Time-asymptotic wave propagation in collisionless plasmas

Carlo Lancellotti
Department of Mathematics, City University of New York–CSI, Staten Island, New York 10314, USA

J. J. Dorning
Engineering Physics Program, University of Virginia, Charlottesville, Virginia 22903-2442, USA
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We report the results of a new, systematic study of nonlinear longitudinal wave propagation in a collisionless plasma. Based on the decomposition of the electric field E into a transient part T and a time-asymptotic part A, we show that A is given by a finite superposition of wave modes, whose frequencies obey a Vlasov dispersion relation, and whose amplitudes satisfy a set of nonlinear algebraic equations. These time-asymptotic mode amplitudes are calculated explicitly, based on approximate solutions for the particle distribution functions obtained by linearizing only the term that contains T in the Vlasov equation for each particle species, and then integrating the resulting equation along the nonlinear characteristics associated with A, which are obtained via Hamiltonian perturbation theory. For "linearly stable" initial Vlasov equilibria, we obtain a *critical initial amplitude* (or threshold), separating the initial conditions that produce Landau damping to zero ($A\equiv 0$) from those that lead to nonzero multiple-traveling-wave time-asymptotic states via nonlinear particle trapping ($A\not\equiv 0$). These theoretical results have important implications about the stability of spatially uniform plasma equilibria, and they also explain why large-scale numerical simulations in some cases lead to zero-field final states whereas in others they yield nonzero multiple-traveling-wave final states.

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I. INTRODUCTION

One of the oldest and most fundamental problems in plasma kinetic theory is that of small-amplitude longitudinal wave propagation in a collisionless plasma described by the Vlasov-Poisson (VP) equations

$$\frac{\partial f_\alpha}{\partial t} + v\,\frac{\partial f_\alpha}{\partial x} + \frac{q_\alpha}{m_\alpha}E\,\frac{\partial f_\alpha}{\partial v} = 0, \tag{1a}$$

$$\frac{\partial E}{\partial x} = 4\pi \sum_\alpha q_\alpha \int dv\, f_\alpha, \tag{1b}$$

where $f_\alpha(x,v,t)$ is the distribution function for particle species α, $\alpha = 1,...,N_S$, and E is the self-consistent longitudinal electric field. This well-known model has played a major role in the analysis of plasma instabilities and wave propagation in a wide variety of settings, ranging from astrophysical, solar, and magnetospheric plasmas to laboratory and fusion plasmas. However, due to the extreme analytical difficulties associated with the nonlinear Vlasov equation, much of the classic theory of plasma waves [1–4] has been based on the analysis of the linearized VP system. The most famous result of the linear theory is Landau damping: according to Landau's solution of the initial value problem, every small perturbation of a Maxwellian electron plasma with a fixed ion background decays exponentially because of collisionless absorption of electric field energy by the resonant particles, i.e., the particles traveling at velocities close to the phase velocity of a wave mode. In the early 1960s, Landau's result was generalized by others [5–7] who proved that the solutions to the linearized VP system exhibit Landau damping whenever the equilibrium distribution is single humped, i.e., when it has only one maximum, as indeed is the case for the Maxwellian. Accordingly, the equilibria in this class are often called "linearly stable."

An unfortunate fact is that the linearization of the VP system is, in general, not uniformly valid in time [7] even for small initial perturbations. Thus, the validity of most of the linear results is limited to a relatively short time scale. This is the case, in particular, for Landau damping; as early as 1965 O'Neil [8] argued that nonlinear effects can prevent the complete damping of a sinusoidal wave and lead to a fixed-amplitude time-asymptotic traveling-wave mode, sustained by the oscillations of the particles trapped in the wave's potential well very much like the well-known Bernstein-Greene-Kruskal (BGK) [9] nonlinear traveling-wave solutions. The time scale on which these nonlinear effects become important can be estimated as the time scale on which a particle crosses the potential well and reaches a turning point. Close to the bottom of the well, the field is well approximated by the harmonic field $E(x,t) = -(ek\epsilon/m)x$ (for a plasma of electrons with charge e and mass m), so that the "trapping" (or "bounce") time is $\tau_b = \sqrt{m/ek\epsilon}$. On the other hand, Landau damping takes place on a different time scale, which can be estimated as $\tau_L = 1/\gamma_L$ where γ_L is a typical Landau damping coefficient. O'Neil [8] observed that there are two limiting cases.

(1) If $\tau_L \ll \tau_b$, the field is damped before nonlinear effects become relevant. The wave dies away before it can significantly distort the single-particle trajectories (and the background equilibrium), and the linear theory is basically accurate.

(2) If $\tau_L \gg \tau_b$, the reverse is true: the wave is not damped before nonlinear effects become important; rather, these effects appear quickly and modify the distribution function in the resonant region, invalidating the linear theory. This sec-

ond case was the object of O'Neil's study; since Landau damping has very little time to affect the wave amplitude, he argued that, to first order in τ_b/τ_L, the constant-amplitude field $E(x,t)=\sin(kx-\omega t)$ can be used to compute the particle trajectories. This can be done in terms of Jacobi elliptic functions and leads to a wave amplitude that is initially damped according to Landau, but later grows back as the trapped particles start transferring energy back to the wave. Then, damping and growing alternate until the field settles to some finite amplitude smaller (in the case of initial damping) than its initial value but different from zero. The decrease of the amplitude oscillations occurs because, as the trapped particles oscillate on orbits of different frequencies, they lose phase coherence until there is no net density flux in phase space across the wave phase velocity.

The limitation of O'Neil's analysis, of course, is that the particle trajectories are calculated by assigning *a priori* the electric field to be a single sinusoidal wave of fixed amplitude. However, in many practical cases amplitude variations and the presence of multiple wave modes do affect the particle trajectories significantly. To account for changing amplitudes, others [10–13] have carried out semi-self-consistent computations, which apply O'Neil's general method to various more sophisticated *Ansätze* for the field. This latter is assumed, again, to be a monochromatic wave, but the amplitude is allowed a slow variation; the crucial point is then the solution of the resulting Newton equations via various asymptotic methods (averaging, adiabatic invariants, etc.). These studies essentially confirm O'Neil's basic result; however, their value is also limited by the restrictive *Ansatz* on the field, which prevents a fully self-consistent treatment of the VP initial value problem. In fact, no progress at all has been made toward a satisfactory self-consistent analysis of the nonlinear VP initial value problem in the general case, which includes τ_b and τ_L being of the same order. In particular, nothing is known about the transition between the initial conditions that lead to a zero electric field and those that lead to nonzero time-asymptotic fields via particle trapping. Results for this transition are included in this paper.

Recently, rigorous nonlinear analyses [14–16] based on BGK representations [9] have shown that collisionless plasmas can sustain small-amplitude waves near single-humped equilibria, in spite of the predictions of the linear theory. The question is whether, and how, periodic traveling-wave solutions of this kind [14,15] can be generated from various initial conditions. Recently, this question has been the subject of much work, both analytical [17–21] and numerical [22–24], and also of some controversy [25]. In order to clarify some of the outstanding issues, in this paper we report the results of a detailed analysis, some of which were summarized earlier in a brief communication [18].

Our analysis is based on four key steps. (a) The decomposition of the electric field E into a transient part T and a time-asymptotic part A such that $E(x,t)=A(x,t)+T(x,t)$. This representation makes it possible to decompose the full nonlinear VP problem itself into a transient part and a time-asymptotic part (Sec. II A). (b) The linearization of the Vlasov equation with respect only to $T(x,t)$ but not $A(x,t)$, i.e.,

$$\frac{\partial f_\alpha}{\partial t}+v\frac{\partial f_\alpha}{\partial x}+\frac{q_\alpha}{m_\alpha}A\frac{\partial f_\alpha}{\partial v}=-\frac{q_\alpha}{m_\alpha}T\frac{\partial \mathcal{F}_\alpha}{\partial v}, \tag{2}$$

where $\mathcal{F}_\alpha(x,v)$ is the initial distribution function (Sec. II B). (c) The formal solution of Eq. (2) in terms of the nonlinear characteristics corresponding to A (Sec. II C). (d) The utilization of the solution of Eq. (2) to obtain algebraic equations for the mode amplitudes of $A(x,t)$ (Secs. II D and II E). Step (d) is somewhat laborious: it involves focusing on the time-asymptotic part of the problem and studying it as a bifurcation problem for small-amplitude multiple-wave solutions near the basic solution $A\equiv 0$ which corresponds to complete Landau damping. We find that, if any nonzero solution for A "bifurcates" from $A\equiv 0$ (for a "critical" initial condition), then it is given, at leading order, by a finite superposition of wave modes whose phase velocities satisfy a Vlasov dispersion relation and whose amplitudes can be obtained from a finite dimensional system of nonlinear algebraic equations that depend on T and $\mathcal{F}_\alpha$. It is crucial that the nonlinear characteristics for a field $A(x,t)$ of this kind can be determined explicitly via Hamiltonian perturbation theory [15]. The explicit calculation of these characteristics, and thus the development of the equations for the mode amplitudes of $A(x,t)$ is carried out in Sec. III, in the case of a two-wave final state. Because the coefficients in these equations still depend on T and $\mathcal{F}_\alpha$, in Sec. IV we have to complete the analysis by turning to the transient part of the problem and introducing a standard perturbation technique to determine T near a bifurcation point of the time-asymptotic problem. This approach to the determination of T exploits the fact that its decay properties neutralize most of the secularities that in the past have plagued attempts at perturbative solutions for the complete field E.

In this way, we obtain two main results: (a) the threshold (or "critical initial amplitude") below which initial field amplitudes are damped to zero and above which they evolve to nonzero time-asymptotic solutions; and (b) the dependence of the time-asymptotic field amplitude upon the initial field amplitude when the latter is above, but close to, the threshold value. In particular, we analyze a case of very general interest, a sinusoidal initial perturbation [8,10,11,13,26,27] and obtain a complete picture of the time-asymptotic evolution of this type initial condition for various initial distribution functions. Interestingly, our analytical calculation of the critical initial amplitude (as briefly summarized in [18]) has been confirmed by recent numerical simulations [24].

II. TIME-ASYMPTOTIC ANALYSIS

In order to obtain the long-time solution to the nonlinear VP equations, in this section we develop an approximation scheme that yields the long-time electric field formally, in the sense that the time-asymptotic solution depends on the details of the transient evolution of the field, which is determined later (in Sec. IV).

A. *A-T* decomposition

As already discussed, a linearly stable spatially periodic initial disturbance in a collisionless plasma will either undergo Landau damping to zero, before the trapping effects become important, or evolve to a nonzero time-asymptotic state. In the second scenario, particle trapping will create a flat spot on the distribution function at the phase velocity of each wave mode that is not Landau damped. This implies that, in the time-asymptotic limit, these modes will become unable to exchange energy with the resonant particles, and will essentially travel with constant amplitude; thus, the electric field will tend asymptotically to a nonlinear superposition of traveling waves very much like the multiple "BGK-like" modes recently described by Buchanan and Dorning [15]. This suggests that we look for solutions in which the electric field $E(x,t)$ is an *asymptotically almost periodic* (AAP) function of t [28]. For such solutions, we can write $E(x,t)$ as the sum of a *transient part* and a *time-asymptotic part*

$$E(x,t)=T(x,t)+A(x,t), \tag{3}$$

where both $T(x,t)$ and $A(x,t)$ are spatially periodic and $\lim_{t\to\infty}T(x,t)=0$; and $A(x,t)$ is an almost periodic [29] function of t, i.e., a general superposition of modes of the form $A(x,t)=\Sigma_{k,\omega_i}a_{k,\omega_i}e^{ikx-i\omega_i t}$, where the frequencies ω_i can take a countable set of real values, unlike the wave numbers k, which are restricted to integer values by the requirement of exact (spatial) periodicity. The amplitudes a_{k,ω_i} are the *Fourier-Bohr coefficients* of $A(x,t)$, given by

$$a_{k,\omega_i}=\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^{\sigma}dt\,\frac{1}{2\pi}\int_{-\pi}^{+\pi}dx\,e^{-ikx+i\omega_i t}E(x,t), \tag{4}$$

which combines a standard Fourier transform in space with a *Bohr transform* in time, in which the averaging operator $\lim_{\sigma\to\infty}(1/\sigma)\int_0^{\sigma}dt$ replaces the usual Fourier integral $\int_{-\infty}^{+\infty}dt$. This integral transform filters out the transient phenomena and retains only the time-asymptotic behavior; thus, the Fourier-Bohr series of an AAP function is a "projector" P_a that separates its transient and time-asymptotic parts. When E is of the form in Eq. (3), clearly $P_aE=A$ and $(I-P_a)E=T$. Thus, applying P_a and $(I-P_a)$ to Eq. (1b) yields a set of coupled equations for A and T:

$$\frac{\partial A}{\partial x}=4\pi P_a\sum_{\alpha} q_{\alpha}\int dv\, f_{\alpha}(A+T), \tag{5a}$$

$$\frac{\partial T}{\partial x}=4\pi(I-P_a)\sum_{\alpha} q_{\alpha}\int dv\, f_{\alpha}(A+T). \tag{5b}$$

From the definition of $T(x,t)$, a solution to Eqs. (5) must satisfy the additional condition $\lim_{t\to\infty}T(x,t)=0$. Equation 5(a) will be called the *time-asymptotic equation*, while Eq. 5(b) will be the *transient equation*. The notation $f_{\alpha}(A+T)$ here emphasizes that the f_{α} depend nonlinearly on A and T via the Vlasov equation.

B. Transient linearization

The *A*-*T* decomposition makes it possible to obtain an approximate solution for $f_{\alpha}(A+T)$ via *transient linearization*, i.e., the linearization of the Vlasov equation only with respect to T, keeping the nonlinearity in the term that contains A. As long as T decays fast enough (before the distribution function deviates substantially from the initial condition), we can expect the transiently linearized approximation to the Vlasov equation to be uniformly valid in time, unlike the standard linearized problem. Moreover, if the characteristics associated with A can be computed, the transiently linearized Vlasov equation can be solved analytically. Writing Eq. (1a) in terms of $A+T$ and introducing the "linearization" of $T\partial f_{\alpha}/\partial v$ about the initial distribution yields

$$\frac{\partial f_{\alpha}}{\partial t}+v\frac{\partial f_{\alpha}}{\partial x}+\frac{q_{\alpha}}{m_{\alpha}}A\frac{\partial f_{\alpha}}{\partial v}=-\frac{q_{\alpha}}{m_{\alpha}}T\frac{\partial \mathcal{F}_{\alpha}}{\partial v}, \tag{6}$$

where $\mathcal{F}_{\alpha}(x,v)\equiv f_{\alpha}(x,v,0)=F_{\alpha}(v)+h_{\alpha}(x,v)$. Here, the initial condition $\mathcal{F}_{\alpha}$ has been written as the sum of its spatially uniform part F_{α} (which will be taken to be a Vlasov equilibrium) and a perturbation h_{α} which will play the role of a running parameter in our analysis.

In the standard linear Vlasov equation the complete field E interacts at all times with the fixed background distribution F_{α}. That approximation is not uniformly valid in the asymptotic time limit, since the nonlinear distribution f_{α} becomes qualitatively different from F_{α}. Conversely, the linearization in Eq. (6) involves only the transient; hence, it does not require $\mathcal{F}_{\alpha}$ to be a uniformly good approximation to f_{α} as $t\to\infty$, as long as $T\to 0$ "fast enough" in that limit. In fact, the full nonlinear interaction between the asymptotic field A and the distribution function is maintained through the term $A\partial f_{\alpha}/\partial v$. Clearly, Eq. (6) can be solved exactly whenever the characteristics $[x_{\tau}^{A}(x,v,t),v_{\tau}^{A}(x,v,t)]$ can be determined and leads to the number density in Eq. (7). Interestingly, Eq. (6) includes as special cases both O'Neil's strong-trapping scenario [8] and linear Landau damping [2]. In the first case, under O'Neil's assumption that the transient part of the field has negligible effects on particle trajectories, $T=0$ and Eq. (6) reduces to the nonlinear Vlasov equation with $E=A$, which O'Neil solved analytically for a single-mode sinusoidal wave. This case also includes all the exact BGK [9,14,16] and BGK-like [15] solutions. At the opposite extreme, whenever the electric field is damped to zero before nonlinear effects become relevant, $A=0$ and Eq. (6) becomes a linearized Vlasov equation with $E=T$, leading to Landau's exponentially damped solution for the electric field [which he obtained under the even stronger assumption that $f_{\alpha}(x,v,0)$ could be approximated by $F_{\alpha}(v)$ [2]].

Whereas the traditional linearization relies on the assumed small amplitude of the electric field, the transient linearization introduced here is based on an assumption about the decay rate of T, not about its amplitude. Specifically, it requires that $\tau_{\mathrm{trans}}\ll\tau_{\mathrm{dyn}}$, where τ_{trans} is the time scale over which T becomes negligibly small, and τ_{dyn} is the time it takes the nonlinear dynamics to make the distribution function f_{α} significantly different from the initial distribution $\mathcal{F}_{\alpha}$. These time scales are defined more precisely in Appendix A, where a detailed discussion of the error involved in the transient linearization is presented. It is interesting to compare the condition $\tau_{\mathrm{trans}}\ll\tau_{\mathrm{dyn}}$, which involves the decay rate of T only, with O'Neil's condition $\tau_L\ll\tau_b$ for the validity of the standard linear theory. Whereas O'Neil considered the damping rate of the whole field E, here the asymptotic part A has been subtracted. Hence, this condition can be satisfied even when the complete field E does not damp at all, e.g., for a BGK traveling mode of the Holloway-Dorning type [14], where $T=0$ and therefore $\tau_{\rm trans}=0$. In this case $\tau_{\rm trans}\ll\tau_{\rm dyn}$ is trivially satisfied, whereas $\tau_L=\infty\gg\tau_b$, i.e., it is a trapping-dominated situation that is completely outside the domain of validity of the linear theory.

C. Critical initial conditions

Equation (6) can be solved formally in terms of the A characteristics $[x_\tau^A(x,v,t),v_\tau^A(x,v,t)]$, which associate with every phase point (x,v) at time t a "starting point" $[x_\tau^A,v_\tau^A]$ at $\tau\leqslant t$ along the trajectory determined by A. Integrating in v then gives

$$\int_{\mathbb{R}}dv\, f_\alpha(x,v,t)=\int_{\mathbb{R}}dv\, \mathcal{F}_\alpha(x_0^A,v_0^A)-\frac{q_\alpha}{m_\alpha}\int_{\mathbb{R}}dv\int_0^t d\tau\left\{T\frac{\partial\mathcal{F}_\alpha}{\partial v}\right\}_{[x_\tau^A,v_\tau^A]} \tag{7}$$

for the number densities, which will be the foundation for our analysis of the nonlinear VP initial value problem. Substituting Eq. (7) into Eq. (5a) and Fourier transforming gives (for $k\neq0$)

$$\begin{aligned}A_k(t)=&\frac{2}{ik}\sum_\alpha q_\alpha P_a\int_{-\pi}^{+\pi}dx\, e^{-ikx}\\&\times\int_{\mathbb{R}}dv\,\mathcal{F}_\alpha(x_0^A(x,v,t),v_0^A(x,v,t))\\&-\frac{2}{ik}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}P_a\int_{-\pi}^{+\pi}dx\, e^{-ikx}\\&\times\int_{\mathbb{R}}dv\int_0^t d\tau\left\{T\frac{\partial\mathcal{F}_\alpha}{\partial v}\right\}_{[x_\tau^A(x,v,t),v_\tau^A(x,v,t)]}\end{aligned} \tag{8}$$

(where here and below the $\pm\pi$ in the limits of integration have units of length, i.e., the wavelength is taken as unity). This equation contains the characteristics for the field A, which, of course, is still unknown. Our strategy is to show that, *at the transition between the initial conditions that lead to complete Landau damping* ($A\equiv0$) *and those that lead to nonzero small-amplitude solutions for A, the general solution for A can be determined a priori*. Such a general solution will depend on a finite set of unknown amplitudes that will satisfy certain nonlinear algebraic equations derived from Eq. (8).

A preliminary step toward determining the general form of small-amplitude solutions for A is to observe that Eq. (8) in isolation has the exact solution $A(x,t)\equiv0$ independent of the transient field T and of the initial distribution function $\mathcal{F}_\alpha(x,v)$. The proof of this result is somewhat tedious and is given in Appendix B. Schematically, if we draw a graph in the h_α-A plane showing the time-asymptotic field vs the initial disturbance, the solutions with $A(x,t)\equiv0$ constitute a trivial branch that coincides with the horizontal axis. These solutions will be called *vanishing asymptotic states* and do not necessarily correspond to solutions of the complete system of equations for A and T, Eqs. (5a) and (5b). In fact, it is not necessarily true that when $A\equiv0$ is substituted into Eq. (5b) the resulting equation $\partial T/\partial x=4\pi\Sigma_\alpha q_\alpha\int dv\, f_\alpha(T)$ will possess a solution T that tends to zero as $t\to\infty$. If this does happen, we will call the corresponding "point" on the fundamental branch an *accessible vanishing asymptotic state* (AVAS). A trivial example is given by the origin $A\equiv0$, $h_\alpha\equiv0$ (from which $T\equiv0$ follows). These states correspond to initial conditions such that the field is completely Landau damped before trapping effects come into play. As discussed in the Introduction, there are other initial conditions (of larger amplitude) for which we expect the trapping effects to lead to $A\neq0$; in these cases, the solution $A=0$ to the time-asymptotic equation is not accessible and does not correspond to the solution to the complete VP initial value problem. The nonzero solutions can be imagined as a nontrivial branch in the h_α-A plane, which bifurcates from the trivial solutions $A\equiv0$ at some *critical initial condition* $h_\alpha\equiv h_\alpha^0$ as the initial condition h_α is changed. Physically, h_α^0 marks the transition between the two classes of initial conditions. The determination of h_α^0 will follow from the nonlinear analysis below; we now proceed to show how the general form of the solution for A can be determined for initial conditions near a generic (given) critical initial state $A\equiv0$, $T\equiv T_0$, $h_\alpha\equiv h_\alpha^0$.

