pmatrix} \cosh(\chi/2) & -e^{\eta}\sinh(\chi/2) \\ -e^{-\eta}\sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}, \tag{24}$$

with

$$\cosh(\chi/2)=\cosh\lambda\cos\theta,$$

$$e^{2\eta}=\frac{\cosh\lambda\sin\theta+\sinh\lambda}{\sinh\lambda-\cosh\lambda\sin\theta}. \tag{25}$$

Conversely, $\lambda$ and $\theta$ can be written in terms of $\chi$ and $\eta$ as

$$\cosh\lambda=(\cosh\eta)\sqrt{\cosh^2(\chi/2)-\tanh^2\eta},$$

$$\cos\theta=\frac{\cosh(\chi/2)}{(\cosh\eta)\sqrt{\cosh^2(\chi/2)-\tanh^2\eta}}. \tag{26}$$

An interesting case is when $\sinh\lambda-\cosh\lambda\sin\theta$ becomes zero and $\eta$ becomes very large. If we insist that

$$e^{\eta}\sin(\phi/2)=u, \tag{27}$$

remain finite, then $\phi/2$ must become very small. On the right-hand side

$$u=2\sinh\lambda, \quad \text{with} \quad \sin\theta=\tanh\lambda. \tag{28}$$

The net result is that both sides take the form

$$\begin{pmatrix} 1 & -2\sinh\lambda \\ 0 & 1 \end{pmatrix}. \tag{29}$$

In their recent paper [7], Baskal and Kim studied in detail the transition from Eq. (22) to Eq. (24) through Eq. (29), and showed that the one-lens camera goes through this transition as we try to focus the image. Mathematically, the system goes through group contraction processes. In the present paper, we show that the same contraction process can be achieved in multilayer optics.

## IV. CYCLIC REPRESENTATION OF THE *S* MATRIX

It was noted in Sec. II that each cycle consists of

$$(SR_1S^{-1}R_2), \tag{30}$$

with

$$R_1=R(\phi_1) \quad \text{and} \quad R_2=R(\phi_2), \tag{31}$$

of Eq. (14), respectively. The squeeze matrix $S$ is given in Eq. (13). For the layer consisting of $N$ cycles, let us consider the chain

$$M_2^N=(SR_1S^{-1}R_2)(SR_1S^{-1}R_2) \times(SR_1S^{-1}R_2)\cdots(SR_1S^{-1}R_2). \tag{32}$$

According to Eq. (20), we can now write $SR_1S^{-1}$ in the above expression as

$$SR_1S^{-1}=R_3\ XR_3, \tag{33}$$

with

$$R_3=\begin{pmatrix} \cos(\phi_3/2) & -\sin(\phi_3/2) \\ \sin(\phi_3/2) & \cos(\phi_3/2) \end{pmatrix}, \quad X=\begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}, \tag{34}$$

and

$$\cosh\lambda=(\cosh\eta)\sqrt{1-\cos^2(\phi_1/2)\tanh^2\eta},$$

$$\cos\phi_3=\frac{\cos(\phi_1/2)}{(\cosh\eta)\sqrt{1-\cos^2(\phi_1/2)\tanh^2\eta}}. \tag{35}$$

The parameters $\lambda$ and $\phi_3$ are determined from $\eta$ and $\phi_1$ which are the input parameters from the optical properties of the media.

The chain of Eq. (32) becomes

$$M_2^N=(R_3XR_3R_2)(R_3XR_3R_2)(R_3XR_3R_2)\cdots(R_3XR_3R_2). \tag{36}$$

Let us next introduce the rotation matrix $R(\alpha)$ as

$$R(\alpha)=(R_2)^{1/2}R_3, \tag{37}$$

with

$$\alpha=\phi_3+\tfrac{1}{2}\phi_2, \tag{38}$$

where $\phi_2$ is an input parameter. Since $\phi_3$ is determined by $\eta$ and $\phi_1$, the rotation angle $\alpha$ is determined by the three input parameters, namely, $\eta$, $\phi_1$, and $\phi_2$.