D. Time-asymptotic linearization

In order to study solutions for A near $A\equiv0$, $T\equiv T_0$, $h_\alpha\equiv h_\alpha^0$, we first consider the time-asymptotic equation linearized about this critical initial state. This linearization of Eq. (8) requires some mathematical care (see Appendix C) and yields

$$a_{k,\omega_i}\mathcal{D}(k,\omega_i)=0, \tag{9}$$

where $\mathcal{D}(k,\omega_i)$ is the *Vlasov dielectric function*

$$\mathcal{D}(k,\omega_i)\equiv1-\frac{4\pi}{k}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}P\int_{\mathbb{R}}dv\,\frac{F_\alpha^{T_0\prime}(v)}{\omega_i-kv}. \tag{10}$$

Here $\mathcal{D}(k,\omega_i)$ is not determined by the Vlasov equilibrium $F_\alpha(v)$ that appears in the initial distribution function; rather, it contains a time-asymptotic equilibrium $F_\alpha^{T_0}$ which includes the effects of the transient field T_0 that evolves from the critical initial state [see Eq. (B4) in Appendix B for a precise definition of $F_\alpha^{T_0}$]. Of course, T_0 has to be obtained from the transient equation, Eq. (5b), with $A\equiv0$; however, there is an important case in which $\mathcal{D}(k,\omega_i)$ turns out not to depend on T_0, namely, when the problem is *reflection symmetric*, i.e., even in x and v. Reflection-symmetric initial conditions occur in many interesting problems [15,27] and lead to reflection-symmetric solutions at all times. In these cases, it easy to show [33] that $F_\alpha^{T_0}(v)=F_\alpha(v)$. Here, we will only

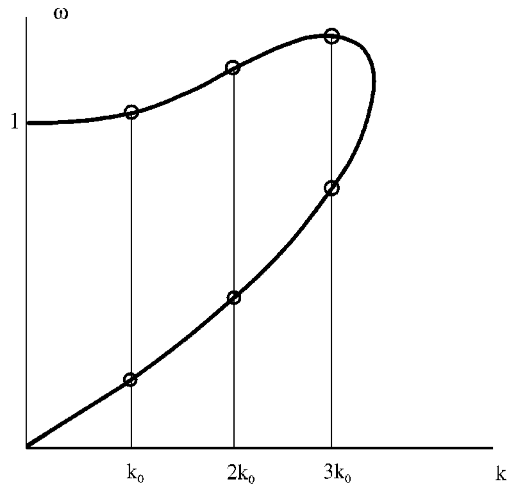

FIG. 1. The Vlasov dispersion curve for a Maxwellian *e-p* plasma with $T_e = T_p$, and the roots $\omega(k)$ of the corresponding dispersion relation for a basic wave number k_0 and its harmonics; k is in units of the inverse Debye length $k_D \equiv 1/\lambda_D$ and ω is in units of the plasma frequency ω_p.

consider initial conditions of this kind. This also has the advantage that we need to consider only spatial Fourier modes with $k \neq 0$ since, if the spatially uniform ($k=0$) component in the electric field is initially zero, it will remain zero for all times.

Equation (9) can be solved immediately: it requires that $a_{k,\omega_i}=0$ except for all k and ω_i that satisfy $\mathcal{D}(k,\omega_i)=0$ which is the *Vlasov dispersion relation* determined by $F_\alpha^{T_0}$. The properties of the Vlasov dispersion relation are well known, especially for physically relevant Vlasov equilibria. For instance, for a Maxwellian the Vlasov dispersion curve can be plotted easily. It shows (Fig. 1) some qualitative features that hold in great generality, even though for a modified Vlasov equilibrium, like $F_\alpha^{T_0}$, it may be necessary to compute the exact roots of $\mathcal{D}$ numerically [14]. Specifically, the dispersion curve shows a *cutoff wave number* k_d such that for $k > k_d$ Eq. (10) has no solution. Moreover, given any wave number $k \leqslant k_d$, Eq. (10) defines implicitly a finite number N_k of simple real roots $\omega_1(k), \omega_2(k), \dots, \omega_{N_k}(k)$, which means that the general solution of the linearized time-asymptotic problem will be given by a finite superposition of wave modes of the form

$$A(x,t) = \sum_{k \leqslant k_d} \sum_{j=1}^{N_k} a_{k,\omega_j} e^{i[kx-\omega_j(k)t]}. \tag{11}$$

More precisely, since the basic wave number here is $k=1$, there will be $\varrho = [k_d]$ (the integer part of k_d) possible wave numbers before cutoff, leading to a total of $N = \Sigma_{k=1}^{\varrho} N_k$ possible modes.

A crucial fact about Eq. (9) is that the Fourier-Bohr transformation has introduced the limit $t \to \infty$ before the linearization was carried out, so that the resulting linear equation is uniformly valid in time, unlike the standard linearized VP system. This is apparent in the fact that the linearization in Eq. (9) is not about the initial equilibrium $F_\alpha(v)$ but about the "final" equilibrium $F_\alpha^{T_0}(v)$, and it is important for the study of the nonlinear problem. Indeed, many nonlinear studies of the VP system [30–32] have been greatly hampered by troublesome singularities that do not appear to be intrinsically related to the physical nature of the problem but only to the nonuniformity in time of the standard linearization.

E. Dimensional reduction

These previous results make it possible to reduce the nonlinear time-asymptotic equation, Eq. (8), to a finite-dimensional system of nonlinear algebraic equations for the N amplitudes a_{k,ω_i} in Eq. (11). This is done by simply substituting Eq. (11) in the right side of Eq. (8),[1] and taking the Bohr transform with respect to $\omega_1(k), \omega_2(k), \dots, \omega_{N_k}(k)$ for $k=1,\dots,\varrho$. This is equivalent, of course, to projecting the nonlinear equation onto the null space of the linearized operator given by Eq. (9), and yields N equations:

$$\begin{aligned} a_{k,\omega_i} = {} & \frac{2}{ik} \sum_\alpha q_\alpha \lim_{\sigma\to\infty} \frac{1}{\sigma} \int_0^\sigma dt\, e^{i\omega_i t} \int_{-\pi}^{+\pi} dx\, e^{-ikx} \\ & \times \int_{\mathbb{R}} dv\, \mathcal{F}_\alpha(x_0^A(x,v,t), v_0^A(x,v,t)) \\ & - \frac{2}{ik} \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \lim_{\sigma\to\infty} \frac{1}{\sigma} \int_0^\sigma dt\, e^{i\omega_i t} \int_{-\pi}^{+\pi} dx\, e^{-ikx} \\ & \times \int_{\mathbb{R}} dv \int_0^t d\tau \left\{ T \frac{\partial \mathcal{F}_\alpha}{\partial v} \right\}_{[x_\tau^A(x,v,t), v_\tau^A(x,v,t)]}, \end{aligned} \tag{12}$$

where the right side is determined by the N amplitudes a_{k,ω_i} through the characteristics associated with A. In the next section we show how, near a critical initial state (where $a_{k,\omega_i} \equiv 0 \;\forall\; k,\omega_i$), these characteristics can be determined via Hamiltonian perturbation theory.

In summary, *the time-asymptotic equation has been reduced to a finite-dimensional problem for the (small) amplitudes of a set of traveling-wave modes that satisfy a Vlasov dispersion relation*. Both the dispersion relation and the am-

[1]By substituting the linear solution for A, Eq. (11), into the nonlinear time-asymptotic equation, Eq. (8), we neglect possible higher-order terms in A corresponding to wave modes $a_{k,\omega_i} e^{ikx-\omega_i t}$ such that $\mathcal{D}(k,\omega_i) \neq 0$. The validity (at leading order) of this approximation will become apparent in the course of our nonlinear analysis, in which we shall obtain the scalar equation for the amplitude a of a two-mode time-asymptotic field with equal mode amplitudes, Eq. (24). That calculation could be extended to include higher-order wave modes such that $\mathcal{D}(k,\omega_i) \neq 0$. It is then easy to see that these modes must be $O(a^{3/2})$ and will generate terms of the same order in the charge density. In principle, these terms are comparable to other terms that we are going to keep in Eq. (12); however, these latter terms are orthogonal to the wave modes corresponding to $\mathcal{D}(k,\omega_i)=0$, and will disappear under the action of the Fourier-Bohr transform on the right side of Eq. (12). Thus, the leading-order contribution to Eq. (12) from the modes such that $\mathcal{D}(k,\omega_i) \neq 0$ will turn out to be of order a^2, and negligible. This specific result has been proved previously in a more general context [20].

plitudes depend on the initial condition and on the transient field. In particular, nonzero small-amplitude solutions for the wave modes may appear when the amplitude of the initial disturbance is increased through certain critical values. In other words, these nonzero undamped solutions may bifurcate (as the initial condition is changed) from the trivial solution branch $A \equiv 0$, which corresponds to a completely Landau-damped electric field as $t \to \infty$. In fact, analogous results have been established for the original (not transiently linearized) VP system [20]. In that case, however, it is too difficult to carry out an explicit calculation of the time-asymptotic wave amplitudes, because of the complicated interaction between the transient electric field and the distribution function. Conversely, for the transiently linearized equations developed here the complete calculation can be carried out explicitly, as is done below.

III. THE TWO-WAVE TIME-ASYMPTOTIC PROBLEM

Let us now consider a sequence of physical problems in which a transition occurs from the strongly Landau damped regime ($A \equiv 0$) to the O'Neil regime where the nonlinear effects sustain small-amplitude wave propagation. We expect that, as the amplitude of the initial disturbance is increased (or dF_α/dv is decreased at the ''right'' phase velocity), at some point a ''first'' undamped nonzero time-asymptotic state will branch off the zero-field solution. It is logical to assume that this phenomenon will not take place for all the wave numbers and frequencies at the same transition point. Rather, according to the basic insights from the standard linear theory, the modes with the lowest wave number and highest phase velocity damp most slowly. Hence, for a single-humped (symmetric) equilibrium, the first nonzero state should be a pair of Langmuir modes on the upper branch of the Vlasov dispersion relation (Fig. 1), with a ''fundamental'' wave number $k = k_0$ determined by the initial condition; thus, A has the two-wave form

$$\begin{aligned} A(x,t) &= 2a \sin k_0 x \cos \omega t \\ &= a \sin(k_0 x - \omega t) + a \sin(k_0 x + \omega t), \end{aligned} \tag{13}$$

where k_0 and ω satisfy $\mathcal{D}(k_0, \omega) = 0$.[2] For this two-wave case Eqs. (12) reduce to one equation for a, and [from Eq. (13)] the asymptotic field belongs to the one-dimensional space spanned by $\sin k_0 x \cos \omega t$. Hence, the projection procedure reduces to a cosine Fourier-Bohr transform (by symmetry), and Eq. (12) becomes

$$\begin{aligned} 2a = {} & \frac{8}{k_0} \sum_\alpha q_\alpha \lim_{\sigma \to \infty} \frac{1}{\sigma} \int_0^\sigma dt \cos \omega t \int_{-\pi}^{+\pi} dx \cos k_0 x \\ & \times \int_{\mathbb{R}} dv\, \mathcal{F}_\alpha(x_0^A(x,v,t), v_0^A(x,v,t)) \\ & - \frac{8}{k_0} \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \lim_{\sigma \to \infty} \frac{1}{\sigma} \int_0^\sigma dt \cos \omega t \int_{-\pi}^{+\pi} dx \cos k_0 x \\ & \times \int_{\mathbb{R}} dv \int_0^t d\tau \left[T \frac{\partial \mathcal{F}_\alpha}{\partial v} \right]_{[x_\tau^A(x,v,t), v_\tau^A(x,v,t)]}, \end{aligned} \tag{14}$$

where A is given by Eq. (13).

The characteristics $[x_\tau^A, v_\tau^A]$ are determined by the dynamics of particles moving in small-amplitude two-wave fields, which have been studied by Buchanan and Dorning [15]. They constructed small-amplitude multiple-wave solutions to the VP system by extending the invariants originally used to generate BGK solutions [9]. The latter are based on an exact invariant of the motion for the single-wave potential, namely, the single-particle energy. Buchanan and Dorning found approximate invariants for multiple-wave systems and used them to obtain generalized BGK solutions. The simplest case is precisely the two-wave field [Eq. (13)]. In the upper half phase plane, where one of the two waves is located, the effect of the other wave (in the lower half plane) can be viewed as a small perturbation of the unperturbed particle motion driven by the first wave. Application of Lie transforms shows that the energy invariant for mode + in isolation (''+'' and ''−'' correspond to the upper and lower half phase planes), $\mathcal{E}_\alpha^{(+)} = \frac{1}{2} m_\alpha (v - v_p)^2 + (q_\alpha a/k_0) \cos(k_0 x - \omega t)$, is modified by the presence of mode − to become [15]

$$\overline{\mathcal{E}}_\alpha^{(+)} = \mathcal{E}_\alpha^{(+)} + \frac{q_\alpha a}{k_0} \frac{v - v_p}{v + v_p} \cos(k_0 x + \omega t) + O\left(\frac{a^2}{(v + v_p)^2} \right), \tag{15}$$

where $v_p \equiv \omega/k_0$. The denominator $v + v_p$ makes this invariant invalid in the phase region of the second wave; in this region, however, the same procedure yields the analogous invariant $\overline{\mathcal{E}}_\alpha^{(-)}$, which is obtained by switching v_p to $-v_p$ and ω to $-\omega$ in Eq. (15). By combining $\overline{\mathcal{E}}_\alpha^{(+)}$ and $\overline{\mathcal{E}}_\alpha^{(-)}$, it is possible to construct a global first order invariant, whose level curves are shown in Fig. 2(a), which gives all the information we need about the particles' motion. Of course, this invariant is not exact, as should be expected since the Hamiltonian system corresponding to Eq. (13) is not integrable. In fact, there are small regions in the phase plane where no invariant curves exist, because the nonlinear resonances between the particles and the waves generate chaotic trajectories. These *stochastic layers* are thin regions that separate bounded and unbounded trajectories [see Fig. 2(b)]. By invoking the classic Kolmogorov-Arnold-Moser (KAM) theorem, though, Buchanan and Dorning [15] showed that these layers are exponentially small in a and can be neglected in a study of the self-consistent VP system.

Below we adapt Buchanan and Dorning's technique and explicitly calculate $[x_\tau^A, v_\tau^A]$. The restriction to a two-wave field is not essential, but the generalization to the N-wave case, which is straightforward, becomes increasingly tedious

[2] Without loss of generality we take $a \geqslant 0$, since its sign can be changed arbitrarily by introducing a constant phase shift π in Eq. (13). This phase will be left undetermined until later, when the analysis of the transient problem will determine the phase that ''connects'' $A(x,t)$ to the initial condition.

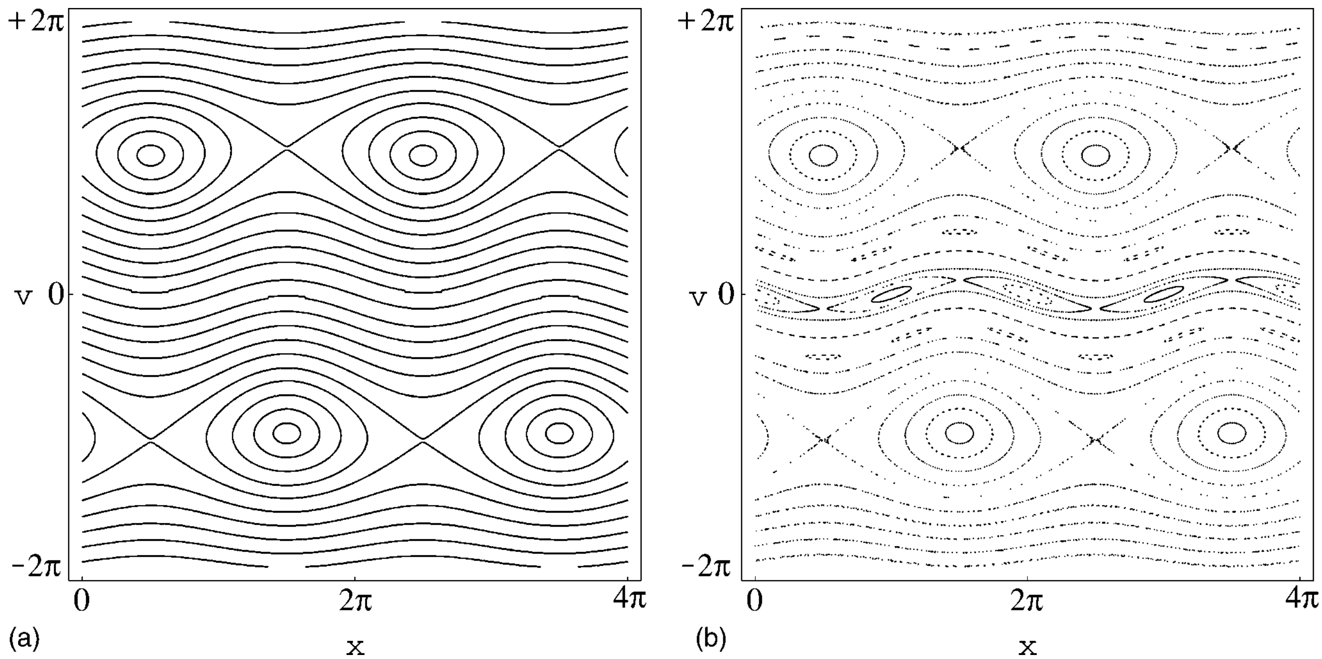

FIG. 2. (a) Buchanan and Dorning's [15] approximate invariant curves for a two-wave system. (b) Level curves, stochastic layers, and higher-order resonance islands for the same system.

as N grows. Hence, it will be omitted in order to make the analysis as clear as possible. For the same purpose, we shall focus our nonlinear analysis of Eq. (14) on one of the simplest, yet most fundamental and widely studied [8,10,11,13,26,27], plasma-wave problems, the evolution of a sinusoidal initial perturbation $E(x,0)=\hat{a}_0 \sin k_0 x$.[3] The corresponding initial distributions are $\mathcal{F}_\alpha(x,v)=F_\alpha(v)+\epsilon h_\alpha(v)\cos k_0 x$, where F_α are (normalized) initial Vlasov equilibria, h_α are given (normalized) even functions of v with strong-decay properties as $|v|\to\infty$, and ϵ is related to $\hat{a}_0$ via the Poisson equation. These initial distributions clearly are reflection symmetric and the initial field has no spatially uniform part. Thus, as anticipated above, the $k=0$ spatial Fourier component of E will be zero at all times, and we can restrict our study to $k\neq 0$. For fixed F_α and h_α, the complete initial condition can be parametrized by ϵ. Hence, Eq. (14) becomes an algebraic equation for the scalar unknown a in terms of the scalar parameter ϵ [and the transient field T, which will be determined later from Eq. (5b) for a given a]. Notwithstanding these simplifications, this initial value problem includes all the essential features of the more general case (i.e., a generic spatially periodic initial perturbation). The analysis can be extended to more general problems, but only at the price of a considerable increase in the algebra.

[3]This implies that we are taking the leading-order time-asymptotic field [Eq. (13)] to have the same wave number as the initial perturbation, which is fully justified based on the analysis of the transient equation in the next section. All the higher harmonics will be indexed by $k=lk_0$ with $l=1,2,...$ (since k_0 is the fundamental wave number).

A. O'Neil terms

Our analysis of Eq. (14) begins with the first term on the right side, which corresponds to the asymptotic evolution of the initial distribution as though it were forced by a field of fixed amplitude, with no transient part. This term and those that follow from it will be called "O'Neil" terms because their structure is loosely analogous to that of quantities that arose in O'Neil's calculation [8], although that work was restricted to a single sinusoidal wave of constant amplitude. Conversely, the second term on the right side of Eq. (14) corresponds to the (linearized) effects of the transient field $T(x,t)$ on the distribution function; for $a=0$ this term is essentially the quantity that arises in Landau's solution [2]. Thus, it and its descendants will be called "Landau" terms.

We evaluate the O'Neil terms in three steps. (1) We calculate explicitly the characteristics $[x_\tau^A, v_\tau^A]$ in terms of Jacobi elliptic functions, via the Buchanan-Dorning technique [15]. (2) Then, we take the limit as $\sigma\to\infty$ by extending an idea of O'Neil's, who noted [8] that the distribution function (corresponding, in his case, to the evolution along the trajectories in a single sinusoidal wave) can be replaced in the time-asymptotic limit by a coarse-grained version, which is obtained by averaging on each energy level of the wave. In our case, we show that the limit as $\sigma\to\infty$ can be carried out by *averaging on each energy level of the time-asymptotic field*. In practice, we do this by transforming the integrations to action-angle variables and replacing the time averages by phase-space averages over the lines of constant action. (3) Finally, we expand the resulting expression asymptotically in terms of half-integer powers of the small time-asymptotic amplitude a. The details of this fairly complicated calculation are given in Appendix D. The resulting expansion for the first term on the right side of Eq. (14) is

$$2[\chi_1 a^{3/2}+\chi_2\epsilon a^{1/2}+\chi_3\epsilon a^{3/2}+aK_0(k_0,\omega)]+O(a^2), \tag{16}$$

where

$$\chi_1\equiv-\sigma_1\sum_\alpha\left[\frac{|q_\alpha|^5}{k_0^5 m_\alpha^3}\right]^{1/2}\left[\frac{d^2F_\alpha}{dv^2}(v_p)-\frac{1}{2v_p}\frac{dF_\alpha}{dv}(v_p)\right], \tag{17}$$

$$\chi_2\equiv-\sigma_1\sum_\alpha s_\alpha\left[\frac{|q_\alpha|^3}{k_0^3 m_\alpha}\right]^{1/2}h_\alpha(v_p), \tag{18}$$

$$\chi_3\equiv-\sigma_2\sum_\alpha s_\alpha\left[\frac{|q_\alpha|^5}{k_0^5 m_\alpha^3}\right]^{1/2}\times\left\{\frac{d^2h_\alpha}{dv^2}(v_p)+\frac{1}{v_p}\frac{dh_\alpha}{dv}(v_p)+\frac{1}{2v_p^2}h_\alpha(v_p)\right\}, \tag{19}$$

$$K_0(k_0,\omega)=\frac{4\pi}{k_0^2}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}P\int_{\mathbb{R}}dv\,\frac{F'_\alpha(v)}{v_p-v}, \tag{20}$$

and $\sigma_1=20.67$ and $\sigma_2=0.53$ are numerical constants.

B. Landau terms

Because of symmetry, we can write T in the last term in Eq. (14) as a Fourier sine series $T(x,t)=\Sigma_{n=1}^{\infty}T_{nk_0}(t)\sin nk_0x$. Then, a calculation very similar to the one that led to Eq. (16), also given in Appendix D, reduces the Landau terms to

$$a^{1/2}2\Gamma(\epsilon,T)+a^{3/2}2\Sigma(\epsilon,T)+O(a^2), \tag{21}$$

where

$$\Gamma(\epsilon,T)\equiv\sum_\alpha\left[\frac{|q_\alpha|^5}{k_0^3 m_\alpha^3}\right]^{1/2}\times\left\{\rho_1(T)\frac{dF_\alpha}{dv}(v_p)+\frac{\epsilon}{2}\rho_2(T)\frac{dh_\alpha}{dv}(v_p)\right\}, \tag{22}$$

$$\Sigma(\epsilon,T)=\sum_\alpha\left[\frac{|q_\alpha|^7}{k_0^5 m_\alpha^5}\right]^{1/2}\left\{2\rho_3(T)\frac{d^3F_\alpha}{dv^3}(v_p)+\epsilon\rho_4(T)\frac{d^3h_\alpha}{dv^3}(v_p)-\frac{\lambda_2(T)}{4v_p}\frac{d^2F_\alpha}{dv^2}(v_p)-\epsilon\frac{\lambda_4(T)}{8v_p}\frac{d^2h_\alpha}{dv^2}(v_p)-\frac{\lambda_1(T)}{8v_p^2}\frac{dF_\alpha}{dv}(v_p)-\epsilon\frac{\lambda_3(T)}{16v_p^2}\frac{dh_\alpha}{dv}(v_p)\right\}. \tag{23}$$

The operators $\rho_i(T)$ and $\lambda_i(T)$, $i=1,\ldots,4$, are defined, respectively, in Eqs. (D55) and (D57) and are linear in T.

C. The amplitude bifurcation problem

Combining Eqs. (14), (16), and (21) gives

$$a-\chi_1 a^{3/2}-\chi_2\epsilon a^{1/2}-\chi_3\epsilon a^{3/2}-aK_0(k_0,\omega)=-a^{1/2}\Gamma(\epsilon,T)-a^{3/2}\Sigma(\epsilon,T)+O(a^2), \tag{24}$$

an explicit relationship between the "final" field amplitude a and the initial amplitude ϵ. From it we easily determine the nature of the transition between field solutions that are Landau damped to zero and those that approach two-wave time-asymptotic states with amplitude a. By construction, k_0 and ω satisfy the time-asymptotic Vlasov dispersion relation $1-K_0(k_0,\omega)=0$. Thus, Eq. (24) reduces to

$$a^{1/2}[\chi_2\epsilon-\Gamma(\epsilon,T)]+a^{3/2}[\chi_1+\chi_3\epsilon-\Sigma(\epsilon,T)]=O(a^2). \tag{25}$$

Near any accessible vanishing asymptotic state (critical or not), we write T as $T_0+\delta T$ where T_0 is the transient field corresponding to the AVAS itself. Since both Γ and Σ are linear in T, Eq. (25) becomes

$$a^{1/2}[\chi_2\epsilon-\Gamma(\epsilon,T_0)]-a^{1/2}\Gamma(\epsilon,\delta T)+a^{3/2}[\chi_1+\chi_3\epsilon-\Sigma(\epsilon,T_0)]=O(a^2,a^{3/2}|\delta T|). \tag{26}$$

The possible bifurcation values, i.e., values of the parameter ϵ where nonzero solutions cross ("branch off") the trivial solution $a^{1/2}=0$, can be found by "lifting" the branch $a^{1/2}=0$ (here simply dividing by $a^{1/2}$) and setting $a=\delta T=0$ and $\epsilon=\epsilon_0$, where ϵ_0 is the amplitude of the initial perturbation corresponding to the AVAS; since $\Gamma(\epsilon_0,0)=0$ this gives the bifurcation condition

$$\chi_2\epsilon_0=\Gamma(\epsilon_0,T_0), \tag{27}$$

which also will be called the *threshold equation*, since $\epsilon=\epsilon_0$ represents a critical initial amplitude, i.e., a value of ϵ where the time-asymptotic behavior of the electric field may change from that of a vanishing state to that of a nonzero state (or vice versa). This equation has a clear physical meaning: it expresses the balance between the effects of the initial transient [contained in Γ, which depends on $(dF_\alpha/dv)(v_p)$ and $(dh_\alpha/dv)(v_p)$], and the long-time trapped-particle effects generated by the initial perturbation [contained in χ_2, which depends on $h_\alpha(v_p)$]. In the case of single-humped unperturbed equilibria, Γ measures the strength of the Landau damping rate, while χ_2 expresses the ability of the initial perturbation to generate a plateau at the phase velocity via particle trapping.

Taking ϵ_0 to be known from Eq. (27), a local analysis can be performed to determine the bifurcating solution branch. Expansion of Eq. (26) about $\epsilon=\epsilon_0$ yields

$$[\chi_2-\Gamma_\epsilon(\epsilon_0,T_0)]\Delta\epsilon-\Gamma(\epsilon_0,\delta T)+a[\chi_1+\chi_3\epsilon_0-\Sigma(\epsilon_0,T_0)]=O(a^{3/2},a|\delta T|,a\Delta\epsilon,|\delta T|\Delta\epsilon), \tag{28}$$

where $\Delta\epsilon\equiv\epsilon-\epsilon_0$ and

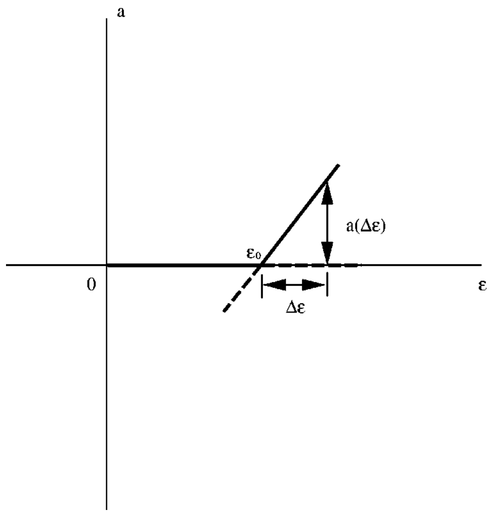

FIG. 3. Local analysis at a critical initial amplitude ϵ_0 in the ϵ-a plane.

$$\Gamma_\epsilon(\epsilon_0,T_0)\equiv\frac{\partial\Gamma}{\partial\epsilon}(\epsilon_0,T_0)=\frac{1}{2}\sum_\alpha\left[\frac{|q_\alpha^5|}{k_0^3m_\alpha^3}\right]^{1/2}\rho_2(T_0)\frac{dh_\alpha}{dv}(v_p). \tag{29}$$

Equation (28) gives an explicit approximation to Eq. (26); thus it could be solved to provide an approximate solution to Eq. (14). In fact, it shows the qualitative dependence of a on ϵ: clearly, a undergoes a transcritical bifurcation at the critical initial amplitude ϵ_0, with the nonzero solutions for a crossing the basic branch $a=0$ at a finite angle. In order to have quantitative results, however, we have to calculate the coefficients in Eqs. (27) and (28), which depend on $T(x,t)$, treated thus far as a parameter; indeed, Eq. (14) must be understood as part of a coupled system of equations for a and T, Eqs. (5). Hence, we now turn to the transient part of the VP problem.

IV. TRANSIENT FIELD EXPANSION

Substituting Eq. (7) into Eq. (5b) yields the transient equation, which we solve via a perturbation expansion of $T(x,t)$ in powers of the deviation of the initial condition from the AVAS under consideration. Because $A(x,t)$ has been subtracted from $E(x,t)$ and its nonlinear effects on the particle trajectories computed analytically, the perturbation solution for $T(x,t)$ will be spared the disruptive secularities that arise in standard perturbation treatments of the VP system which use the complete field $E(x,t)$.

Here we shall not assume *a priori* [as in Eq. (13), see footnote 3] that the time-asymptotic field has the same wave number k_0 as the initial condition; rather, we shall write

$$\begin{aligned}A(x,t)&=2a\sin k_fx\cos\omega t\\&=a\sin(k_fx+\omega t)+a\sin(k_fx-\omega t)\end{aligned} \tag{30}$$

and then prove that indeed $k_f=k_0$. Below, the transient equation will be expanded in the neighborhood of a given critical amplitude ϵ_0 in the small amplitudes a and $\Delta\epsilon$ (Fig. 3). However, a itself ultimately depends on $\Delta\epsilon$, so that *both* T and a must be expanded in $\Delta\epsilon$. Since the expansion of the time-asymptotic equation generated half-integer powers of a, half-integer powers of $\Delta\epsilon$ may be needed (at higher order) to have consistent expansions for a and T. Thus, our focus being on first order terms, we introduce the truncated expansion for T:

$$T(x,t)=T^{(0)}(x,t)+\Delta\epsilon T^{(1)}(x,t)+O(\Delta\epsilon^{3/2}). \tag{31}$$

Substituting Eq. (31) into Eq. (28) [with $T_0\equiv T^{(0)}$ and $\delta T\equiv\Delta\epsilon T^{(1)}+O(\Delta\epsilon^{3/2})$] gives[4]

$$a=-\frac{[\chi_2-\Gamma_\epsilon(\epsilon_0,T^{(0)})-\Gamma(\epsilon_0,T^{(1)})]}{[\chi_1+\chi_3\epsilon_0-\Sigma(\epsilon_0,T^{(0)})]}\Delta\epsilon+O(\Delta\epsilon^{3/2}). \tag{32}$$

A. Small critical initial amplitudes

We first calculate the transient field *along* the basic branch $a=0$, which provides $T_0\equiv T^{(0)}$ and also is necessary to determine ϵ_0 from Eq. (27). Even for $a=0$, the leading-order expansion of the transient equation is rather tedious; hence, it is developed in Appendix E 1 which leads to the following equation for the Fourier-Laplace-transformed zeroth-order transient electric field $\tilde{T}_k^{(0)}(p)$:

$$\begin{aligned}D_k(p)\tilde{T}_k^{(0)}(p)-\epsilon_0C_k(p)[\tilde{T}_{k+k_0}^{(0)}(p)+\tilde{T}_{k-k_0}^{(0)}(p)]\\=\epsilon_0\delta_{k,k_0}N_k(p),\end{aligned} \tag{33}$$

where $N_k(p)$ and $D_k(p)$ are the same quantities that appear, respectively, in the numerator and the denominator in Landau's solution to the standard linearized initial value problem [2] (for the initial condition $\mathcal{F}_\alpha$),

$$N_k(p)=-\frac{4\pi i}{k^2}\sum_\alpha q_\alpha\int_{\mathbb{R}}dv\,\frac{h_\alpha(v)}{v-ip/k}, \tag{34}$$

$$D_k(p)=1+\frac{4\pi}{k^2}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\int_{\mathbb{R}}dv\,\frac{F'_\alpha(v)}{v-ip/k}, \tag{35}$$

and

$$C_k(p)\equiv-\frac{2\pi}{k^2}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\int_{\mathbb{R}}dv\,\frac{h'_\alpha(v)}{v-ip/k}. \tag{36}$$

[4]The sign of a [taken to be positive after postulating an appropriate phase shift in Eq. (13)] depends here on the signs of $\Delta\epsilon$ and its coefficient. In turn, $\Delta\epsilon$ must have the same sign as ϵ_0, when studying the transition from completely Landau damped solutions to small-amplitude nonzero time-asymptotic solutions. Nevertheless, if a in Eq. (32) is negative, one can immediately find a solution with $a>0$ simply by considering a perturbation $-\Delta\epsilon$ to a critical amplitude $-\epsilon_0$. This is equivalent to applying the coordinate transformation $x\to-x$, $v\to-v$, corresponding to the reflection symmetry.

Equation (33) is very similar to the equation that results from Landau's solution to the linear VP system [2]. The main difference is that, because of the spatially nonuniform part of the initial distribution $\mathcal{F}_\alpha$, the equations for different values of k are coupled. Hence, Eq. (33) is an infinite system and its solution will require an approximation. In general, numerical approximation could be used, which then would lead to a solution of Eq. (27) for general (not small) critical amplitudes. Here, however, we are interested in small initial perturbations, i.e., $\epsilon \ll 1$; hence $\epsilon_0 \ll 1$. Thus, further expanding $\tilde{T}_k^{(0)}$ in powers of ϵ_0,

$$\tilde{T}_k^{(0)} = \epsilon_0 \tilde{T}_k^{(0,1)} + \epsilon_0^2 \tilde{T}_k^{(0,2)} + \cdots + \epsilon_0^m \tilde{T}_k^{(0,m)} + \cdots, \tag{37}$$

substituting into Eq. (33), and solving at each order in ϵ_0 straightforwardly yields

$$\begin{aligned}
\tilde{T}_k^{(0,1)}(p) &= \delta_{k,k_0} \frac{N_{k0}(p)}{D_{k_0}(p)},\\
\tilde{T}_k^{(0,2)}(p) &= \delta_{k+k_0,k_0} \frac{N_{k+k_0}(p) C_k(p)}{D_{k+k_0}(p) D_k(p)} + \delta_{k-k_0,k_0} \frac{N_{k-k_0}(p) C_k(p)}{D_{k-k_0}(p) D_k(p)}\\
&= \delta_{k,2k_0} \frac{N_{k_0}(p) C_{2k_0}(p)}{D_{k_0}(p) D_{2k_0}(p)},\\
\tilde{T}_k^{(0,3)}(p) &= \delta_{k,3k_0} \frac{N_{k_0}(p) C_{2k_0}(p) C_{3k_0}(p)}{D_{k_0}(p) D_{2k_0}(p) D_{3k_0}(p)} + \delta_{k,k_0} \frac{N_{k_0}(p) C_{k_0}(p) C_{2k_0}(p)}{D_{k_0}(p)^2 D_{2k_0}(p)}.
\end{aligned} \tag{38}$$

The expressions for $T_k^{(0,m)}$ can be obtained by inverting the Laplace transforms. [In particular, $\tilde{T}_k^{(0,1)}(p)$ yields Landau's classic damped solution.]

Clearly, there are initial conditions for which this solution is not acceptable, because it does not vanish as $t \to \infty$. For linearly stable initial equilibria F_α, the standard results for the zeros of $D_k(p)$ ensure that $\tilde{T}_k^{(0)}(p)$ have no poles with real parts $\geqslant 0$, so $\lim_{t\to\infty} T_k^{(0)}(t) = 0$. But for initial conditions such that the standard linear theory predicts traveling-wave or growing-wave solutions, $\lim_{t\to\infty} T_k^{(0)}(t) \neq 0$. This is perfectly reasonable, because in these cases we expect that the asymptotic amplitude of the waves will be nonzero, no matter how small the initial amplitude is; thus, no solutions for T can be found along the axis $a=0$ and the vanishing asymptotic states are not accessible, which is consistent with our earlier remark that the solution $A \equiv 0$ for Eq. (5a) does not necessarily correspond to a solution of the complete system Eqs. (5) for both A and T.

The perturbation solution for $T^{(0)}$, Eq. (38), enables us to compute explicitly the quantities that appear in the threshold equation (for $k_f = k_0$), Eq. (27). The details, given in Appendix E 2, lead to the following equation for the (small) critical amplitude ϵ_0:

$$\begin{aligned}
\chi_2 \epsilon_0 = {} & \epsilon_0 \sigma_1 S_{1,1}^{(0,1)} \sum_\alpha \left[\frac{|q_\alpha|^5}{k_0^3 m_\alpha^3} \right]^{1/2} \frac{dF_\alpha}{dv}(v_p)\\
& - 4\epsilon_0^2 \left\{ \sigma_2 S_{2,2}^{(0,2)} \sum_\alpha s_\alpha \left[\frac{|q_\alpha|^5}{k_0^3 m_\alpha^3} \right]^{1/2} \frac{dF_\alpha}{dv}(v_p) + \frac{\sigma_2}{2} S_{1,2}^{(0,1)} \sum_\alpha s_\alpha \left[\frac{|q_\alpha|^5}{k_0^3 m_\alpha^3} \right]^{1/2} \frac{dh_\alpha}{dv}(v_p) \right\} + O(\epsilon_0^3).
\end{aligned} \tag{39}$$

Clearly, this equation has the fundamental zero-field solution $\epsilon_0^{(0)} = 0$, corresponding to a zero initial amplitude. In this case Eq. (5b) has the trivial solution $T_0 \equiv 0$, and the threshold equation is obviously satisfied, since $\Gamma(0,0) = 0$. This just corresponds to the trivial solution $E = A = T \equiv 0$, $f_\alpha \equiv F_\alpha$. More interestingly, Eq. (39) also yields

$$\epsilon_0^{(1)} = \frac{\sigma_1 \Sigma_\alpha s_\alpha [|q_\alpha|^3/m_\alpha]^{1/2} h_\alpha(v_p) + \sigma_1 S_{1,1}^{(0,1)} \Sigma_\alpha [|q_\alpha|^5/m_\alpha^3]^{1/2} (dF_\alpha/dv)(v_p)}{4\sigma_2 S_{2,2}^{(0,2)} \Sigma_\alpha s_\alpha [|q_\alpha|^5/m_\alpha^3]^{1/2} (dF_\alpha/dv)(v_p) + 2\sigma_2 S_{1,2}^{(0,1)} \Sigma_\alpha s_\alpha [|q_\alpha|^5/m_\alpha^3]^{1/2} (dh_\alpha/dv)(v_p)}, \tag{40}$$

a nonzero critical amplitude, at which a nontrivial solution branch for a crosses the axis $a=0$ (Fig. 3). Since Eq. (40) was derived via an expansion for small ϵ_0, consistency requires that $\epsilon_0^{(1)}$ be close to $\epsilon_0^{(0)} = 0$. Whenever the initial condition is such that $\epsilon_0^{(1)}$ in Eq. (40) does *not* satisfy $|\epsilon_0^{(1)}| \ll 1$, there is no (nonzero) small-amplitude critical point corresponding to $k_f = k_0$, and all small sinusoidal perturbations to a linearly stable Vlasov equilibrium undergo Landau damping to zero. Of course, this does not preclude the possibility of a nonsmall critical amplitude.

B. The critical amplitude $\epsilon_0^{(0)} = 0$

We next determine the transient field for solutions with $a \neq 0$. We begin from the "trivial" critical amplitude $\epsilon_0^{(0)} = 0$, and we seek small-amplitude time-asymptotic solutions close to it, and look for a branch of time-asymptotic amplitudes that bifurcates from $\epsilon_0=0$, $a=0$ in the ϵ-a plane. Whether such a branch exists is physically important because it determines the time-asymptotic stability of the plasma. Indeed, such a branch would imply that there are arbitrarily small initial conditions that do not damp and lead to time-asymptotic multiple-mode traveling-wave solutions of the Buchanan-Dorning [15] type (in the coarse-grained sense described above). If no branch bifurcates from $\epsilon_0^{(0)}$, it will be necessary to consider the nonzero critical amplitude $\epsilon_0^{(1)}$ in Eq. (40) as another possible branching point for nonzero time-asymptotic solutions, since in this case initial conditions with amplitudes $\epsilon<\epsilon_0^{(1)}$ would have to be damped to zero. Even then, though, the analysis at $\epsilon_0=\epsilon_0^{(0)}$ will be important, because it will provide the leading order term in the expansion of the transient equation at $\epsilon_0=\epsilon_0^{(1)}$ with respect to the small critical amplitude $\epsilon_0=\epsilon_0^{(1)}\ll 1$.

For $\epsilon_0=0$ and $T_0\equiv 0$ several terms in the transient equation are zero, and a tedious but straightforward extension [33] of the calculation that led to Eq. (33) yields [for Re(p) $\geqslant 0$]

$$D_k(p)\tilde{T}_k^{(1)}(p)=\delta_{k,k_0}N_k(p)-\delta_{k,k_f}D_k(p)\tilde{A}_k^{(0)}(p), \quad (41)$$

where $A_k^{(0)}(p)$ is the Fourier-Laplace transform of A in Eq. (30) for $\epsilon_0=0$, i.e.,

$$\tilde{A}_k^{(0)}(p)\equiv\int_0^\infty dt\, e^{-pt}A_k(t)=\delta_{k,k_f}\left[\frac{a_0}{(p+ik_f v_p)}+\frac{a_0}{(p-ik_f v_p)}\right] \quad (42)$$

with

$$a_0\equiv-\frac{1}{\chi_1}[\chi_2-\Gamma(0,T^{(1)})], \quad (43)$$

which follows from evaluating Eq. (32) at $\epsilon_0=0$ [where $T^{(0)}\equiv 0$ and the $\Delta\epsilon$ has canceled with ϵ's multiplying the first two terms in Eq. (41)]. Equation (41) is similar to the equation that arises in Landau's analysis [2]. In fact, it could be obtained formally from Landau's solution by simply decomposing the field into $T+A$, moving A to the right side, and assuming it to be of the form in Eq. (30). However, Eq. (41) is an equation for T and a, not E and must be solved simultaneously with Eq. (25) in such a way that $\lim_{t\to\infty}T(x,t)=0$.