In terms of $R=R(\alpha)$, the chain of Eq. (36) becomes

$$M_2^N=R_3R^{-1}(RXR)(RXR)(RXR)\cdots(RXR)R^{-1}R_3R_2. \tag{39}$$

Since $R_3R^{-1}=R_2^{-1/2}$ and $R^{-1}R_3R_2=R_2^{1/2}$ from Eq. (38),

$$M_2^N=(R_2)^{-1/2}(RXR)(RXR)(RXR)\cdots(RXR)(R_2)^{1/2}. \tag{40}$$

The $(R_2)^{1/2}$ factors in this expression indicate that the cycle starts in the middle of the second medium, as illustrated in Fig. 1.

According to Eqs. (20) and (22), we can now write $RXR$ as

$$RXR=\begin{pmatrix} \cosh\lambda\cos\alpha & -(\cosh\lambda\sin\alpha+\sinh\lambda) \\ \cosh\lambda\sin\alpha-\sinh\lambda & \cosh\lambda\cos\alpha \end{pmatrix}. \tag{41}$$

According to the formulas given in Sec. III, especially Eq. (20), $RXR$ can also be written as

$$RXR=ZAZ^{-1}, \tag{42}$$

with

$$Z=\begin{pmatrix} e^{\xi/2} & 0 \\ 0 & e^{-\xi/2} \end{pmatrix}. \tag{43}$$

Now the $2\times2$ matrix $A$ can take one of the following forms.

If the off-diagonal elements of the matrix of Eq. (41) have opposite signs, the $A$ matrix becomes

$$A=\begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix}, \tag{44}$$

with

$$\cos(\phi/2)=\cosh\lambda\cos\alpha,$$

$$e^{2\xi}=\frac{\cosh\lambda\sin\alpha+\sinh\alpha}{\cosh\lambda\sin\alpha-\sinh\lambda}. \tag{45}$$

If, on the other hand, the off-diagonal elements of the matrix $RXR$ have the same sign, the matrix $A$ should be written as

$$A=\begin{pmatrix} \cosh(\chi/2) & -\sinh(\chi/2) \\ -\sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}, \tag{46}$$

with

$$\cosh(\chi/2)=\cosh\lambda\cos\alpha,$$

$$e^{2\xi}=\frac{\cosh\lambda\sin\alpha+\sinh\lambda}{\sinh\lambda-\cosh\lambda\sin\alpha}. \tag{47}$$

We note from Eqs. (44) and (46) that the matrix $A$ takes circular or hyperbolic forms depending on the sign of the lower-left element of Eq. (41), which is

$$\sinh\lambda-(\sin\alpha)\cosh\lambda, \tag{48}$$

and note that this expression can change from a positive to negative number continuously as the parameters $\lambda$ and $\alpha$ vary. These two parameters are determined from the reflection and transmission properties of the media.

While expression of Eq. (48) makes continuous transition, it has to go through zero. If it vanishes,

$$RXR=\begin{pmatrix} 1 & -2\sinh\lambda \\ 0 & 1 \end{pmatrix}. \tag{49}$$

The transition of $A$ from Eq. (44) to Eq. (46) through this process has been discussed in detail in Ref. [7] in connection with the contraction of Wigner's little groups.

As we noted in Sec. II, the matrix $A$ has the desired cyclic property. Thus,

$$M_2^N=(R_2)^{-1/2}[(ZAZ^{-1})(ZAZ^{-1})(ZAZ^{-1})\cdots(ZAZ^{-1})]\times(R_2)^{1/2}. \tag{50}$$

Consequently,

$$M_2^N=(R_2)^{-1/2}[Z\,A^N\,Z^{-1}](R_2)^{1/2}. \tag{51}$$

If $A$ takes the form of Eq. (44),

$$A^N=\begin{pmatrix} \cos(N\phi/2) & -\sin(N\phi/2) \\ \sin(N\phi/2) & \cos(N\phi/2) \end{pmatrix}. \tag{52}$$