If $k_f\neq k_0$, for $k=k_f$ Eq. (41) gives $\tilde{T}_{k_f}^{(1)}(p)=-\tilde{A}_{k_f}^{(0)}(p)$, which is clearly unacceptable. Hence, the initial value problem cannot have a nonzero time-asymptotic solution of the form in Eq. (30) unless $k_f=k_0$, as anticipated in the previous section (see footnote 3). Thus, let $k_f=k_0$; if $k\neq k_f=k_0$, Eq. (41) implies that $\tilde{T}_k^{(1)}(p)\equiv 0$, so that the transient has no leading-order component with that wave number. We conclude that the only relevant spatial Fourier mode in T must correspond to $k=k_f=k_0$. Thus, the solution to Eq. (41) is

$$\tilde{T}_{k_0}^{(1)}(p)=\frac{N_{k_0}(p)}{D_{k_0}(p)}-\tilde{A}_{k_0}^{(0)}(p). \quad (44)$$

Applying Landau's procedure [2] to this equation, the transient field T is obtained as a sum of exponential terms, each corresponding to a pole of $[N_{k_0}^+(p)/D_{k_0}^+(p)]-\tilde{A}_{k_0}^{(0)}(p)$ where $N_{k_0}^+$ and $D_{k_0}^+$ are the analytic continuations of N_{k_0} and D_{k_0} [2]. Under Landau's assumptions [2] on the analyticity of F_α and h_α, this function has *two* kinds of pole: (a) the poles associated with the zeros of $D_{k_0}^+$, and (b) the poles of $\tilde{A}_{k_0}^{(0)}(p)$ at $p=\pm ik_0v_p$ [from Eq. (42)]. The crucial point is that the extra poles, corresponding to $\tilde{A}_{k_0}^{(0)}(p)$, lie on the imaginary axis. But since T is the *transient* field, it cannot include undamped terms, and a necessary condition for the existence of solutions to Eq. (41) is that *the residues of* $[N_{k_0}^+(p)/D_{k_0}^+(p)]-\tilde{A}_{k_0}^{(0)}(p)$ *at* $p=\pm ik_0v_p$ *must be zero*. We classify the situations that lead to this condition according to the nature of the roots of $D_{k_0}^+$.

(I) First, all the roots of $D_{k_0}^+$ have Re(p)<0. Then, for the residues at $p=\pm ik_0v_p$ to be zero a_0 must equal zero. Hence, all time-asymptotic solutions with initial amplitudes in the neighborhood of $\epsilon_0=0$ coincide with the branch $a=0$ [with errors of order $O(e^{3/2})$] and correspond to complete Landau damping, notwithstanding the trapping effects. Then nonzero solutions for the time-asymptotic electric field will be possible only for initial amplitudes greater than some nonzero critical value, like $\epsilon_0^{(1)}$ in Eq. (40). This situation will be considered in the next subsection.

(II) Next, $D_{k_0}^+$ has roots with Re(p)>0. Then the solutions for T contain growing modes, which is not acceptable. Thus, for the choices of F_α that lead to this situation, there is no small-amplitude time-asymptotic solution in the neighborhood of $\epsilon_0=0$, not even $a=0$. This leads us to the conjecture that in these linearly unstable cases the time-asymptotic field amplitude tends to a nonzero value as $\epsilon\to 0$. This problem, although interesting, will not be further pursued here.

(III) Finally, $D_{k_0}^+$ has poles with Re(p)$\leqslant$0. This includes poles on the imaginary axis, corresponding to a pair of eigenvalues embedded in the Van Kampen–Case [3,4] continuous spectrum for the linearized VP system. These poles produce undamped terms that must cancel with those coming from $\tilde{A}_{k_0}^{(0)}(p)$, in order to have a valid solution. Evaluating the Landau dispersion relation on the imaginary axis and separating the real and imaginary parts gives

$$1+\frac{4\pi}{k_0^2}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\mathrm{P}\int_{\mathbb{R}}dv\frac{F'_\alpha(v)}{v-\lambda/k_0}=0, \quad (45)$$

$$\sum_\alpha\frac{q_\alpha^2}{m_\alpha}F'_\alpha(\lambda/k_0)=0, \quad (46)$$

where $\lambda=ip$, $\lambda\in\mathbb{R}$. Equation (45) is the time-asymptotic Vlasov dispersion relation (for symmetric initial conditions and under the approximation of transient linearization). The residues at $p=\pm ik_0v_p$ can be zero only if (i) $\lambda=\pm k_0v_p$ satisfies Eqs. (45) and (46), and (ii) the residues of $N_{k_0}^+/D_{k_0}^+$ at $p=\pm ik_0v_p$ are equal to those of $\tilde{A}_{k_0}^{(0)}(p)$, i.e., equal to a_0.

In the context of the two-wave case, we consider the situation in which Eq. (45) has just one pair[5] of simple roots on the imaginary axis $p=\mp ik_0v_p$ or $\lambda=\pm k_0v_p$, which satisfy

$$\sum_\alpha \frac{q_\alpha^2}{m_\alpha}F'_\alpha(\pm v_p)=0. \tag{47}$$

A Taylor expansion [6] of $D^+_{k_0}$ about $\pm ik_0v_p$ yields the residues at these poles:

$$\mathrm{Res}\left(\frac{N^+_{k_0}}{D^+_{k_0}},\pm ik_0v_p\right)$$
$$=-\frac{k_0\Sigma_\alpha q_\alpha[\mathrm{P}\int_{\mathbb{R}}dv\ h_\alpha(v)/(v\pm v_p)+i\pi h_\alpha(v_p)]}{\Sigma_\alpha(q_\alpha^2/m_\alpha)[\mathrm{P}\int_{\mathbb{R}}dv\ F''_\alpha(v)/(v\pm v_p)+i\pi F''_\alpha(v_p)]}. \tag{48}$$

(Here and below the symmetries of F_α, h_α, F'_α, etc., allow us to replace $\pm v_p$ by v_p in their arguments.) This expression must be set equal to a_0, which is obtained from Eqs. (43), (17), (18), and (22). The expression for a_0 can be made somewhat simpler if Eq. (47) is replaced by the slightly stronger condition $(dF_\alpha/dv)(v_p)=0$ $(\alpha=1,2,...,N_S)$. In many physical situations the two will be equivalent (e.g., in an ion-electron plasma, in which the charge-mass ratio for the electrons is much larger than for all the other species combined). Then, Eq. (43) [with Eqs. (17), (18), and (22)] yields

$$a_0=-\frac{k_0\Sigma_\alpha s_\alpha[|q_\alpha|^3/m_\alpha]^{1/2}h_\alpha(v_p)}{\Sigma_\alpha[|q_\alpha|^5/m_\alpha^3]^{1/2}F''_\alpha(v_p)}. \tag{49}$$

For $[N^+_{k_0}(p)/D^+_{k_0}(p)]-\tilde{A}^{(0)}_{k_0}(p)$ to have zero residues at $\pm ik_0v_p$, Eqs. (48) and (49) must be equal, up to a possible change in the sign of a_0 due to the choice of the phase of $A(x,t)$ in Eq. (13).[6] Setting the real part of Eq. (48) equal to a_0 and the imaginary part equal to zero yields the twofold condition

$$\pm\frac{\Sigma_\alpha s_\alpha[|q_\alpha|^3/m_\alpha]^{1/2}h_\alpha(v_p)}{\Sigma_\alpha[|q_\alpha|^5/m_\alpha]^{1/2}F''_\alpha(v_p)}=\frac{\Sigma_\alpha q_\alpha h_\alpha(v_p)}{\Sigma_\alpha(q_\alpha^2/m_\alpha)F''_\alpha(v_p)}=\frac{\Sigma_\alpha q_\alpha\mathrm{P}\int_{\mathbb{R}}dv\ h_\alpha(v)/(v-v_p)}{\Sigma_\alpha(q_\alpha^2/m_\alpha)\mathrm{P}\int_{\mathbb{R}}dv\ F''_\alpha(v)/(v-v_p)}, \tag{50}$$

where the sign of the left-most term must be chosen to be equal to the sign of the middle term to ensure that the nonlinear solution connects smoothly to the linearized initial solution, from which it evolves after particle trapping becomes significant.

When both Eqs. (47) and (50) are satisfied, Eq. (41) has an acceptable solution for $T^{(1)}_{k_f}$, which goes to zero as $t\to\infty$. This solution is basically Landau's solution for the given initial condition, minus the "undamped" terms that correspond to the poles on the imaginary axis. Indeed, the inverse Laplace transform of Eq. (44) yields a sum of exponentials corresponding to all the roots of Eq. (50). The "linear" contributions due to these two roots are replaced exactly by the time-asymptotic field *A*, which is given by Eq. (30) with $a=a_0\Delta\epsilon$. Equation (50) ensures that the time-asymptotic wave amplitude is equal to the amplitude of the corresponding modes in the linear theory. Thus, as long as Eq. (50) is satisfied the electric field solution from the nonlinear theory is actually the same (for case III) as that from the linear analysis, not only initially but at all times. Clearly, this happens only because we are considering the special case of initial Vlasov equilibria with zero derivatives at the phase velocities. Nevertheless, the solution for the distribution function, inside the v integrals in Eq. (7), is completely different from the solution to the linearized problem; in particular, it contains the trapping effects via the characteristics for the time-asymptotic field *A*. Thus, Eq. (50) gives a *nonlinear* criterion that determines which small-amplitude initial conditions lead to time-asymptotic traveling-wave solutions. According to the standard linear theory, this happens whenever Eqs. (45) and (46) hold. Unfortunately this is only an initial time result, and it is not at all clear from the linear theory whether the Landau modes corresponding to poles on the imaginary axis can keep traveling unchanged in the nonlinear regime, i.e., when the trapping effects become relevant. Equation (50), however, ensures precisely that the undamped modes generated by the poles $p=\pm ik_fv_p$ are consistent with the nonlinear dynamics, and will keep traveling at the same amplitude in the time-asymptotic limit.

An important example of an initial condition that satisfies Eq. (50) is provided by Buchanan and Dorning's undamped two-wave BGK-like solutions [15] (see Appendix F here). Thus, those solutions are a special case of the solutions developed here. What characterizes BGK and "BGK-like" solutions is that the distribution function is constant along the level curves of suitable invariants for single-particle motion (such as the energy in true BGK solutions or Buchanan and

[5]In special cases Eq. (45) could have N (>1) pairs of roots on the imaginary axis. Then, the *Ansatz* would be extended to include up to $2N$ waves. If conditions (i) and (ii) were satisfied at all $2N$ poles, we would associate a time-asymptotic wave mode with each root of $D^+_{k_0}$ on the imaginary axis. However, here we focus on the two-wave case, both because cases with $N>1$ are exceptional and because the extension to those cases is straightforward, although very tedious.

[6]We remarked in footnotes 2 and 4 that $a=a_0\Delta\epsilon$ can always be made positive via an appropriate choice of the phase of $A(x,t)$, and that this might require us to replace $\Delta\epsilon$ by $-\Delta\epsilon$ if $a_0<0$. However, until this point it was not clear why the sign of *a* should be chosen positive or negative. Now, the transient analysis gives the natural criterion to determine the phase of $A(x,t)$ (and the sign of *a*) for a given initial condition: the phase of $A(x,t)$ must be chosen so that it is consistent with the phase of the corresponding linear modes (with phase velocities $\pm v_p$), whose amplitude is given by the real part of Eq. (48). Indeed, we know that these linear modes correctly describe the *initial* wave propagation, and that $A(x,t)$ must "replace" them in the nonlinear regime. Hence, the sign of a_0 must be the same as the sign of the real part of Eq. (48).

Dorning's two-wave approximate invariants in BGK-like solutions), so that there is no energy exchange between the plasma and the field. Equation (50) provides a generalized BGK condition on the initial distribution in the sense that, even when the resulting solution is not really BGK-like, the resonant wave-particle interactions are sufficiently "gentle" that small-amplitude wave propagation still occurs. These solutions provide the only nontrivial solution branch through the origin in the ϵ-a plane.

Interestingly, in the time-asymptotic limit the distribution function becomes macroscopically equivalent to the coarse-grained BGK-like distribution obtained by averaging f_α over the two-wave approximate invariant curves. By "macroscopically equivalent" we mean that, when integrated over phase space, the two distribution functions produce the same macroscopic quantities (see Appendix D, Proposition 1). In this coarse-grained sense, the solutions that we have obtained evolve in time to reach BGK-like states as the outcome of the non-linear dynamics, whereas the undamped solutions obtained by Buchanan and Dorning [15] have to be set up by an initial distribution function that already has exactly the structure of a BGK-like solution.

C. The nonzero critical amplitude $\epsilon_0^{(1)}$

In case I above we established that whenever the initial Vlasov equilibrium is linearly stable the only solution branch through $\epsilon_0=\epsilon_0^{(0)}=0$ is the basic branch $a=0$. Hence, we must raise the question whether there are nonzero time-asymptotic solutions that branch from $a=0$ at a *nonzero* critical initial amplitude. Equation (40) gives the small nonzero critical initial amplitude $\epsilon_0^{(1)}$, which is a possible threshold separating the initial conditions that result in Landau damping to zero from those that lead to traveling-wave solutions with $a\neq 0$. To find such nonzero time-asymptotic solutions, we now study the transient equation in the neighborhood of $\epsilon_0^{(1)}$. This is also done via a perturbation expansion, first in powers of a and $\Delta\epsilon$, and then in powers $\epsilon_0^{(1)}$ [to invert the linear operator on the left side of Eq. (33)]. Fortunately, all the leading-order information can be obtained from a simple extension of the analysis carried out in Sec. IV B for the "trivial" critical initial amplitude $\epsilon_0^{(0)}=0$.

At zeroth order in $\Delta\epsilon$, of course, we have the purely transient field $T_0\equiv T^{(0)}$, which was already computed in Eq. (38), but now with $\epsilon_0=\epsilon_0^{(1)}$. At first order in $\Delta\epsilon$ we expand the first order term $\tilde{T}_k^{(1)}$ in powers of $\epsilon_0^{(1)}$, in the form

$$\tilde{T}_k^{(1)}=\tilde{T}_k^{(1,0)}+\epsilon_0^{(1)}\tilde{T}_k^{(1,1)}+\cdots. \tag{51}$$

The equation for $\tilde{T}_k^{(1,0)}$ is identical to Eq. (41) (which corresponds to the limit $\epsilon_0^{(1)}\to 0$). Hence, the same arguments used before show that solutions with $a_0\neq 0$ are possible only if $k=k_0=k_f$, and that $\tilde{T}_{k_0}^{(1,0)}(p)$ must be obtained from the analysis of the poles of the function $[N_{k_0}^+(p)/D_{k_0}^+(p)]-\tilde{A}_{k_0}^{(0)}(p)$. Again, $\tilde{A}_{k_0}^{(0)}(p)$ has poles $\pm ik_0v_p$, whose residues must be zero to have a valid solution for $T_{k_0}^{(1,0)}(t)$. Since we are considering initial conditions near linearly stable equilibria, the residue cannot be exactly zero if $a_0\neq 0$, just as in case I above. However, it can be approximately zero, with the same accuracy in $\epsilon_0^{(1)}$ as $\tilde{T}_k^{(1,0)}$ itself. The necessary condition for this is that $\pm ik_0v_p$ must be roots of the Landau dispersion relation $D_{k_0}^+(p)=0$ at zero order in $\epsilon_0^{(1)}$. (They cannot be exact roots because in the case being considered all the roots have negative real parts.) Hence,

$$\begin{aligned}D_{k_0}^+(\pm ik_0v_p)&\\ &=1+\frac{4\pi}{k_0^2}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\left[\mathrm{P}\int_{\mathbb{R}}dv\,\frac{F'_\alpha(v)}{(v\pm v_p)}+i\pi F'_\alpha(\mp v_p)\right]\\ &=O(\epsilon_0^{(1)}).\end{aligned} \tag{52}$$

Since $\pm v_p$ satisfy the Vlasov dispersion relation, this is equivalent to

$$\sum_\alpha\frac{q_\alpha^2}{m_\alpha}F'_\alpha(\mp v_p)=O(\epsilon_0^{(1)}). \tag{53}$$

Obviously, this is a generalization of Eq. (47): it indicates that for there to be a small critical amplitude, the derivative of the initial Vlasov equilibrium at the phase velocity must be small, so that Landau damping is weak and even a small initial disturbance has the possibility of trapping particles. In what follows, Eq. (53) will be taken to be a consequence of the stronger condition

$$\frac{dF_\alpha}{dv}(\pm v_p)=O(\epsilon_0^{(1)})\quad \alpha=1,2,\ldots,N_s. \tag{54}$$

Then, the residue calculation for $N_{k_0}^+/D_{k_0}^+$ is the same as in case III, since Eq. (47) holds at zeroth order and leads to the same expression as in Eq. (48). Similarly, a_0 has the same expression at zeroth order in Eq. (49). Hence, the same argument as in the case $\epsilon_0=0$ shows that the initial distribution must satisfy (now at leading order) the conditions in Eq. (50), in order for small-amplitude nonlinear wave propagation to be possible in the neighborhood of the small critical amplitude $\epsilon_0^{(1)}$. Then, the solution for T is

$$T(x,t)=\epsilon_0^{(1)}T^{(0,1)}(x,t)+\Delta\epsilon T^{(1,0)}(x,t)+O(\epsilon_0^{(1)}\Delta\epsilon), \tag{55}$$

where $T^{(0,1)}$ is Landau's damping solution for the given initial condition and $T^{(1,0)}$ is the same Landau solution minus the contributions from the poles that correspond (at zeroth order) to the time-asymptotic phase velocities $\pm v_p$.

The field A is given, of course, by the two-wave solution in Eq. (30), with $k_f=k_0$ and amplitude a obtained from Eq. (32) (see Appendix E 3 for the detailed calculation). To first order in both $\Delta\epsilon$ and ϵ_0

$$a=\Bigg\{-\frac{k_0\Sigma_\alpha s_\alpha[|q_\alpha|^3/m_\alpha]^{1/2}h_\alpha(v_p)}{\Sigma_\alpha[|q_\alpha|^5/m_\alpha^3]^{1/2}F''_\alpha(v_p)}+\frac{\Sigma_\alpha[|q_\alpha^5|/k_0^3m_\alpha^3]^{1/2}[2s_\alpha\epsilon_0^{(1)}\sigma_2S_{1,2}^{(0,1)}h'_\alpha(v_p)-\sigma_1S_{1,1}^{(1,0)}F'_\alpha(v_p)]}{\sigma_1\Sigma_\alpha[|q_\alpha|^5/k_0^5m_\alpha^3]^{1/2}F''_\alpha(v_p)}$$
$$+\frac{\chi_2\chi_3\epsilon_0^{(1)}+\chi_2\Sigma_\alpha[|q_\alpha|^5/k_0^5m_\alpha^3]^{1/2}\{(\sigma_1/2v_p)F'_\alpha(v_p)-\epsilon_0^{(1)}(|q_\alpha|/m_\alpha)[\sigma_2S_{1,1}^{(0,1)}F'''_\alpha(v_p)-\sigma_1S_{1,3}^{(0,1)}(1/4v_p)F''_\alpha(v_p)]\}}{[\sigma_1\Sigma_\alpha[|q_\alpha|^5/k_0^5m_\alpha^3]^{1/2}F''_\alpha(v_p)]^2}\Bigg\}\Delta\epsilon, \tag{56}$$

where the $S_{i,j}^{(m,n)}$ are given by Eqs. (E9), (E10), and (E14). Equation (56) gives the nonzero final amplitude that $E=A+T$ reaches via particle trapping after slow initial Landau damping. In the limit $\epsilon_0^{(1)}\to 0$ this branch reduces to Eq. (49) [recall $F'_\alpha(v_p)=O(\epsilon_0^{(1)})$], i.e., a solution branch coming from the origin in the ϵ-a plane and corresponding to a (possibly infinitesimal) ''flat spot'' on the initial Vlasov equilibrium. The corresponding distribution function is again given by the integrand in Eq. (7) and is equivalent to a BGK-like distribution in the sense discussed in the previous section; namely, as $t\to\infty$ f_α generates the same macroscopic quantities as the coarse-grained function which is obtained by averaging f_α in phase space along the ''invariant curves'' for the field A. In this sense, these time-asymptotic solutions to the VP system can be viewed as a generalization of the multiple-traveling-wave BGK-like solutions of Buchanan and Dorning [15]. The solutions obtained here, of course, exist only for initial field amplitudes above the threshold value given by Eq. (40) and have time-asymptotic amplitudes given by Eq. (56).

V. CONCLUSION

Particle trapping effects are ubiquitous in plasma physics. Even in simple situations, however, the theoretical understanding of these effects is severely limited by our inability to analyze (except possibly by large-scale numerical simulations) the nonlinear equations that model self-consistent field-particle interactions. In this article we have studied what is possibly the most fundamental example of nonlinear wave-particle dynamics, the one associated with longitudinal waves in a collisionless plasma. This classic problem, often referred to as ''nonlinear Landau damping,'' is fairly well understood in the two limiting cases of strong Landau damping [2] and weakly damped trapping-dominated wave propagation [8,9]. Conversely, there is very little understanding of the difficult ''intermediate'' regime in which the two time scales associated with linear Landau damping and with particle trapping are of the same order of magnitude; hence, our goal has been to study this intermediate regime. The fact that the same type of initial disturbance (e.g., a single-mode sine wave) can, depending on its amplitude, be Landau damped to zero or evolve to traveling-wave behavior [8] suggests that there must be a threshold (a ''critical initial amplitude'') separating the initial conditions that lead to these two very different time-asymptotic states. The existence of such a threshold is strongly supported by numerical simulations showing that small-amplitude electric fields are damped to zero, whereas initial conditions of larger amplitude evolve to nonzero multiple-wave final states [26,27]. In this article we have developed analytical expressions both for the threshold and for the amplitude of the time-asymptotic superimposed traveling waves that evolve from initial conditions with amplitudes just above the threshold.

In the first part of the paper (Sec. II) we developed a general technique for studying self-consistent wave-particle dynamics. This technique, which has wider applicability, is based on two essential ideas. The first follows from the observation that, for initial conditions with amplitude just above the threshold, the time-asymptotic electric field is given by a superposition of small-amplitude BGK-like wave modes, each of which satisfies a Vlasov dispersion relation. Thus, the problem of solving the Vlasov-Poisson equation at long times can be reduced to the determination of a finite number of time-asymptotic amplitudes (one for a symmetric pair of waves). The second essential idea is that, since the general form of the time-asymptotic field is a discrete superposition of small-amplitude waves, the long-time solutions to the Vlasov-Poisson equations can be approximated via ''transient linearization,'' i.e., by linearizing only the interaction between the distribution function and the transient part of the electric field, while keeping the full nonlinear wave-particle interaction in the limit $t\to\infty$. Under this approximation, the equations can be solved exactly via Hamiltonian perturbation theory.

The detailed solution, given in Sec. III for the important case of a sinusoidal initial disturbance [8], shows that, as the initial amplitude ϵ of the perturbation is increased through a certain threshold, the time-asymptotic wave amplitude a changes *transcritically* (i.e., at a finite angle) from zero (complete Landau damping) to a nonzero value (traveling-wave propagation); the dependence of the final amplitude a on the initial amplitude ϵ near the threshold is given by Eq. (32). The threshold itself satisfies a scalar equation, Eq. (27), which depends on the details of the initial distribution function. Naturally, the equations for the threshold and the time-asymptotic field amplitude have coefficients that depend on the transient part of the electric field; hence, these general equations must be combined with analysis of the transient behavior. That analysis was presented in the last part of the paper (Sec. IV) for the more restricted but very important case in which the critical initial amplitude (or threshold) itself is small. Physically, this occurs when linear Landau damping is weak, so that even a small initial perturbation from zero is sufficient to cause particle trapping and evolution to self-sustained traveling-wave modes. In this case, the transient behavior could be analyzed via straightforward perturbation expansions, which led to two types of long-time solutions.

The solutions of the first type (Sec. IV B) are those in which the threshold in the initial amplitude is actually zero. These solutions, for arbitrarily small perturbations of linearly stable equilibria, are non-BGK-like solutions that are not Landau damped to zero; rather, they evolve to undamped multiple traveling waves. (They include as a special case undamped small-amplitude multiple-wave BGK-like solutions reported previously [15].) The solutions of the second type (Sec. IV C) branch from the trivial zero-field time-asymptotic solution at a nonzero threshold and lead to nonzero final electric field states given by a nonlinear superposition of traveling-wave modes. The analysis yielded completely explicit results for the threshold Eq. (40) and for the final amplitude of the time-asymptotic field generated by initial perturbations just above the threshold Eq. (56). Interestingly, recent large-scale numerical simulations [24] have already confirmed these results, which were summarized earlier in a brief communication [18].

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APPENDIX A: RANGE OF VALIDITY OF THE TRANSIENT LINEARIZATION

To establish the range of validity of the transiently linearized Vlasov equation, Eq. (6), we consider the error introduced in replacing f_α by $\mathcal{F}_\alpha$ in Eq. (7):

$$R_\alpha(x,t)=\frac{q_\alpha}{m_\alpha}\int_{\mathbb{R}}dv\int_0^t d\tau\left\{T\left[\frac{\partial f_\alpha}{\partial v}-\frac{\partial \mathcal{F}_\alpha}{\partial v}\right]\right\}_{[x_\tau^A(x,v,t),v_\tau^A(x,v,t)]}. \tag{A1}$$

The Fourier coefficients of the "residual" become, after exchanging the order of integration, performing the area-preserving transformation of the (x,v) integration variables to $(\overline{x},\overline{v})=[x_\tau^A(x,v,t),v_\tau^A(x,v,t)]$, and integrating by parts,

$$R_{\alpha,k}(t)=\frac{q_\alpha}{m_\alpha}\frac{k}{i\pi}\int_0^t d\tau\int_{-\pi}^{+\pi}d\overline{x}\int_{\mathbb{R}}d\overline{v}\frac{\partial x^A}{\partial \overline{v}}(\overline{x},\overline{v},\tau)\times e^{-ikx^A(\overline{x},\overline{v},\tau)}T(\overline{x},\tau)[f_\alpha(\overline{x},\overline{v},\tau)-\mathcal{F}_\alpha(\overline{x},\overline{v})]. \tag{A2}$$

Here $x^A(x,v,\tau)$ is the "direct" trajectory, i.e., the position at time τ of a particle starting from the point $\overline{x},\overline{v}$ at time zero. For the multiple-wave time-asymptotic fields that we consider, the function $x^A(\overline{x},\overline{v},t)$ can be computed explicitly via the Buchanan-Dorning perturbation method [15], in which the phase plane is divided into separate regions so that the problem in each region can be reduced to motion in an autonomous, integrable system. Then the function x^A can be obtained explicitly in terms of elliptic functions and $(\partial x^A/\partial\overline{v})(\overline{x},\overline{v},t)=d_\alpha^A(\overline{x},\overline{v},t)t$ where $d_\alpha^A(\overline{x},\overline{v},t)$ is a uniformly bounded function whose detailed expression is not important here.

To estimate the order of magnitude of $R_{\alpha,k}$, we restrict the domain of the $\overline{v}$ integration to a finite interval $\Theta\subset\mathbb{R}$ that represents the "width" of the distribution functions (i.e., the temperature of the plasma). Since $\mathcal{F}_\alpha$ and f_α are assumed to have strong decay properties, Θ can be chosen so that the error introduced by the restriction of the domain is of higher order. Likewise, assuming $tT(x,t)$ to be integrable in t, there must be a positive δ such that $\forall\ x$

$$\int_0^\delta|tT(x,t)|dt \gg \int_\delta^{+\infty}|tT(x,t)|dt. \tag{A3}$$

Clearly, δ is smaller if the rate of decay of T is greater. When $t>\delta$, replacing $\mathbb{R}$ by Θ, and using Eq. (A3) and the uniform boundedness of $f_\alpha-\mathcal{F}_\alpha$ and d_a^A, Eq. (A2) can be well approximated as

$$R_{\alpha,k}(t)\sim\frac{q_\alpha}{m_\alpha}\frac{k}{i\pi}\int_0^\delta d\tau\int_{-\pi}^{+\pi}d\overline{x}\int_\Theta d\overline{v}\frac{\partial x^A}{\partial\overline{v}}(\overline{x},\overline{v},\tau)\times e^{-ikx^A(\overline{x},\overline{v},\tau)}T(\overline{x},\tau)[f_\alpha(\overline{x},\overline{v},\tau)-\mathcal{F}_\alpha(\overline{x},\overline{v})]. \tag{A4}$$

Here $f_\alpha(\overline{x},\overline{v},t)-\mathcal{F}_\alpha(\overline{x},\overline{v})=\mathcal{F}_\alpha(x_0^E,v_0^E)-\mathcal{F}_\alpha(\overline{x},\overline{v})$ where $[x_0^E(\overline{x},\overline{v}),v_0^E(\overline{x},\overline{v})]$ are the inverse trajectories (for the species α, with index omitted) associated with the total electric field E. From the Newton equations it follows immediately that, for $\tau\in(0,\delta)$,

$$|x_0^E-\overline{x}|\leqslant\delta|\overline{v}|+\frac{q_\alpha}{m_\alpha}\delta^2|E|,\quad |v_0^E-\overline{v}|\leqslant\frac{q_\alpha}{m_\alpha}\delta|E|, \tag{A5}$$

where $|E|\equiv\sup_{x,t}|E(x,t)|$. Then, from the mean value theorem

$$|f_\alpha(\overline{x},\overline{v},\tau)-\mathcal{F}_\alpha(\overline{x},\overline{v})|\leqslant\beta_{x,\alpha}\left(\delta|\overline{v}|+\frac{q_\alpha}{m_\alpha}\delta^2|E|\right)+\beta_{v,\alpha}\frac{q_\alpha}{m_\alpha}\delta|E|, \tag{A6}$$

where $\beta_{\eta,\alpha}\equiv\sup_{x\in[0,2\pi]}\sup_{v\in\Theta}|(\partial\mathcal{F}_\alpha/\partial\eta)(x,v)|,\ \ \eta=x,v$. Hence, the order of magnitude of $R_{\alpha,k}$ is

$$|R_{\alpha,k}(t)|\sim\frac{kq_\alpha}{m_\alpha}\delta^2|\Theta||T||d_\alpha^A|\left[\beta_{x,\alpha}\left(\delta|\Theta|+\frac{q_\alpha}{m_\alpha}\delta^2|E|\right)+\beta_{v,\alpha}\frac{q_\alpha}{m_\alpha}\delta|E|\right], \tag{A7}$$

where $|T|\equiv\sup_{x,t}|T(x,t)|$ and $|d_\alpha^A|\equiv\sup_{x,v,t}|d_\alpha^A(x,v,t)|$.

The same procedure applied to the approximate expression in Eq. (7)

$$Q_\alpha(x,t) \equiv \frac{q_\alpha}{m_\alpha}\int_{\mathbb{R}} dv \int_0^t d\tau \left\{ T \frac{\partial \mathcal{F}_\alpha}{\partial v} \right\}_{[x_\tau^A(x,v,t),v_\tau^A(x,v,t)]} \tag{A8}$$

leads, for the Fourier coefficients, to

$$|Q_{\alpha,k}(t)| \sim \frac{kq_\alpha}{m_\alpha}\delta^2|\Theta||T||d_\alpha^A|. \tag{A9}$$

Clearly, for Eq. (7) to be a good approximation to the full Vlasov equation, $|R_\alpha| \ll |Q_\alpha|$, or, from Eqs. (A7) and (A9),

$$\beta_{x,\alpha}\delta|\Theta| + \beta_{x,\alpha}\delta^2\frac{q_\alpha}{m_\alpha}|E| + \beta_{v,\alpha}\delta\frac{q_\alpha}{m_\alpha}|E| \ll 1 \tag{A10}$$

($\alpha = 1,\ldots,N_S$). Physically, δ is the time scale for the decay of the transient T, whereas $\beta_{x,\alpha}|\Theta|$, $(q_\alpha/m_\alpha)\beta_{x,\alpha}|E|$, and $(q_\alpha/m_\alpha)\beta_{v,\alpha}|E|$ measure the various effects that make f_α drift away from the initial condition $\mathcal{F}_\alpha$. In particular, $\beta_{x,\alpha}|\Theta|$ corresponds to the zero-field advection (in space) of the spatially nonuniform part of the initial distribution, which becomes relevant on the time scale $\tau_{ax,\alpha} \approx 1/\beta_{x,\alpha}|\Theta|$. The term $(q_\alpha/m_\alpha)\beta_{x,\alpha}|E|$ measures the action of the field E on the position of the particles and corresponds to the trapping time scale $\tau_{bx,\alpha} \approx \sqrt{m_\alpha/q_\alpha\beta_{x,\alpha}|E|}$. Finally, $(q_\alpha/m_\alpha)\beta_{v,\alpha}|E|$ expresses the deviation imposed by E on the velocity of the particles (i.e., the advection in velocity) on the time scale $\tau_{bv,\alpha} \approx m_\alpha/q_\alpha\beta_{v,\alpha}|E|$. Defining $\tau_{ax} \equiv \min_\alpha \tau_{ax,\alpha}$, $\tau_{bx} \equiv \min_\alpha \tau_{bx,\alpha}$, and $\tau_{bv} \equiv \min_\alpha \tau_{bv,\alpha}$, Eq. (A10) can be broken into the three conditions

$$\delta \ll \tau_{ax}, \quad \delta \ll \tau_{bx}, \quad \delta \ll \tau_{bv} \tag{A11}$$

for the validity of the transient linearization. They have been derived for $t > \delta$, but the development can be adapted for $t < \delta$. One obtains conditions just like those in Eqs. (A11), but with δ replaced by t; since $t < \delta$, it follows that Eqs. (A11) provide a sufficient condition for the accuracy of the transient linearization *at all times*. Finally, $\tau_{\rm trans} \ll \tau_{\rm dyn}$ follows by defining $\tau_{\rm trans} \equiv \delta$ and $\tau_{\rm dyn} \equiv \min(\tau_{ax}, \tau_{bx}, \tau_{bv})$.

In practice, τ_{ax}, τ_{bx}, and τ_{bv} (and thus $\tau_{\rm dyn}$) will usually be determined by the electrons. Interestingly, the definitions of the field-effect time scales τ_{bx} and τ_{bv} include the parameters $\beta_{x,\alpha}$ and $\beta_{v,\alpha}$, which depend, respectively, on the spatial and velocity gradients of the initial distribution, whereas O'Neil's τ_b contains the typical wave number k and therefore depends only on the spatial gradient. In most physical situations, $\beta_{v,\alpha} \sim 1$ and $\beta_{x,\alpha} \sim k|E|$; thus, all three time scales on the right sides of Eqs. (A11) are proportional to $1/|E|$. Hence, in the small-amplitude case $\tau_{\rm dyn}$ is larger than O'Neil's trapping time τ_b, which is proportional to $1/|E|^{1/2}$.

APPENDIX B: VANISHING TIME-ASYMPTOTIC FIELD SOLUTIONS

To prove that $A \equiv 0$ is a solution to Eq. (8), whether or not it can be reached from a given initial condition, we note that for $A(x,t) \equiv 0$ the $[x_\tau^A, v_\tau^A]$ are $x_\tau^0(x,v,t) = x - v(t-\tau)$, $v_\tau^0(x,v,t) = v$, and the solution to Eq. (6) is

$$f_\alpha^T(x,v,t) = \mathcal{F}_\alpha(x-vt,v) - \frac{q_\alpha}{m_\alpha}\int_0^t d\tau \left\{ T\frac{\partial \mathcal{F}_\alpha}{\partial v}\right\}_{[x-v(t-\tau),v]}. \tag{B1}$$

Under the assumption that $tT(x,t)$ is integrable at infinity, by writing the interval of integration as $[0,+\infty)$ minus $[t,+\infty)$ Eq. (B1) can be written as

$$f_\alpha^T(x,v,t) = \mathcal{F}_\alpha^T(x-vt,v) + g_\alpha^T(x,v,t), \tag{B2}$$

where $g_\alpha^T(x,v,t) \to 0$ uniformly as $t \to \infty$ and

$$\mathcal{F}_\alpha^T(x,v) \equiv \mathcal{F}_\alpha(x,v) - \frac{q_\alpha}{m_\alpha}\int_0^\infty d\tau \left\{ T\frac{\partial \mathcal{F}_\alpha}{\partial v}\right\}_{[x+v\tau,v]}. \tag{B3}$$

From Eq. (B2) it follows that, in the time-asymptotic limit, f_α^T is macroscopically equivalent to a spatially uniform Vlasov equilibrium, in the sense that there is an equilibrium $F_\alpha^T(v)$ such that f_α^T and F_α^T generate the same macroscopic quantities. To be precise, consider any integral of the form $\int_{\mathbb{R}} du\, \mathcal{G}(v,u) f_\alpha^T(x,u,t)$, which could be a charge or current density $[\mathcal{G}(v,u) = 1,u]$, or any higher moment $[\mathcal{G}(v,u) = u^n]$, or a filtered distribution function [26]. Substituting f_α^T from Eq. (B2) and taking the spatial Fourier transform, it is easy to see that, for $k \neq 0$, $\int_{\mathbb{R}} du\, \mathcal{G}(v,u) f_{\alpha,k}^T(u,t) \to 0$ as $t \to \infty$ by the Riemann-Lebesgue lemma [because $f_{\alpha,k}^T(u,t) = \mathcal{F}_{\alpha,k}^T(u) e^{ikut}$ plus terms derived from the transient functions g_α, which vanish in the time-asymptotic limit]. Hence, $\int_{\mathbb{R}} du\, \mathcal{G}(v,u) f_\alpha^T(x,u,t) \to \int_{\mathbb{R}} du\, \mathcal{G}(v,u) F_\alpha^T(u)$ as $t \to \infty$, where we have introduced the time-asymptotic equilibrium

$$F_\alpha^T(v) = \mathcal{F}_{\alpha,0}^T(v) = F_\alpha(v) - \frac{q_\alpha}{m_\alpha}\frac{1}{2\pi}\int_{-\pi}^{+\pi} dx \int_0^\infty d\tau \left\{ T\frac{\partial \mathcal{F}_\alpha}{\partial v}\right\}_{[x+v\tau,v]}. \tag{B4}$$

In particular, the charge density for f_α^T on the right side of Eq. (8) is now equal to the charge density for F_α^T. By definition, the initial Vlasov equilibrium $F_\alpha(v)$ has zero charge density, as does the other term in Eq. (B4) (this follows from using spatial periodicity to eliminate the shift vt that arises in the limits of integration in dx and an integration by parts). Hence, $F_\alpha^T(v)$ is a charge-neutral equilibrium and Eq. (8) is identically satisfied by $A \equiv 0$.

APPENDIX C: LINEARIZATION OF THE TIME-ASYMPTOTIC EQUATION

To develop the linearization of the time-asymptotic equation about $A \equiv 0$, we substitute into Eq. (5a) for f_α

$$f_\alpha(x,v,t) = \mathcal{F}_\alpha(x-vt,v) - \frac{q_\alpha}{m_\alpha}\int_0^t d\tau \left\{ A\frac{\partial f_\alpha}{\partial v}\right\}_{[x-v(t-\tau),v]} - \frac{q_\alpha}{m_\alpha}\int_0^t d\tau \left\{ T\frac{\partial \mathcal{F}_\alpha}{\partial v}\right\}_{[x-v(t-\tau),v]}, \tag{C1}$$

which is obtained by integrating the transiently linearized Vlasov equation along straight line trajectories ($A \equiv 0$). The first and third terms on the right side generate vanishing quantities by the Riemann-Lebesgue lemma; hence, Eq. (5a) becomes

$$A_k(t) = -\frac{2}{ik}\sum_\alpha \frac{q_\alpha^2}{m_\alpha} P_a \int_{-\pi}^{+\pi} dx\, e^{-ikx} \int_{\mathbb{R}} dv \times \int_0^t d\tau \left\{ A \frac{\partial f_\alpha}{\partial v} \right\}_{[x-v(t-\tau),v]}. \tag{C2}$$

By periodicity, the integrand can be shifted by $v(t-\tau)$ in the x variable; then, taking the Bohr transform of both sides yields

$$a_{k,\omega_i} = -\frac{2}{ik}\sum_\alpha \frac{q_\alpha^2}{m_\alpha} \lim_{\sigma\to\infty} \frac{1}{\sigma} \int_0^\sigma dt\, e^{i\omega_i t} \int_{-\pi}^{+\pi} dx \int_{\mathbb{R}} dv \times \int_0^t d\tau\, e^{-ikx-ikv(t-\tau)} A(x,\tau) \frac{\partial f_\alpha}{\partial v}(x,v,\tau). \tag{C3}$$

Integrating by parts in t, introducing principal value integrals, and noting the cancellation of the associated half residues gives

$$\begin{aligned} a_{k,\omega_i} = & -\frac{2}{k}\sum_\alpha \frac{q_\alpha^2}{m_\alpha} \lim_{\sigma\to\infty} \frac{1}{\sigma} \mathrm{P}\int_{\mathbb{R}} dv \frac{e^{i\omega_i \sigma}}{\omega_i - kv} \int_0^\sigma dt\, e^{-ikv(\sigma-t)} \\ & \times \int_{-\pi}^{+\pi} dx\, e^{-ikx} A(x,t) \frac{\partial f_\alpha}{\partial v}(x,v,t) \\ & + \frac{2}{k}\sum_\alpha \frac{q_\alpha^2}{m_\alpha} \lim_{\sigma\to\infty} \frac{1}{\sigma} \mathrm{P}\int_{\mathbb{R}} dv \int_0^\sigma dt \frac{e^{i\omega_i t}}{\omega_i - kv} \\ & \times \int_{-\pi}^{+\pi} dx\, e^{-ikx} A(x,t) \frac{\partial f_\alpha}{\partial v}(x,v,t). \end{aligned} \tag{C4}$$

The first term on the right side can be simplified by noting that it contains (except for vanishing terms) the right side of Eq. (C1) evaluated at $t=\sigma$. Hence,

$$\begin{aligned} a_{k,\omega_i} = & -\frac{2}{k}\sum_\alpha q_\alpha \lim_{\sigma\to\infty} \frac{1}{\sigma} \mathrm{P}\int_{\mathbb{R}} dv \frac{e^{i\omega_i \sigma}}{\omega_i - kv} f_{\alpha,k}(v,\sigma) \\ & + \frac{2}{k}\sum_\alpha \frac{q_\alpha^2}{m_\alpha} \lim_{\sigma\to\infty} \frac{1}{\sigma} \mathrm{P}\int_{\mathbb{R}} dv \int_0^\sigma dt \frac{e^{i\omega_i t}}{\omega_i - kv} \\ & \times \int_{-\pi}^{+\pi} dx\, e^{-ikx} A(x,t) \frac{\partial f_\alpha}{\partial v}(x,v,t). \end{aligned} \tag{C5}$$

The first term on the right side is zero, since $1/\sigma$ multiplies bounded functions of σ. Finally, another change in integration order yields

$$a_{k,\omega_i} = \frac{2}{k}\sum_\alpha \frac{q_\alpha^2}{m_\alpha} \lim_{\sigma\to\infty} \frac{1}{\sigma} \int_0^\sigma dt \int_{-\pi}^{+\pi} dx\, e^{-ikx+i\omega_i t} A(x,t) \times \mathrm{P}\int_{\mathbb{R}} dv \frac{f'_{\alpha,k}}{\omega_i - kv}, \tag{C6}$$

which can be linearized about $A\equiv 0$ by simply replacing $f'_\alpha(A+T)$ by its value at $A\equiv 0$. This gives the linearized time-asymptotic equation

$$a_{k,\omega_i} = \frac{2}{k}\sum_\alpha \frac{q_\alpha^2}{m_\alpha} \lim_{\sigma\to\infty} \frac{1}{\sigma} \int_0^\sigma dt \int_{-\pi}^{+\pi} dx\, e^{-ikx+i\omega_i t} A(x,t) \times \mathrm{P}\int_{\mathbb{R}} dv \frac{f^{T_0\prime}_{\alpha,k}}{\omega_i - kv}, \tag{C7}$$

where $f_\alpha^{T_0}$ is the distribution function at the critical state, Eq. (B1) (with $T=T_0$).