For $A$ given in Eq. (46),

$$A^N=\begin{pmatrix} \cosh(N\chi/2) & -\sinh(N\chi/2) \\ -\sinh(N\chi/2) & \cosh(N\chi/2) \end{pmatrix}. \tag{53}$$

As Eq. (49),

$$(RXR)^N=\begin{pmatrix} 1 & -2N\sinh\lambda \\ 0 & 1 \end{pmatrix}. \tag{54}$$

Then, the calculation of $(M_2)^N$ for the $N$-layer case is straightforward. We can now compute the matrix $(M_1)^N$ using the conjugate transformation of Eq. (17). Let us write our result in $2\times2$ matrices:

$$M_2^N=\left[\begin{pmatrix} \cos(\phi_2/4) & -\sin(\phi_2/4) \\ \sin(\phi_2/4) & \cos(\phi_2/4) \end{pmatrix}\begin{pmatrix} e^{\xi/2} & 0 \\ 0 & e^{-\xi/2} \end{pmatrix}\right]\times\begin{pmatrix} \cos(N\phi/2) & -\sin(N\phi/2) \\ \sin(N\phi/2) & \cos(N\phi/2) \end{pmatrix}\left[\begin{pmatrix} e^{-\xi/2} & 0 \\ 0 & e^{\xi/2} \end{pmatrix}\times\begin{pmatrix} \cos(\phi_2/4) & \sin(\phi_2/4) \\ -\sin(\phi_2/4) & \cos(\phi_2/4) \end{pmatrix}\right] \tag{55}$$

for $A$ of Eq. (44). For $A$ of Eq. (46),

$$M_2^N=\left[\begin{pmatrix}\cos(\phi_2/4) & -\sin(\phi_2/4)\\ \sin(\phi_2/4) & \cos(\phi_2/4)\end{pmatrix}\begin{pmatrix}e^{\xi/2} & 0\\ 0 & e^{-\xi/2}\end{pmatrix}\right]\times\begin{pmatrix}\cosh(N\chi/2) & -\sinh(N\chi/2)\\ -\sin(N\chi/2) & \cos(N\chi/2)\end{pmatrix}\left[\begin{pmatrix}e^{-\xi/2} & 0\\ 0 & e^{\xi/2}\end{pmatrix}\times\begin{pmatrix}\cos(\phi_2/4) & \sin(\phi_2/4)\\ -\sin(\phi_2/4) & \cos(\phi_2/4)\end{pmatrix}\right]. \quad (56)$$

If the lower-left element given in Eq. (48) vanishes, we have to go back to Eqs. (40) and (49), and write

$$M_2^N=\begin{pmatrix}\cos(\phi_2/4) & -\sin(\phi_2/4)\\ \sin(\phi_2/4) & \cos(\phi_2/4)\end{pmatrix}\begin{pmatrix}1 & -2N\sinh\lambda\\ 0 & 1\end{pmatrix}\times\begin{pmatrix}\cos(\phi_2/4) & \sin(\phi_2/4)\\ -\sin(\phi_2/4) & \cos(\phi_2/4)\end{pmatrix}. \quad (57)$$

As we noted in Sec. II, we use $M_2$ and $M_2^N$ for mathematical convenience. In the real world, we have to use $M_1$ and $M_1^N$. It is not difficult to write this expression using the conjugate transformation of Eq. (16). It can be written as

$$M_1^N=\left[\begin{pmatrix}e^{-i\phi_2/4} & 0\\ 0 & e^{i\phi_2/4}\end{pmatrix}\begin{pmatrix}\cosh(\xi/2) & \sinh(\xi/2)\\ \sinh(\xi/2) & \cosh(\xi/2)\end{pmatrix}\right]\times\begin{pmatrix}e^{-iN\phi/2} & 0\\ 0 & e^{iN\phi/2}\end{pmatrix}\left[\begin{pmatrix}\cosh(\xi/2) & -\sinh(\xi/2)\\ -\sinh(\xi/2) & \cosh(\xi/2)\end{pmatrix}\times\begin{pmatrix}e^{i\phi_2/4} & 0\\ 0 & e^{-i\phi_2/4}\end{pmatrix}\right] \quad (58)$$