Substituting $f_\alpha^{T_0}$ from Eq. (B2) into Eq. (C7) yields Eq. (9) as the linearized equation since the term containing the transient $g_\alpha^{T_0}$ gives no contribution to the time average and all the spatial Fourier components of $\mathcal{F}_\alpha^{T_0}$ with $k\neq 0$ generate oscillatory terms in the principal value integral, which go to zero as $t\to\infty$ by the Riemann-Lebesgue lemma [20]. Thus, the only nonzero contribution comes from the spatially uniform part of $f_\alpha^{T_0\prime}$, i.e., the time-asymptotic equilibrium $F_\alpha^{T_0}$ in Eq. (B4).

APPENDIX D: EXPANSION OF THE TIME-ASYMPTOTIC EQUATION

The details of the analysis that results in Eq. (24), the leading-order expansion of the two-wave time-asymptotic equation, Eq. (14), in terms of a, are given here.

1. O'Neil terms

We start from the first term on the right side in Eq. (14) (O'Neil terms). To calculate the characteristics $[x_0^A, v_0^A]$, following Ref. [15], we divide the phase plane into two halves, one for each wave mode, and perform a sequence of canonical transformations that transform the dynamics in each half plane into those for a single sinusoidal wave, which can be calculated in terms of elliptic integrals. Instead of computing $[x_0^A, v_0^A]$ directly, we shall use these canonical transformations to change the integration variables, working our way backward. All such changes of variables will have Jacobian determinants equal to 1 (at least to first order in a) because the corresponding coordinate transformations are (approximately) canonical, i.e., area preserving. We first consider the half plane $v \geqslant 0$ that corresponds to the wave with $v_p = +\omega/k_0$. [We shall not explicitly indicate all the infinitesimal transformations in the lower limit of integration in v as the integration variables are transformed, because in the end all the contributions will combine into integrals on the whole v axis. We shall simply write a plus sign on top of the integration symbol, to indicate a domain of integration that corresponds to a positive v semiaxis in the (x,v) coordinates.]

The first change of variables moves the problem to the wave frame ($p=m_\alpha v$, and the α is suppressed)

$$\theta = k_0 x - \omega t, \quad J = \frac{1}{k_0}(p - m_\alpha v_p). \tag{D1}$$

Correspondingly, the "backward" values $[x_\tau^A, v_\tau^A]$ will be transformed to $[\theta_\tau^A, J_\tau^A]$, where

$$\theta_\tau^A(\theta,J,t) = k_0 x_\tau^A(x,v,t) - \omega\tau,$$

$$J_\tau^A(\theta,J,t) = \frac{m_\alpha}{k_0}[v_\tau^A(x,v,t) - v_p] \tag{D2}$$

(α suppressed). Hence, the first term on the right side of Eq. (14) becomes

$$\frac{8}{k_0}\sum_\alpha \frac{q_\alpha}{m_\alpha}\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\sigma dt\cos\omega t\int_{-k_0\pi}^{+k_0\pi}d\theta$$
$$\times\cos(\theta+\omega t)\int^+ dJ\left\{F_\alpha\left(v_p + \frac{k_0}{m_\alpha}J_0^A(\theta,J,t)\right)\right.$$
$$\left.+\epsilon h_\alpha\left(v_p+\frac{k_0}{m_\alpha}J_0^A(\theta,J,t)\right)\cos\theta_0^A(\theta,J,t)\right\}, \tag{D3}$$

where the shift $-\omega t$ in the θ integration limits has been eliminated by periodicity.

The second coordinate change is [15]

$$\bar\theta = \theta - \frac{q_\alpha k_0 a}{m_\alpha}\frac{1}{[(k_0^2/m_\alpha)J+2\omega]^2}\sin[\theta+2\omega t] + O(a^2),$$

$$\bar J = J - \frac{q_\alpha a}{k_0}\frac{1}{[(k_0^2/m_\alpha)J+2\omega]}\cos[\theta+2\omega t]+O(a^2). \tag{D4}$$

For $\cos\theta_\tau^A$ and J_τ^A, the analog of Eq. (D2) is found by evaluating the inverse equations at $t=\tau$,

$$\cos\theta_\tau^A = \cos\bar\theta_\tau^A - \frac{q_\alpha a}{k_0 m_\alpha}\frac{1}{[2v_p+(k_0/m_\alpha)\bar J_\tau^A]^2}$$
$$\times\sin\bar\theta_\tau^A\sin[\bar\theta_\tau^A+2\omega\tau]+O(a^2), \tag{D5}$$

$$J_\tau^A = \bar J_\tau^A + \frac{q_\alpha a}{k_0^2}\frac{1}{[2v_p+(k_0/m_\alpha)\bar J_\tau^A]}\cos[\bar\theta_\tau^A+2\omega\tau]+O(a^2). \tag{D6}$$

Then Eq. (D3) becomes [with error $O(a^2)$]

$$\frac{8}{k_0}\sum_\alpha\frac{q_\alpha}{m_\alpha}\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\sigma dt\cos\omega t\int_{-k_0\pi}^{+k_0\pi}d\bar\theta\int^+ d\bar J\cos\left(\bar\theta+\omega t+\frac{q_\alpha a}{k_0 m_\alpha}\frac{\sin[\bar\theta+2\omega t]}{[2v_p+(k_0/m_\alpha)\bar J]^2}\right)$$
$$\times\left\{F_\alpha(v_p+(k_0/m_\alpha)\bar J_0^A)+\epsilon h_\alpha(v_p+(k_0/m_\alpha)\bar J_0^A)\cos\bar\theta_0^A - \frac{q_\alpha\epsilon a}{k_0 m_\alpha}h_\alpha(v_p+(k_0/m_\alpha)\bar J_0^A)\frac{\sin^2\bar\theta_0^A}{[2v_p+(k_0/m_\alpha)\bar J_0^A]^2}\right.$$
$$\left.+\frac{q_\alpha\epsilon a}{k_0 m_\alpha}\frac{dh_\alpha}{dv}(v_p+(k_0/m_\alpha)\bar J_0^A)\frac{\cos^2\bar\theta_0^A}{[2v_p+(k_0/m_\alpha)\bar J_0^A]}+\frac{q_\alpha a}{k_0 m_\alpha}\frac{dF_\alpha}{dv}(v_p+(k_0/m_\alpha)\bar J_0^A)\frac{\cos\bar\theta_0^A}{[2v_p+(k_0/m_\alpha)\bar J_0^A]}\right\}. \tag{D7}$$

Here, $(\bar\theta_0^A, \bar J_0^A)$ (with subscripts corresponding to $\tau=0$) have arguments $(\bar\theta,\bar J,t)$; we have also ignored the effect of Eq. (D4) on the limits of the integration in space (since there also is periodicity in $\bar\theta$).

In $(\bar\theta,\bar J)$ the Hamiltonian is [15]

$$\bar H(\bar J,\bar\theta) \equiv \frac{k_0^2}{2m_\alpha}\bar J^2 + \frac{q_\alpha a}{k_0}\cos\bar\theta = E, \tag{D8}$$

which corresponds to particle motion in a single sinusoidal wave with amplitude $A=q_\alpha a$. When $q_\alpha<0$ this is the Hamiltonian for a nonlinear pendulum. When $q_\alpha>0$ it is the Hamiltonian for an upside-down pendulum, corresponding to the fact that the positive particles oscillate in the downward trough of the wave. Clearly, the Hamiltonian with $q_\alpha>0$ is transformed into the Hamiltonian with $q_\alpha<0$ simply by shifting the spatial variable $\bar\theta$ by π. Hence, we shall study only the case $q_\alpha<0$, and transform $\bar\theta$ to $\bar\theta+\pi$ in the $q_\alpha>0$ terms in Eq. (D7). For $q_\alpha<0$ the trajectories $[\bar\theta_0^A(\bar\theta,\bar J,t),\bar J_0^A(\bar\theta,\bar J,t)]$ that correspond to the Hamiltonian in Eq. (D8) can be obtained explicitly [33] as

$$\bar\theta_0^A(\bar\theta,\bar J,t) = 2\,\mathrm{am}\left[F\left(\frac{\bar\theta}{2}\middle| m\right) - \frac{t}{\kappa\tau_b}, m\right], \tag{D9}$$

$$\bar J_0^A(\bar\theta,\bar J,t) = \frac{2m_\alpha}{k_0^2}\frac{1}{\tau_b\kappa}\mathrm{dn}\left[\frac{t}{\kappa\tau_b} - F\left(\frac{\bar\theta}{2}\middle| m\right), m\right], \tag{D10}$$

where $F(y|m)$ is the incomplete elliptic integral of the first kind and

$$\tau_b = \sqrt{m_\alpha/|q_\alpha|k_0 a}, \quad m=\kappa^2 = \frac{a}{(k_0^3/4|q_\alpha|m_\alpha)\bar J^2 + a\sin^2\bar\theta/2}. \tag{D11}$$

By definition, κ takes the same sign as $\bar{J}$. The inverse trajectories depend on the particle species via τ_b, κ, and m; however, we have suppressed the α indices. Using $\bar{\theta}\to\bar{\theta}+\pi$ when $q_\alpha>0$ in Eq. (D7) yields

$$\frac{8}{k_0}\sum_\alpha \frac{s_\alpha q_\alpha}{m_\alpha}\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\sigma dt\cos\omega t\int_{-k_0\pi}^{+k_0\pi}d\bar{\theta}\int^{+}d\bar{J}\cos\left\{\bar{\theta}+\omega t+\frac{s_\alpha q_\alpha a}{k_0 m_\alpha}\frac{\sin[\bar{\theta}+2\omega t]}{[2v_p+(k_0/m_\alpha)\bar{J}]^2}\right\}$$
$$\times\left\{F_\alpha(v_p+(k_0/m_\alpha)\bar{J}_0^A)+\epsilon s_\alpha h_\alpha(v_p+(k_0/m_\alpha)\bar{J}_0^A)\cos\bar{\theta}_0^A-\frac{q_\alpha\epsilon a}{k_0 m_\alpha}h_\alpha(v_p+(k_0/m_\alpha)\bar{J}_0^A)\frac{\sin^2\bar{\theta}_0^A}{[2v_p+(k_0/m_\alpha)\bar{J}_0^A]^2}\right.$$
$$\left.+\frac{q_\alpha\epsilon a}{k_0 m_\alpha}\frac{dh_\alpha}{dv}(v_p+(k_0/m_\alpha)\bar{J}_0^A)\frac{\cos^2\bar{\theta}_0^A}{[2v_p+(k_0/m_\alpha)\bar{J}_0^A]}+\frac{s_\alpha q_\alpha a}{k_0 m_\alpha}\frac{dF_\alpha}{dv}(v_p+(k_0/m_\alpha)\bar{J}_0^A)\frac{\cos\bar{\theta}_0^A}{[2v_p+(k_0/m_\alpha)\bar{J}_0^A]}\right\},\tag{D12}$$

where $s_\alpha\equiv-q_\alpha/|q_\alpha|$ mark the terms whose sign changes for $q_\alpha>0$ and the shifted quantities are still denoted by $\bar{\theta},\bar{\theta}_0^A$. When we substitute Eqs. (D9) and (D10) into Eq. (D12) we obtain a completely explicit expression.

Even though it appears quite complicated, Eq. (D12) can be simplified by exploiting the time-asymptotic properties of the inverse phase flow $[\bar{\theta}_0(\bar{\theta},\bar{J},t),\bar{J}_0(\bar{\theta},\bar{J},t)]$. Since the potential well in Eq. (D8) is not harmonic, particles with different energies oscillate at different frequencies and "mix up" the initial distribution. At long times, as filamentation grows, we can expect the $\bar{\theta}$-$\bar{J}$ integration in Eq. (D12) to average away all the high-frequency terms and leave only some coarse-grained component. In practice, after performing a straightforward expansion of the trigonometric terms appearing in the first line, Eq. (D12) can be written as a linear combination of terms of the form

$$\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\sigma dt\,w(t)\int_{-k_0\pi}^{+k_0\pi}d\bar{\theta}$$
$$\times\int^{+}d\bar{J}\,\mathcal{K}(\bar{\theta},\bar{J})\mathcal{G}_0[\bar{\theta}_0(\bar{\theta},\bar{J},t),\bar{J}_0(\bar{\theta},\bar{J},t)],\tag{D13}$$

where w and $\mathcal{K}$ are continuous, bounded, and periodic in t and $\bar{\theta}$, respectively, and

$$\mathcal{G}_0(\bar{\theta},\bar{J})\equiv F_\alpha(v_p+(k_0/m_\alpha)\bar{J})+\epsilon s_\alpha h_\alpha(v_p+(k_0/m_\alpha)\bar{J})\cos\bar{\theta}-\frac{q_\alpha\epsilon a}{k_0 m_\alpha}h_\alpha(v_p+(k_0/m_\alpha)\bar{J})\frac{\sin^2\bar{\theta}}{[2v_p+(k_0/m_\alpha)\bar{J}]^2}$$
$$+\frac{q_\alpha\epsilon a}{k_0 m_\alpha}\frac{dh_\alpha}{dv}(v_p+(k_0/m_\alpha)\bar{J})\frac{\cos^2\bar{\theta}}{[2v_p+(k_0/m_\alpha)\bar{J}]}+\frac{s_\alpha q_\alpha a}{k_0 m_\alpha}\frac{dF_\alpha}{dv}(v_p+(k_0/m_\alpha)\bar{J})\frac{\cos\bar{\theta}}{[2v_p+(k_0/m_\alpha)\bar{J}]}.\tag{D14}$$

Then the following general result applies.

Proposition 1. Given an autonomous, periodic, and integrable one-degree-of-freedom system with "inverse characteristics" $[\bar{\theta}_\tau(\bar{\theta},\bar{J},t),\bar{J}_\tau(\bar{\theta},\bar{J},t)]$, and two functions $\mathcal{K}$ and $\mathcal{G}_0$ as in Eq. (D13),

$$\lim_{t\to\infty}\int_{-k_0\pi}^{+k_0\pi}d\bar{\theta}\int_{\bar{\Omega}}d\bar{J}\,\mathcal{K}(\bar{\theta},\bar{J})\mathcal{G}_0[\bar{\theta}_0(\bar{\theta},\bar{J},t),\bar{J}_0(\bar{\theta},\bar{J},t)]$$
$$=\int_{-k_0\pi}^{+k_0\pi}d\bar{\theta}\int_{\bar{\Omega}}d\bar{J}\,\mathcal{K}(\bar{\theta},\bar{J})\bar{\mathcal{G}}_0(\bar{\theta},\bar{J}),\tag{D15}$$

where $\bar{\Omega}$ is an interval in $\mathbb{R}$ and $\bar{\mathcal{G}}_0(\bar{\theta},\bar{J})$ is the average in phase space of the function $\mathcal{G}_0(\bar{\theta},\bar{J})$ along the curves of constant energy of the system.

This is shown easily [33] by transforming to action-angle variables and invoking the Riemann-Lebesgue lemma. Equation (D15) and Fréchet's lemma [28] imply that Eq. (D13) can be rewritten as

$$\int_{-k_0\pi}^{+k_0\pi}d\bar{\theta}\int^{+}d\bar{J}\,\mathcal{K}(\bar{\theta},\bar{J})\bar{\mathcal{G}}_0(\bar{\theta},\bar{J})\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\sigma dt\,w(\omega t).\tag{D16}$$

Then, tedious manipulations reduce Eq. (D12) to

$$\frac{4}{k_0}\sum_\alpha \frac{s_\alpha q_\alpha}{m_\alpha}\int_{-k_0\pi}^{+k_0\pi} d\overline{\theta}\int^{+} d\overline{J} \times\left\{\cos\overline{\theta}-\frac{s_\alpha q_\alpha a}{2k_0 m_\alpha}\frac{1}{[2v_p+(k_0/m_\alpha)\overline{J}]^2}\right\}\overline{\mathcal{G}_0}(\overline{\theta},\overline{J}). \tag{D17}$$

The terms in Eq. (D17) that depend on the asymptotic field amplitude a only through the averaging process are

$$\frac{4}{k_0}\sum_\alpha \frac{s_\alpha q_\alpha}{m_\alpha}\int_{-k_0\pi}^{+k_0\pi} d\overline{\theta}\int^{+} d\overline{J}\cos\overline{\theta}\overline{\mathcal{H}_0}(\overline{\theta},\overline{J}), \tag{D18}$$

where $\overline{\mathcal{H}_0}(\overline{\theta},\overline{J})$ is obtained by averaging

$$\mathcal{H}_0(\overline{\theta},\overline{J})\equiv F_\alpha\left(v_p+\frac{k_0}{m_\alpha}\overline{J}\right)+\epsilon s_\alpha h_\alpha\left(v_p+\frac{k_0}{m_\alpha}\overline{J}\right)\cos\overline{\theta} \tag{D19}$$

on the pendulum energy levels. These expressions correspond to particle dynamics *in a single sinusoidal wave*, while the other terms in Eq. (D17), temporarily set aside, represent the effects of the second wave, with phase velocity $-\omega/k_0$. To compute the phase-plane averages in Eq. (D18), we transform from $(\overline{\theta},\overline{J})$ to the action-angle variables $(\tilde{\theta},\tilde{J})$ for the nonlinear pendulum, where the averaging can be performed more easily. For $m<1$ (untrapped particles), we define $\xi\equiv\overline{\theta}/2$; then $\tilde{\theta}=[\pi/K(m)]F(\xi|m)$ and $\tilde{J}=(4/\pi)(m_\alpha/k^2)\times(1/\tau_b\kappa)E(m)$, where K and E are the complete elliptic integrals of the first and second kinds, respectively. Hence,

$$\xi=\mathrm{am}\left[\frac{\tilde{\theta}K(m)}{\pi},m\right],\quad \cos\overline{\theta}=\cos 2\xi=1-2\,\mathrm{sn}^2\left[\frac{\tilde{\theta}K(m)}{\pi},m\right] \tag{D20}$$

and

$$\overline{J}=\frac{2m_\alpha}{k_0^2}\dot{\xi}=\frac{2m_\alpha}{k_0^2}\frac{1}{\tau_b\kappa}\sqrt{1-\kappa^2\sin^2\xi} =\frac{2m_\alpha}{k_0^2}\frac{1}{\tau_b\kappa}\mathrm{dn}\left[\frac{\tilde{\theta}K(m)}{\pi},m\right] \tag{D21}$$

where m, κ, and τ_b were defined in Eq. (D11) and $\dot{\xi}=(1/\tau_b\kappa)\sqrt{1-\kappa^2\sin^2\xi}$ follows from the conservation of energy. Then,

$$\overline{\mathcal{H}_0}(\overline{\theta},\overline{J})=\frac{1}{2\pi}\int_0^{2\pi}d\tilde{\theta}\left\{F_\alpha\left(v_p+\frac{2}{k_0}\frac{1}{\tau_b\kappa}\mathrm{dn}[\tilde{\theta}K(m)/\pi,m]\right)+\epsilon s_\alpha h_\alpha\left(v_p+\frac{2}{k_0}\frac{1}{\tau_b\kappa}\mathrm{dn}[\tilde{\theta}K(m)/\pi,m]\right)\times\{1-2\,\mathrm{sn}^2[\tilde{\theta}K(m)/\pi,m]\}\right\}. \tag{D22}$$

For $m>1$ we define ζ such that $\sin\overline{\zeta}=\kappa\sin\xi$; then the action-angle variables are [33] $\tilde{\theta}=[\pi/2K(1/m)]F(\zeta|1/m)$, $\tilde{J}=(8/\pi)(m_\alpha/\tau_b k^2)[E(1/m)-(1-1/m)K(1/m)]$. Hence,

$$\zeta=\mathrm{am}\left[K\left(\frac{1}{m}\right)\frac{2\tilde{\theta}}{\pi},\frac{1}{m}\right],$$

$$\cos\overline{\theta}=\cos 2\xi=1-\frac{2}{\kappa^2}\sin^2\zeta=1-\frac{2}{\kappa^2}\mathrm{sn}^2\left[K\left(\frac{1}{m}\right)\frac{2\tilde{\theta}}{\pi},\frac{1}{m}\right] \tag{D23}$$

and

$$\overline{J}=\frac{2m_\alpha}{k_0^2}\frac{1}{\tau_b\kappa}\sqrt{1-\kappa^2\sin^2\xi} =\frac{2m_\alpha}{k_0^2}\frac{1}{\tau_b\kappa}\cos\zeta =\frac{2m_\alpha}{k_0^2}\frac{1}{\tau_b\kappa}\mathrm{cn}\left[K\left(\frac{1}{m}\right)\frac{2\tilde{\theta}}{\pi},\frac{1}{m}\right]. \tag{D24}$$

Then

$$\overline{\mathcal{H}_0}(\overline{\theta},\overline{J})=\frac{1}{2\pi}\int_0^{2\pi}d\tilde{\theta}\left\{F_\alpha\left(v_p+\frac{2}{k_0}\frac{1}{\tau_b\kappa}\mathrm{cn}\left[K\left(\frac{1}{m}\right)\frac{2\tilde{\theta}}{\pi},\frac{1}{m}\right]\right)+\epsilon s_\alpha h_\alpha\left(v_p+\frac{2}{k_0}\frac{1}{\tau_b\kappa}\mathrm{cn}\left[K\left(\frac{1}{m}\right)\frac{2\tilde{\theta}}{\pi},\frac{1}{m}\right]\right)\times\left(1-\frac{2}{\kappa^2}\mathrm{sn}^2\left[K\left(\frac{1}{m}\right)\frac{2\tilde{\theta}}{\pi},\frac{1}{m}\right]\right)\right\}. \tag{D25}$$

Rescaling the integration variables in Eqs. (D22) and (D25) and substituting into Eq. (D18) yields

$$\frac{4}{k_0}\sum_\alpha \frac{s_\alpha q_\alpha}{m_\alpha}\int_{-k_0\pi}^{+k_0\pi} d\overline{\theta}\int_{m(\overline{J},\overline{\theta})<1}^{+} d\overline{J}\cos\overline{\theta} \times\frac{1}{2K(m)}\int_0^{2K(m)}dz\left\{F_\alpha\left(v_p+\frac{2}{k_0}\frac{1}{\tau_b\kappa}\mathrm{dn}[z,m]\right)+\epsilon s_\alpha h_\alpha\left(v_p+\frac{2}{k_0}\frac{1}{\tau_b\kappa}\mathrm{dn}[z,m]\right)(1-2\,\mathrm{sn}^2[z,m])\right\} \tag{D26}$$

for $m<1$, and for $m>1$

$$\frac{4}{k_0}\sum_\alpha \frac{s_\alpha q_\alpha}{m_\alpha}\int_{-k_0\pi}^{+k_0\pi} d\overline{\theta}\int_{m(\overline{J},\overline{\theta})>1}^{+} d\overline{J}\cos\overline{\theta} \times\frac{1}{4K(1/m)}\int_0^{4K(1/m)}dz\left\{F_\alpha\left(v_p+\frac{2}{k_0}\frac{1}{\tau_b\kappa}\mathrm{cn}\left[z,\frac{1}{m}\right]\right)+\epsilon s_\alpha h_\alpha\left(v_p+\frac{2}{k_0}\frac{1}{\tau_b\kappa}\mathrm{cm}\left[z,\frac{1}{m}\right]\right)\left(1-\frac{2}{\kappa^2}\mathrm{sn}^2\left[z,\frac{1}{m}\right]\right)\right\}. \tag{D27}$$

To expand these quantities in powers of a, which appears in τ_b, κ, and m [Eq. (D11)], we shall break the velocity integrals into regions that correspond to different types of wave-particle interactions, distinguishing between resonant and nonresonant particles based on the size of the coefficient $(\tau_b\kappa)^{-1}$ in the arguments of F_α and h_α. Here $1/\tau_b$ is of order $\sqrt{a}$, whereas $1/|\kappa|$ can vary from 0 at the bottom of the wave's potential well to 1 on the separatrix to ∞ far from the wave in the phase plane. For $1/|\kappa|\lesssim 1$, we perform "resonant" expansions of F_α and h_α around v_p (resonant particles). But when $1/|\kappa|$ becomes larger, the coefficient $(\tau_b\kappa)^{-1}$ is no longer small and we must find a way to expand the functions around the free-streaming particle velocity instead of v_p. This will be the case of nonresonant particles.

a. Resonant particles

The resonant region will be further divided into two subregions, corresponding to the untrapped $(m<1)$ and trapped $(m>1)$ resonant particles.

a1. Resonant untrapped particles $(m<1)$. We shall consider as resonant the phase region such that $1>|\kappa|>a^{1/4}$, i.e., $1/|\tau_b\kappa|<a^{1/4}$. Then Taylor expanding F_α and h_α about v_p in Eq. (D26) gives

$$\frac{4}{k_0}\sum_\alpha \frac{s_\alpha q_\alpha}{m_\alpha}\int_{-k_0\pi}^{+k_0\pi}d\overline{\theta}\int_{I_1}d\overline{J}\cos\overline{\theta}\frac{1}{2K(m)}\int_0^{2K(m)}dz \times\sum_{j=0}^{\infty}\left\{\left[\frac{2}{k_0}\frac{1}{\tau_b\kappa}\right]^j\frac{1}{j!}\frac{d^jF_\alpha}{dv^j}(v_p)\mathrm{dn}^j[z,m] +\epsilon s_\alpha\left[\frac{2}{k_0}\frac{1}{\tau_b\kappa}\right]^j\frac{1}{j!}\frac{d^jh_\alpha}{dv^j}(v_p)\mathrm{dn}^j[z,m] \times(1-2\,\mathrm{sn}^2[z,m])\right\}, \tag{D28}$$

where

$$I_1\equiv\left(-a^{1/4},-\frac{2m_\alpha}{k_0^2\tau_b}\cos\frac{\overline{\theta}}{2}\right)\cup\left(+\frac{2m_\alpha}{k_0^2\tau_b}\cos\frac{\overline{\theta}}{2},+a^{1/4}\right).$$

All terms with j odd are odd functions of $\overline{J}$, since they are odd in κ, and κ has the same sign as $\overline{J}$. Hence, since I_1 is symmetric, these terms vanish. For n even both the $A_n(m)$ and the $B_n(m)$ can be calculated via standard recursive formulas [33]. In Eq. (D28) the small parameter a appears not only in $1/\tau_b$ but also in κ and m (in the product $\tau_b^2\overline{J^2}$). To deal with this we rescale $\overline{J}$ by $2m_\alpha/k_0^2\tau_b$ (and still use $\overline{J}$, κ, and m for the transformed quantities) and arrive [33] at the leading-order terms in Eq. (D28) as

$$\frac{8}{k_0^3}\sum_\alpha s_\alpha q_\alpha\int_{-k_0\pi}^{+k_0\pi}d\overline{\theta}\int_{\tilde{I}_1}d\overline{J}\cos\overline{\theta}\left\{\frac{2}{k_0^2}\frac{1}{\kappa^2\tau_b^3}\frac{E(m)}{K(m)}\frac{d^2F_\alpha}{dv^2}(v_p)+\frac{\epsilon s_\alpha}{\tau_b}\left[1+\frac{2}{m}\left(\frac{E(m)}{K(m)}-1\right)\right]h_\alpha(v_p) +\epsilon s_\alpha\frac{2}{k_0^2}\frac{1}{\kappa^2\tau_b^3}\frac{1}{3}\left[1+\left(\frac{2}{m}-1\right)\left(\frac{E(m)}{K(m)}-1\right)\right]\frac{d^2h_\alpha}{dv^2}(v_p)\right\}, \tag{D29}$$

where $E(m)$ is the complete elliptic integral of the second kind, now $m\equiv\kappa^2=[\overline{J}^2+\sin^2(\overline{\theta}/2)]^{-1}$ and $\tilde{I}_1\equiv(-a^{-1/4},-\cos(\overline{\theta}/2))\cup(+\cos(\overline{\theta}/2),+a^{-1/4})$. Now, κ and m do not depend on a, and the integrand in Eq. (D26) has been written as a series expansion in powers of τ_b^{-1}, i.e., powers of $a^{1/2}$.

The only inconvenience is the powers $a^{-1/4}$ that appear in $\tilde{I}_1$. Let us add and subtract two terms and add a term that integrates to zero in $\overline{\theta}$ to rewrite Eq. (D29) as

$$\frac{8}{k_0^3}\sum_\alpha s_\alpha q_\alpha\int_{-k_0\pi}^{+k_0\pi}d\overline{\theta}\int_{\tilde{I}_1}d\overline{J}\cos\overline{\theta}\left\{\frac{1}{k_0^2}\frac{1}{\tau_b^3}\left[1+\frac{2}{m}\left(\frac{E(m)}{K(m)}-1\right)\right]\frac{d^2F_\alpha}{dv^2}(v_p)+\frac{1}{k_0^2}\frac{1}{\tau_b^3}\left(\frac{2}{m}-1\right)\frac{d^2F_\alpha}{dv^2}(v_p) +\frac{\epsilon s_\alpha}{\tau_b}\left[1+\frac{2}{m}\left(\frac{E(m)}{K(m)}-1\right)\right]h_\alpha(v_p)+\epsilon s_\alpha\frac{1}{k_0^2}\frac{1}{\tau_b^3}\left[\frac{2}{3m}\left[1+\left(\frac{2}{m}-1\right)\left(\frac{E(m)}{K(m)}-1\right)\right]-\frac{1}{4}\right]\frac{d^2h_\alpha}{dv^2}(v_p)\right\}. \tag{D30}$$

All the functions in the square brackets are integrable as $|\overline{J}|\to\infty$, i.e., $|m|\to 0$, since $m\sim\overline{J}^{-2}$ [see [34], 17.3.11 and 17.3.12]; in the last term integrability was obtained by subtracting the spatially uniform term $\epsilon\,[1/(4k_0^2\tau_b^3)](d^2h_\alpha/dv^2)(v_p)$, which vanishes under the $\overline{\theta}$ integration]. These terms will combine with corresponding terms from the nonresonant particle region, allowing us to extend the domain of integration for these functions from $\tilde{I}_1$ to $\overline{I}_1\equiv(-\infty,-\cos(\overline{\theta}/2))\cup(+\cos(\overline{\theta}/2),+\infty)$. For the remaining term in Eq. (D30), if we rescale $\overline{J}$ back to its original definition and substitute the original expression for m from Eq. (D11), the part depending on $\overline{J}$ cancels under the $\overline{\theta}$ integration and (noting that $s_\alpha q_\alpha |q_\alpha| = -q_\alpha^2$) we have

$$-a\frac{4}{k_0^2}\sum_\alpha \frac{q_\alpha^2}{m_\alpha^2}\int_{-k_0\pi}^{+k_0\pi} d\overline{\theta}\int_{I_1} d\overline{J}\cos\overline{\theta}\left(2\sin^2\frac{\overline{\theta}}{2}-1\right)\frac{d^2F_\alpha}{dv^2}(v_p)$$

$$=a\frac{4\pi}{k_0}\sum_\alpha \frac{q_\alpha^2}{m_\alpha^2}\int_{I_1} d\overline{J}\frac{d^2F_\alpha}{dv^2}(v_p)=a\frac{4\pi}{k_0^2}\sum_\alpha \frac{q_\alpha^2}{m_\alpha}\mathrm{P}\int_{I_1} d\overline{J}\frac{1}{\overline{J}}\left\{\frac{dF_\alpha}{dv}(v_p)+\frac{d^2F_\alpha}{dv^2}(v_p)\frac{k_0}{m_\alpha}\overline{J}+\frac{1}{2}\frac{d^3F_\alpha}{dv^3}(v_p)\left[\frac{k_0}{m_\alpha}\overline{J}\right]^2\right\}$$

$$=a\frac{4\pi}{k_0^2}\sum_\alpha \frac{q_\alpha^2}{m_\alpha}\mathrm{P}\int_{I_1} d\overline{J}\frac{1}{\overline{J}}F'_\alpha\left(v_p+\frac{k_0}{m_\alpha}\overline{J}\right)+O(a^{7/4}), \tag{D31}$$

where the first and third terms in the braces (each of which integrates to zero) have been added to form the truncated Taylor series of $F'_\alpha(v_p+(k_0/m_\alpha)\overline{J})$. The last line yields, at leading order, the Vlasov dispersion integral (in the wave frame) restricted to the resonant region. As such, it will combine naturally with corresponding integrals from the other regions in the phase plane.

a2. Trapped particles $(m>1)$. All the trapped particles are resonant and for them $|1/\tau_b\kappa|\ll 1$ always. Thus, we can expand F_α and h_α in Eq. (D27) about v_p, just as in case a1. The analog of Eq. (D28) then is

$$\frac{4}{k_0}\sum_\alpha \frac{s_\alpha q_\alpha}{m_\alpha}\int_{-k_0\pi}^{+k_0\pi} d\tilde{\theta}\int_{I_2} d\overline{J}\frac{\cos\overline{\theta}}{4K(1/m)}\int_0^{4K(1/m)} dz$$
$$\times\sum_{j=0}^{\infty}\left\{\left[\frac{2}{k_0}\frac{1}{\tau_b\kappa}\right]^j\frac{1}{j!}\frac{d^jF_\alpha}{dv^j}(v_p)\mathrm{cn}^j\left[z,\frac{1}{m}\right]\right.$$
$$+\epsilon s_\alpha\left[\frac{2}{k_0}\frac{1}{\tau_b\kappa}\right]^j\frac{1}{j!}\frac{d^jh_\alpha}{dv^j}(v_p)\mathrm{cn}^j\left[z,\frac{1}{m}\right]$$
$$\left.\times\left(1-\frac{2}{\kappa^2}\mathrm{sn}^2\left[z,\frac{1}{m}\right]\right)\right\}, \tag{D32}$$

where $I_2\equiv(-(2m_\alpha/k_0^2\tau_b)\cos(\overline{\theta}/2),+(2m_\alpha/k_0^2\tau_b)\cos(\overline{\theta}/2))$. As in case a1, all the terms with odd j vanish, and the others can be calculated following the same steps. Rescaling $\overline{J}$ by $2m_\alpha/k_0^2\tau_b$, redefining m as in case a1, and introducing the modified domain $\tilde{I}_2\equiv(-\cos(\overline{\theta}/2),+\cos(\overline{\theta}/2))$ yields at leading order

$$\frac{8}{k_0^3}\sum_\alpha s_\alpha q_\alpha\int_{-k_0\pi}^{+k_0\pi} d\overline{\theta}\int_{\tilde{I}_2} d\overline{J}\cos\overline{\theta}\left\{\frac{2}{k_0^2}\frac{1}{\tau_b^3}\right.$$
$$\times\left[\frac{K(1/m)}{K(1/m)}-\frac{m-1}{m}\right]\frac{d^2F_\alpha}{dv^2}(v_p)+\frac{\epsilon s_\alpha}{\tau_b}\left[2\frac{E(1/m)}{K(1/m)}-1\right]$$
$$\times h_\alpha(v_p)+\epsilon s_\alpha\frac{1}{k_0^2}\frac{1}{\tau_b^3}\left\{\frac{2}{3}\left[\frac{1}{m}+\left(\frac{2}{m}-1\right)\left(\frac{E(1/m)}{K(1/m)}-1\right)\right]\right.$$
$$\left.\left.-\frac{1}{4}\right\}\frac{d^2h_\alpha}{dv^2}(v_p)\right\}, \tag{D33}$$

where we have introduced the same constant factor in the last term that was added to the corresponding term in Eq. (D30) to make it integrable in $\overline{J}$. This spatially uniform correction, of course, vanishes under the $\overline{\theta}$ integration. In Eq. (D34) we can write

$$\frac{2}{k_0^2}\frac{1}{\tau_b^3}\left[\frac{E(1/m)}{K(1/m)}-\frac{m-1}{m}\right]$$
$$=\frac{1}{k_0^2}\frac{1}{\tau_b^3}\left[2\frac{E(1/m)}{K(1/m)}-1\right]+\frac{1}{k_0^2}\frac{1}{\tau_b^3}\left(\frac{2}{m}-1\right), \tag{D34}$$

where the first term connects continuously with the corresponding quantity in Eq. (D30), and the second generates the trapped particle contribution to the Vlasov dispersion integral [as in Eq. (D31)].

b. Nonresonant particles

We also must consider the case $|\kappa|<a^{1/4}$, i.e., $1/|\tau_b\kappa| > a^{1/4}$. Since $1/|\tau_b\kappa|$ can become arbitrarily large as the distance from the wave in phase space increases, Taylor expansions around v_p are not appropriate. Instead, since $m=\kappa^2 < \sqrt{a}$, we expand the elliptic function dn, which enables us to expand F_α and h_α around the free-streaming velocity $v_p + (k_0/m_\alpha)\overline{J}$ [33] to obtain

$$a\frac{4\pi}{k_0^2}\sum_\alpha \frac{q_\alpha^2}{m_\alpha}\int_{I_3}^{+} d\overline{J}\frac{1}{\overline{J}}\frac{dF_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\overline{J}\right) \tag{D35}$$

as the leading-order approximation to Eq. (D26). Equation (D35) is the contribution from the nonresonant particles to the Vlasov dispersion integral.

Combining Eqs. (D30), (D34), and (D35) yields the leading-order terms in the expansion of the single-mode O'Neil terms:

$$-a^{3/2}\sigma_1\sum_\alpha\left[\frac{|q_\alpha|^5}{k_0^5m_\alpha^3}\right]^{1/2}\frac{d^2F_\alpha}{dv^2}(v_p)$$
$$+\epsilon a^{1/2}\sigma_1\sum_\alpha q_\alpha\left[\frac{|q_\alpha|}{k_0^3m_\alpha}\right]^{1/2}h_\alpha(v_p)$$
$$+a\frac{4\pi}{k_0^2}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\int^{+}d\overline{J}\frac{1}{\overline{J}}\frac{dF_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\overline{J}\right)$$
$$+\epsilon a^{3/2}\sigma_2\sum_\alpha q_\alpha\left[\frac{|q_\alpha|^3}{k_0^5m_\alpha^3}\right]^{1/2}\frac{d^2h_\alpha}{dv^2}(v_p), \qquad \text{(D36)}$$

where $\sigma_n\equiv 8\int_{-\pi}^{+\pi}d\overline{\theta}\int_{\mathrm{R}}d\overline{J}\cos\overline{\theta}g_n(\overline{\theta,J})$ and the integrals in $\overline{\theta}$ over $[-k_0\pi,+k_0\pi]$ have been written as k_0 times integrals over $[-\pi,+\pi]$. The functions g_1 and g_2 are

$$g_1(\overline{\theta,J})\equiv\begin{cases}1+\dfrac{2}{m}\left(\dfrac{E(m)}{K(m)}-1\right), & m<1,\\[2ex] 2\dfrac{E(1/m)}{K(1/m)}-1, & m>1,\end{cases} \qquad \text{(D37)}$$

$$g_2(\overline{\theta,J})\equiv\begin{cases}\dfrac{2}{3m}\left[1+\left(\dfrac{2}{m}-1\right)\left(\dfrac{E(m)}{K(m)}-1\right)\right]-\dfrac{1}{4}, & m<1,\\[2ex] \dfrac{2}{3m}\left[1+(2-m)\left(\dfrac{E(1/m)}{K(1/m)}-1\right)\right]-\dfrac{1}{4}, & m>1,\end{cases} \qquad \text{(D38)}$$

and $m\equiv\kappa^2=[\overline{J}^2+\sin^2(\overline{\theta}/2)]^{-1}$. Numerical integration yields $\sigma_1=8\times 2.58=20.67$ and $\sigma_2=8\times 0.066=0.53$.

We still must consider the terms in Eq. (D17) that correspond to the interaction between the wave under consideration and the "other" wave:

$$\frac{4}{k_0}\sum_\alpha\frac{s_\alpha q_\alpha}{m_\alpha}\int_{-k_0\pi}^{+k_0\pi}d\overline{\theta}\int^{+}d\overline{J}\Bigg\{\cos\overline{\theta}\overline{\left[\frac{s_\alpha q_\alpha a}{k_0m_\alpha}\frac{dF_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\overline{J}\right)\frac{\cos\overline{\theta}}{[2v_p+(k_0/m_\alpha)\overline{J}]}\right.}$$
$$\overline{\left.+\frac{q_\alpha\epsilon a}{k_0m_\alpha}\frac{dh_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\overline{J}\right)\frac{\cos^2\overline{\theta}}{[2v_p+(k_0/m_\alpha)\overline{J}]}-\frac{q_\alpha\epsilon a}{k_0m_\alpha}h_\alpha\left(v_p+\frac{k_0}{m_\alpha}\overline{J}\right)\frac{\sin^2\overline{\theta}}{[2v_p+(k_0/m_\alpha)\overline{J}]^2}\right]}$$
$$-\frac{s_\alpha q_\alpha a}{2k_0m_\alpha}\frac{1}{[2v_p+(k_0/m_\alpha)\overline{J}]^2}\overline{\left[F_\alpha\left(v_p+\frac{k_0}{m_\alpha}\overline{J}\right)+\epsilon s_\alpha h_\alpha\left(v_p+\frac{k_0}{m_\alpha}\overline{J}\right)\cos\overline{\theta}\right]}+O(a^2), \qquad \text{(D39)}$$

where the terms with overbars must be averaged on the pendulum energy levels. This averaging can be carried out, and the resulting quantities expanded in powers of a, using the above techniques, which entails some tedious algebra but no new ideas. Actually, we need to compute only terms through order $a^{3/2}$, to be consistent with Eq. (D18); we obtain

$$a^{3/2}\frac{\sigma_1}{2v_p}\sum_\alpha\left[\frac{|q_\alpha|^5}{k_0^5m_\alpha^3}\right]^{1/2}\frac{dF_\alpha}{dv}(v_p)$$
$$+\epsilon a^{3/2}\frac{\sigma_2}{2v_p^2}\sum_\alpha q_\alpha\left[\frac{|q_\alpha|^3}{k_0^5m_\alpha^3}\right]^{1/2}\left[h_\alpha(v_p)+2v_p\frac{dh_\alpha}{dv}(v_p)\right]$$
$$-a\frac{4\pi}{k_0}\sum_\alpha\frac{q_\alpha^2}{m_\alpha^2}\int^{+}d\overline{J}\frac{F_\alpha(v_p+(k_0/m_\alpha)\overline{J})}{[2v_p+(k_0/m_\alpha)\overline{J}]^2}. \qquad \text{(D40)}$$

Equations (D36) and (D40) provide the leading-order terms in the expansion of Eq. (D17) in the upper half plane. Since the initial condition is reflection symmetric, the lower half-plane contribution will be identical. Thus, the O'Neil term becomes Eq. (16), where the Vlasov dispersion term $K_0(k_0,\omega)$ is obtained by transforming the integration variable $\overline{J}$ back to v, integrating by parts, and using the symmetry of $F_\alpha(v)$.