if $A$ takes the form of Eq. (44) with a positive value of Eq. (48). If it takes the form of Eq. (46) with a negative value of Eq. (48),

$$M_1^N=\left[\begin{pmatrix}e^{-i\phi_2/4} & 0\\ 0 & e^{i\phi_2/4}\end{pmatrix}\begin{pmatrix}\cosh(\xi/2) & \sinh(\xi/2)\\ \sinh(\xi/2) & \cosh(\xi/2)\end{pmatrix}\right]\times\begin{pmatrix}\cosh(N\chi/2) & i\sinh(N\chi/2)\\ -i\sinh(N\chi/2) & \cosh(N\chi/2)\end{pmatrix}\times\left[\begin{pmatrix}\cosh(\xi/2) & -\sinh(\xi/2)\\ -\sinh(\xi/2) & \cosh(\xi/2)\end{pmatrix}\begin{pmatrix}e^{i\phi_2/4} & 0\\ 0 & e^{-i\phi_2/4}\end{pmatrix}\right]. \quad (59)$$

If the expression of Eq. (48) vanishes,

$$M_1^N=\begin{pmatrix}e^{-i\phi_2/4} & 0\\ 0 & e^{i\phi_2/4}\end{pmatrix}\begin{pmatrix}1-iN\sinh\lambda & iN\sinh\lambda\\ -iN\sinh\lambda & 1+iN\sinh\lambda\end{pmatrix}\times\begin{pmatrix}e^{i\phi_2/4} & 0\\ 0 & e^{-i\phi_2/4}\end{pmatrix}. \quad (60)$$

This is not yet the $S$ matrix. The first and the last layers have boundaries with air or the third medium. It is straightforward to take these boundary conditions into consideration. This procedure was discussed in detail in our previous paper [1].

## V. EXPERIMENTAL POSSIBILITIES

The variables for the $S$ matrix given in Secs. III and IV are determined by the optical parameters, namely, the two phase shifts and one reflection/transmission coefficient. The combinations of these three variables will determine the form of the $S$ matrix, which may take three different forms.

We note first that the $N$ dependence of the $S$ matrix comes from the form of the $A$ matrix or the $RXR$ matrix of Eq. (41). If the optical parameters are such that the $A$ matrix takes the form of Eq. (44), the elements of the $A^N$ matrix of Eq. (52) are bounded and oscillating functions of $N$. If $A$ takes the form of Eq. (46), the $A^N$ matrix becomes Eq. (53). The elements of this matrix are not bounded as $N$ becomes large. Thus, in the real world, $N$ layers can have two different types depending on the form of $A$.

In addition, the optical layers can satisfy the condition that the expression of Eq. (48) be zero:

$$\sinh\lambda-(\sin\alpha)\cosh\lambda=0. \quad (61)$$

Then the $RXR$ matrix takes the form of Eq. (49) and the $N$ dependence is linear. This case can be tested as the optical parameters are varied from positive values of Eq. (48) to a positive value through zero. This condition does not depend on $N$. We have discussed a similar case in our previous paper [1].

In their recent paper [7], Baskal and Kim noted the same transition process for one-lens optics. They noted that the camera focusing mechanism corresponds to contraction of Wigner's little groups. It is interesting to note that the same contraction mechanism exists in $N$-layer optics.

## VI. WIGNER'S LITTLE GROUPS

The algebra of 2×2 matrices is the basic scientific language in ray optics, including polarization optics, interferometers, lens systems, lasers, and multilayer optics. The algebra of 2×2 unimodular matrices is called the group $SL(2,c)$, and is the universal covering group for the six-parameter Lorentz group applicable to Lorentz transformations in the Minkowskian space of one time and three space dimensions. This allows us to study ray optics with the Lorentz group.

There are a number of interesting subgroups of the Lorentz group. Among them is the three-parameter rotation group. There is also a subgroup that shares the same algebraic property as the two-dimensional Euclidean group. There is also a three-parameter subgroup consisting of Lorentz transformations applicable to the Minkowskian space of one time and two space dimensions. In 1939 [5], Wigner observed that these subgroups dictate the internal space-time symmetries of massive, massless, and tachyonic particles, respectively.

These are called Wigner's little groups. In his 1939 paper, Wigner constructed the little group as the maximal subgroup of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. For instance, the four-momentum of a massive particle in its rest frame is invariant under rotations in the three-dimensional space. What happens when the particle moves? The momentum of the particle can be boosted from that of its rest frame. The little group is then a Lorentz-boosted rotation group. In mathematical language, this is a conjugate transformation. In this case, the rotation matrix is sandwiched between a boost matrix and its inverse, as in the case of Eq. (8). In this paper, we exploited this aspect of Wigner's little group.

In addition, each little group contains a one-parameter subgroup. For instance, rotations around the $y$ axis form a one-parameter Abelian group. The slide-rule-like property discussed in this paper comes from this aspect of the little groups.

As for the Sp(2) group, it is gratifying to note that the $S$ matrix formalism, originally formulated in terms of complex matrices, can be transformed into the real-matrix representation of Sp(2) by a conjugate transformation as was noted in Sec. II, and as was discussed in detail in our previous paper [1]. This is equivalent to restricting transformations in the two-dimensional space consisting of boosts along the $z$ and $x$ directions and rotations around the $y$ axis. Under this restriction, the little groups become one-parameter Abelian groups, represented by the matrices given in Eq. (1). We can recover the full little groups by simply adding rotations around the $z$ axis [12].

The little groups can be discussed in the framework of Lie groups and Lie algebras. In this framework, group contractions are strictly singular transformations, and it is not possible to make an analytic continuation from one little group to another. However, we can circumvent this inconvenience by using different parametrization. In this way, it is possible to make the desired analytic continuation. This aspect was noted by Baskal and Kim in their recent paper [3] and is seen again in the present paper.

## VII. CONCLUDING REMARKS

Based on Wigner's little groups, we have developed an algebraic method that allows us to study the cyclic properties of $2\times 2$ $S$ matrices for multilayer optics. Starting from the single-layer $S$ matrix, it is possible to write the $N$-layer matrix by multiplying one of the parameters by $N$. The $N$ dependence is therefore transparent.

This is possible because the core matrices of Wigner's little groups have a slide-rule-like property that allows us to perform multiplications by additions, as noted in Eq. (2). This property is an important element in computer designs.

As was noted in Ref. [7], the transition from Eq. (44) to Eq. (21) corresponds to camera focusing in one-lens optics. From the mathematical point of view, it corresponds to the contraction and expansion of the little groups. From the geometrical point of view, this corresponds to transformation from a circle to a hyperbola. It is interesting to note that we can also perform these operations in multilayer optics. Indeed, as in the case of lens optics [7], multilayer optics can serve as an analog computer for group contractions.

The correspondence between the Lorentz groups O(3,1) and SL($2,c$), the group of $2\times 2$ unimodular matrices, is well known. Since most of the matrices in ray optics are $2\times 2$, the Lorentz group is becoming the major language in this field. Ray optics is the backbone of future technology, and optical devices, such as polarizers, lenses, interferometers, and multilayers, all speak the language of the Lorentz group. Thus, it is possible for the Lorentz group to play computational roles in future generations of computers.

It is a prevailing view in physics, especially in optics, that group theory is only for studying symmetries and is not useful for computational purposes. Indeed, we do not need group theory to carry out the matrix multiplications given in this paper, and we started only with three matrices given in Eqs. (18) and (19). However, we are going through some important theorems in group theory while going through the simple matrix algebras given in this paper. We choose not to elaborate on this point.


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