2. Landau terms

Substituting $T(x,t)=\sum_{n=1}^{\infty}T_{nk_0}(t)\sin nk_0x$ into the second term on the right side in Eq. (14) (Landau terms) and transforming to the wave frame via Eqs. (D1) and (D2) gives

$$\frac{8}{k_0}\sum_\alpha\frac{q_\alpha^2}{m_\alpha^2}\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\sigma dt\cos\omega t\int_{-k_0\pi}^{+k_0\pi}d\theta\cos(\theta+\omega t)$$
$$\times\int^{+}dJ\int_0^t d\tau\sum_n T_{nk_0}(\tau)\sin n[\theta_\tau^A(\theta,J,t)+\omega\tau]$$
$$\times\Bigg\{\frac{dF_\alpha}{dv}\left(v_p+\frac{k_0}{m}J_\tau^A(\theta,J,t)\right)$$
$$+\epsilon\frac{dh_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}J_\tau^A(\theta,J,t)\right)\cos[\theta_\tau^A(\theta,J,t)+\omega\tau]\Bigg\}, \qquad \text{(D41)}$$

where $-\omega t$ in the θ integration limits has been eliminated by periodicity. Then, using Eq. (D4), the inverse relations Eqs. (D5) and (D6), and $\overline{\theta}\to\overline{\theta}+\pi$ when $q_\alpha>0$, Eq. (D41) becomes, with order $O(a^2)$ error,

$$\frac{8}{k_0}\sum_\alpha \frac{q_\alpha^2}{m_\alpha^2}\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\sigma dt\cos\omega t\int_{-k_0\pi}^{+k_0\pi}d\bar\theta\int^{+}d\bar J\cos\left(\bar\theta+\omega t+\frac{s_\alpha q_\alpha a}{k_0 m_\alpha}\frac{\sin[\bar\theta+2\omega t]}{[2v_p+(k_0/m_\alpha)\bar J]^2}\right)\int_0^t d\tau\sum_n T_{nk_0}(\tau)s_{\alpha,n}$$
$$\times\left[\sin n(\bar\theta_\tau^A+\omega\tau)+n\frac{s_\alpha q_\alpha a}{k_0 m_\alpha}\frac{\sin(\bar\theta_\tau^A+2\omega\tau)\cos n(\bar\theta_\tau^A+\omega\tau)}{[2v_p+(k_0/m_\alpha)\bar J_\tau^A]^2}\right]\left\{\frac{dF_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\bar J_\tau^A\right)+\epsilon s_\alpha\frac{dh_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\bar J_\tau^A\right)\right.$$
$$\times\cos(\bar\theta_\tau^A+\omega\tau)+\frac{s_\alpha q_\alpha a}{k_0 m_\alpha}\frac{d^2F_\alpha}{dv^2}\left(v_p+\frac{k_0}{m_\alpha}\bar J_\tau^A\right)\frac{\cos(\bar\theta_\tau^A+2\omega\tau)}{[2v_p+(k_0/m_\alpha)\bar J_\tau^A]}+\frac{\epsilon q_\alpha a}{k_0 m_\alpha}\frac{d^2h_\alpha}{dv^2}\left(v_p+\frac{k_0}{m_\alpha}\bar J_\tau^A\right)$$
$$\left.\times\frac{\cos(\bar\theta_\tau^A+2\omega\tau)\cos(\bar\theta_\tau^A+\omega\tau)}{[2v_p+(k_0/m_\alpha)\bar J_\tau^A]}-\frac{\epsilon q_\alpha a}{k_0 m_\alpha}\frac{dh_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\bar J_\tau^A\right)\frac{\sin(\bar\theta_\tau^A+2\omega\tau)\sin(\bar\theta_\tau^A+\omega\tau)}{[2v_p+(k_0/m_\alpha)\bar J_\tau^A]^2}\right\}, \tag{D42}$$

where $s_{\alpha,n}$ is s_α for n even and unity for n odd. Since the variables with overbars satisfy Eq. (D8), the $[\bar\theta_\tau^A(\bar\theta,\bar J,t),\bar J_\tau^A(\bar\theta,\bar J,t)]$ in Eq. (D42) are easily obtained in terms of elliptic functions.

A straightforward extension [33] of the arguments that led from Eq. (D12) to Eq. (D17) shows that Eq. (D42) can be rewritten as

$$\frac{4}{k_0}\sum_\alpha\frac{q_\alpha^2}{m_\alpha^2}\int_{-k_0\pi}^{+k_0\pi}d\bar\theta\int^{+}d\bar J\left\{\cos\bar\theta-\frac{s_\alpha q_\alpha a}{2k_0 m_\alpha}\frac{1}{[2v_p+(k_0/m_\alpha)\bar J]^2}\right\}\sum_n\bar{\mathcal G}_{0,n}(\bar\theta,\bar J), \tag{D43}$$

where the $\bar{\mathcal G}_{0,n}(\bar\theta,\bar J)$ are obtained trivially (but tediously) by applying standard sum formulas to the trigonometric functions in

$$T_{nk_0}(\tau)s_{\alpha,n}\left[\sin n(\bar\theta+\omega\tau)+n\frac{s_\alpha q_\alpha a}{k_0 m_\alpha}\frac{\sin(\bar\theta+2\omega\tau)\cos n(\bar\theta+\omega\tau)}{[2v_p+(k_0/m_\alpha)\bar J]^2}\right]\left\{\frac{dF_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\bar J\right)+\epsilon s_\alpha\frac{dh_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\bar J\right)\cos(\bar\theta+\omega\tau)\right.$$
$$+\frac{s_\alpha q_\alpha a}{k_0 m_\alpha}\frac{d^2F_\alpha}{dv^2}\left(v_p+\frac{k_0}{m_\alpha}\bar J\right)\frac{\cos(\bar\theta+2\omega\tau)}{[2v_p+(k_0/m_\alpha)\bar J]}+\frac{\epsilon q_\alpha a}{k_0 m_\alpha}\frac{d^2h_\alpha}{dv^2}\left(v_p+\frac{k_0}{m_\alpha}\bar J\right)\frac{\cos(\bar\theta+2\omega\tau)\cos(\bar\theta+\omega\tau)}{[2v_p+(k_0/m_\alpha)\bar J]}$$
$$\left.-\frac{\epsilon q_\alpha a}{k_0 m_\alpha}\frac{dh_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\bar J\right)\frac{\sin(\bar\theta+2\omega\tau)\sin(\bar\theta+\omega\tau)}{[2v_p+(k_0/m_\alpha)\bar J]^2}\right\} \tag{D44}$$

and then (a) integrating the factors depending on τ from zero to infinity, and (b) averaging the terms depending on $(\bar\theta,\bar J)$ on the energy levels of the pendulum.

In Eq. (D43) the single-mode terms, i.e., the terms that do not have an explicit dependence on a coming from multiple-mode effects, are given by

$$\frac{4}{k_0}\sum_\alpha\frac{q_\alpha^2}{m_\alpha^2}\int_{-k_0\pi}^{+k_0\pi}d\bar\theta\int^{+}d\bar J\cos\bar\theta\sum_n\bar{\mathcal H}_{0,n}(\bar\theta,\bar J), \tag{D45}$$

where the $\bar{\mathcal H}_{0,n}(\bar\theta,\bar J)$ are obtained from the terms

$$T_{nk_0}(\tau)s_{\alpha,n}\sin n(\bar\theta+\omega\tau)\left\{\frac{dF_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\bar J\right)+\epsilon s_\alpha\frac{dh_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\bar J\right)\cos(\bar\theta+\omega\tau)\right\} \tag{D46}$$

via the procedure described above. Standard trigonometric formulas (and $s_{\alpha,n+1}=s_\alpha s_{\alpha,n}$) lead to

$$\mathcal H_{0,n}(\bar\theta,\bar J)\equiv s_{\alpha,n}[C_{n,n}\sin n\bar\theta+S_{n,n}\cos n\theta]\frac{dF_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\bar J\right)+s_{\alpha,n+1}\frac{\epsilon}{2}[C_{n,n-1}\sin(n-1)\bar\theta+S_{n,n-1}\cos(n-1)\bar\theta$$
$$+C_{n,n+1}\sin(n+1)\bar\theta+S_{n,n+1}\cos(n+1)\bar\theta]\frac{dh_\alpha}{dv}\left(v_p+\frac{k_0}{m_\alpha}\bar J\right), \tag{D47}$$

where $C_{n,j}\equiv\int_0^\infty d\tau\, T_{nk_0}(\tau)\cos j\omega\tau$ and $S_{n,j}\equiv\int_0^\infty d\tau\, T_{nk_0}(\tau)\sin j\omega\tau$. For reference in Appendix E, these coefficients can be expressed in terms of the Laplace transform $\tilde{T}_n(p)$ of $T_n(t)$ as

$$C_{n,j}=\frac{1}{2}[\tilde{T}_{nk_0}(-ij\omega)+\tilde{T}_{nk_0}(ij\omega)],\quad S_{n,j}=\frac{1}{2i}[\tilde{T}_{nk_0}(-ij\omega)-\tilde{T}_{nk_0}(ij\omega)]. \tag{D48}$$

Now, each $\mathcal{H}_{0,n}$ must be averaged on the energy levels of the nonlinear pendulum, via the techniques above. First, the trigonometric functions of $\overline{\theta}$ are expressed in terms of $\sin\xi$ and $\cos\xi$, where $\xi\equiv\overline{\theta}/2$: $\sin n\overline{\theta}=P_n^S(\sin\xi,\cos\xi)$ and $\cos n\overline{\theta}=P_n^C(\sin\xi,\cos\xi)$, where P_n^S and P_n^C are $(n+1)$th degree polynomials. Then Eq. (D45) becomes

$$\frac{4}{k_0}\sum_\alpha \frac{q_\alpha^2}{m_\alpha^2}\int_{-k_0\pi}^{+k_0\pi}d\overline{\theta}\int_{m(\overline{J},\overline{\theta})<1}^{+}d\overline{J}\cos\overline{\theta}\frac{1}{2K(m)}\int_0^{2K(m)}dz\sum_n s_{\alpha,n}[C_{n,n}P_n^S(\text{sn},\text{cn})+S_{n,n}P_n^C(\text{sn},\text{cn})]$$
$$\times\frac{dF_\alpha}{dv}\left(v_p+\frac{2}{k_0}\frac{1}{\tau_b\kappa}\text{dn}\right)+s_{\alpha,n+1}\frac{\epsilon}{2}\{C_{n,n-1}P_{n-1}^S(\text{sn},\text{cn})+S_{n,n-1}P_{n-1}^C(\text{sn},\text{cn})+C_{n,n+1}P_{n+1}^S(\text{sn},\text{cn})$$
$$+S_{n,n+1}P_{n+1}^C(\text{sn},\text{cn})\}\frac{dh_\alpha}{dv}\left(v_p+\frac{2}{k_0}\frac{1}{\tau_b\kappa}\text{dn}\right) \tag{D49}$$

for $m<1$, where the elliptic functions sn, cn, and dn are understood to take the arguments $[z,m]$. Similarly, we find

$$\frac{4}{k_0}\sum_\alpha \frac{q_\alpha^2}{m_\alpha^2}\int_{-k_0\pi}^{+k_0\pi}d\overline{\theta}\int_{m(\overline{J},\overline{\theta})>1}^{+}d\overline{J}\cos\overline{\theta}\frac{1}{4K(1/m)}\int_0^{4K(1/m)}dz\sum_n s_{\alpha,n}[C_{n,n}P_n^S(\kappa^{-1}\text{sn},\text{dn})+S_{n,n}P_n^C(\kappa^{-1}\text{sn},\text{dn})]$$
$$\times\frac{dF_\alpha}{dv}\left(v_p+\frac{2}{k_0}\frac{1}{\tau_b\kappa}\text{cn}\right)+s_{\alpha+1,n}\frac{\epsilon}{2}\{C_{n,n-1}P_{n-1}^S(\kappa^{-1}\text{sn},\text{dn})+S_{n,n-1}P_{n-1}^C(\kappa^{-1}\text{sn},\text{dn})+C_{n,n+1}P_{n+1}^S(\kappa^{-1}\text{sn},\text{dn})$$
$$+S_{n,n+1}P_{n+1}^C(\kappa^{-1}\text{sn},\text{dn})\}\frac{dh_\alpha}{dv}\left(v_p+\frac{2}{k_0}\frac{1}{\tau_b\kappa}\text{cn}\right) \tag{D50}$$

for $m>1$, where now the elliptic functions take the arguments $[z,1/m]$.

These expressions have the same general structure as Eqs. (D26) and (D27) and can be similarly expanded. In the resonant region we simply expand dF_α/dv and dh_α/dv in Taylor series about v_p, eliminate the odd terms in $\overline{J}$, and rescale $\overline{J}$ to obtain

$$\frac{8}{k_0^3}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\int_{-k_0\pi}^{+k_0\pi}d\overline{\theta}\int_{\tilde{I}_1\cup\tilde{I}_2}d\overline{J}\,\tau_b^{-1}\cos\overline{\theta}\sum_{j=0}^{\infty}\left\{\left[\frac{2}{k_0}\frac{1}{\tau_b\kappa}\right]^{2j}\frac{\tilde{A}_{2j}(m)}{(2j)!}\frac{d^{2j+1}F_\alpha}{dv^{2j+1}}(v_p)+\frac{\epsilon}{2}\left[\frac{2}{k_0}\frac{1}{\tau_b\kappa}\right]^{2j}\frac{\tilde{B}_{2j}(m)}{(2j)!}\frac{d^{2j+1}h_\alpha}{dv^{2j+1}}(v_p)\right\}, \tag{D51}$$

where $\tilde{I}_1$, $\tilde{I}_2$, and m are defined above, and

$$\tilde{A}_l(m)=\begin{cases}\dfrac{1}{2K(m)}\displaystyle\int_0^{2K(m)}dz\,\text{dn}^l[z,m]\sum_n s_{\alpha,n}S_{n,n}P_n^C(\text{sn}[z,m],\text{cn}[z,m]), & m<1,\\[2ex] \dfrac{1}{4K(1/m)}\displaystyle\int_0^{4K(1/m)}dz\,\text{cn}^l\left[z,\frac{1}{m}\right]\sum_n s_{\alpha,n}S_{n,n}P_n^C\left(\frac{1}{\kappa}\text{sn}\left[z,\frac{1}{m}\right],\text{dn}\left[z,\frac{1}{m}\right]\right), & m>1,\end{cases} \tag{D52}$$

$$\tilde{B}_l(m)=\begin{cases}\dfrac{1}{2K(m)}\displaystyle\int_0^{2K(m)}dz\,\text{dn}^l[z,m]\sum_n s_{\alpha,n+1}\{S_{n,n-1}P_{n-1}^C(\text{sn}[z,m],\text{cn}[z,m])\\ \quad+S_{n,n+1}P_{n+1}^C(\text{sn}[z,m],\text{cn}[z,m])\},\quad m<1,\\ \dfrac{1}{4K(1/m)}\displaystyle\int_0^{4K(1/m)}dz\,\text{cn}^l\left[z,\frac{1}{m}\right]\sum_n s_{\alpha,n+1}\left\{S_{n,n-1}P_{n-1}^C\left(\kappa^{-1}\text{sn}\left[z,\frac{1}{m}\right],\text{dn}\left[z,\frac{1}{m}\right]\right)\right.\\ \quad\left.+S_{n,n+1}P_{n+1}^C\left(\kappa^{-1}\text{sn}\left[z,\frac{1}{m}\right],\text{dn}\left[z,\frac{1}{m}\right]\right)\right\},\quad m>1.\end{cases} \tag{D53}$$

Here we have used the fact that the z integrals containing P_n^S are zero, as follows from the symmetries of the elliptic functions (see [34], Fig. 16.1). Indeed, P_n^S and P_n^C are, respectively, odd and even functions of each of their arguments, as can be seen by using the standard trigonometric multiple-angle formulas and mathematical induction. In principle, the z integrals can be computed analytically by the methods introduced above; in practice, as n grows this becomes very burdensome. Fortunately, in many concrete cases, e.g., those discussed in Sec. IV, only very few spatial Fourier modes T_n are non-negligible. As above [see the comments that follow Eq. (D30)], the $\overline{J}$ integration in Eq. (D51) is extended to $\mathbb{R}$ by combining the appropriate resonant and nonresonant quantities, after eliminating from the resonant terms certain spatially uniform quantities that are not integrable at infinity in $\overline{J}$ but vanish under the $\overline{\theta}$ integration.

In the nonresonant region Eq. (D49) is expanded about the free-streaming particle trajectories, exactly as for the corresponding O'Neil terms. However, here the Landau terms give no contribution at leading order [33]. Hence, the leading-order single-mode Landau terms from Eq. (D51) are

$$a^{1/2}\sum_\alpha \left[\frac{|q_\alpha|^5}{k_0^3 m_\alpha^3}\right]^{1/2}\left[\rho_1(T)\frac{dF_\alpha}{dv}(v_p)+\frac{\epsilon}{2}\rho_2(T)\frac{dh_\alpha}{dv}(v_p)\right] + a^{3/2}\sum_\alpha \left[\frac{|q_\alpha|^7}{k_0^5 m_\alpha^5}\right]^{1/2}\left[2\rho_3(T)\frac{d^3F_\alpha}{dv^3}(v_p) + \epsilon\rho_4(T)\frac{d^3h_\alpha}{dv^3}(v_p)\right], \tag{D54}$$

where

$$\rho_i(T)\equiv 8\int_{-\pi}^{+\pi}d\overline{\theta}\int d\overline{J}\cos\overline{\theta}\,R_i(m) \tag{D55}$$

and the $R_i(m)$, $i=1,...,4$, are, respectively, $\tilde{A}_0(m)$, $\tilde{B}_0(m)$, $\tilde{A}_2(m)/m$, and $\tilde{B}_2(m)/m$. The functional dependence of ρ_i on the field T has been indicated explicitly, and $T\equiv 0$ implies $C_{n,j}=S_{n,j}=\tilde{A}_l=\tilde{B}_l=0$, so that $\rho_i(0)=0$, $i=1,...,4$.

The same methods can be used to compute the multiple-mode effects in Eq. (D43). The result to leading order is [33]

$$-a^{3/2}\sum_\alpha \left[\frac{|q_\alpha|^7}{k_0^5 m_\alpha^5}\right]^{1/2}\left[\frac{\lambda_1(T)}{8v_p^2}\frac{dF_\alpha}{dv}(v_p)+\frac{\lambda_2(T)}{4v_p}\frac{d^2F_\alpha}{dv^2}(v_p) + \frac{\epsilon\lambda_3(T)}{16v_p^2}\frac{dh_\alpha}{dv}(v_p)+\frac{\epsilon\lambda_4(T)}{8v_p}\frac{d^2h_\alpha}{dv^2}(v_p)\right], \tag{D56}$$

where

$$\lambda_i(T)\equiv 8\int_{-\pi}^{+\pi}d\overline{\theta}\int_{\mathbb{R}}d\overline{J}\cos\overline{\theta}\tilde{M}_i(m), \tag{D57}$$

$$\tilde{M}_i(m)\equiv\begin{cases}\dfrac{1}{2K(m)}\displaystyle\int_0^{2K(m)}dz\sum_n M_{i,n}(\mathrm{sn}[z,m],\mathrm{cn}[z,m]), & m<1,\\[2ex] \dfrac{1}{4K(1/m)}\displaystyle\int_0^{4K(1/m)}dz\sum_n M_{i,n}(\kappa^{-1}\mathrm{sn}[z,m],\mathrm{dn}[z,m]), & m>1,\end{cases} \tag{D58}$$

and

$$M_{1,n}\equiv s_{\alpha,n}[C_{n,n+2}P_{n+1}^S+S_{n,n+2}P_{n+1}^C-C_{n,n-2}P_{n-1}^S - S_{n,n-2}P_{n-1}^C], \tag{D59a}$$

$$M_{2,n}\equiv s_{\alpha,n}[C_{n,n+2}P_{n+1}^S+S_{n,n+2}P_{n+1}^C+C_{n,n-2}P_{n-1}^S + S_{n,n-2}P_{n-1}^C], \tag{D59b}$$

$$M_{3,n}\equiv s_{\alpha,n+1}[(n+1)C_{n,n+3}P_{n+2}^S+(n+1)S_{n,n+3}P_{n+2}^C -(n-1)C_{n,n-3}P_{n-2}^S-(n-1)S_{n,n-3}P_{n-2}^C +(n-1)(C_{n,n+1}P_n^S+S_{n,n+1}P_n^C) -(n+1)(C_{n,n-1}P_n^S+S_{n,n-1}P_n^C)], \tag{D59c}$$

$$M_{4,n}\equiv s_{\alpha,n+1}[C_{n,n-3}P_{n-2}^S+S_{n,n-3}P_{n-2}^C+C_{n,n+3}P_{n+2}^S +S_{n,n+3}P_{n+2}^C+(C_{n,n+1}+C_{n,n-1})P_n^S +(S_{n,n+1}+S_{n,n-1})P_n^C]. \tag{D59d}$$

Equations (D54) and (D56) can be summed and extended (by symmetry) to the other half phase plane, yielding the total-Landau contribution to the nonlinear Poisson equation given in Eq. (21).

APPENDIX E: TRANSIENT FIELD EXPANSIONS

Here we present the more tedious calculations needed for the transient analysis in Sec. IV. Substituting Eq. (7) into Eq. (5b) and Fourier and Laplace transforming gives

$$\tilde{T}_k(p)=\frac{4}{k}\sum_\alpha q_\alpha\int_0^\infty dt\,e^{-pt}(I-P_a)\int_{-\pi}^{+\pi}dx\cos kx \times\int_{\mathbb{R}}dv\,\mathcal{F}_\alpha(x_0^A(x,v,t),v_0^A(x,v,t)) - \frac{4}{k}\sum_\alpha \frac{q_\alpha^2}{m_\alpha}\int_0^\infty dt\,e^{-pt}(I-P_a)\int_{-\pi}^{+\pi}dx\cos kx \times\int_{\mathbb{R}}dv\int_0^t d\tau\left\{T\frac{\partial\mathcal{F}_\alpha}{\partial v}\right\}_{[x_\tau^A(x,v,t),v_\tau^A(x,v,t)]}, \tag{E1}$$

where $k=lk_0$, $l=1,2,\ldots$, $\mathcal{F}_\alpha(x,v)=F_\alpha(v)+\epsilon h_\alpha(v)\cos k_0x$, and A is given by Eq. (30). It is assumed that T_k and the right side of Eq. (5b) are integrable. To obtain a more explicit expression, we apply the procedure developed for the time-asymptotic equation to the two integral terms on the right side. We carry out the same sequence of transformations in the integration variables that led to Eqs. (D17) and (D43), with k_0 in Eq. (D1) replaced by k_f. The result is

$$\tilde{T}_k(p)=\frac{4}{k}\sum_\alpha\frac{q_\alpha}{m_\alpha}\int_0^\infty dt\,e^{-pt}(I-P_a)\int_{-k_f\pi}^{+k_f\pi}d\bar{\theta}\int^{+}d\bar{J}\cos\frac{k}{k_f}\left(\bar{\theta}+\omega t+\frac{q_\alpha a}{k_f m_\alpha}\frac{\sin[\bar{\theta}+2\omega t]}{[2v_p+(k_0/m_\alpha)\bar{J}]^2}\right)$$
$$\times\Bigg\{\Bigg[F_\alpha(w_0^A)+\epsilon h_\alpha(w_0^A)\cos\frac{k_0}{k_f}\bar{\theta}_0^A+\frac{q_\alpha a}{k_f m_\alpha}\Bigg(\frac{F'_\alpha(w_0^A)\cos\bar{\theta}_0^A}{[v_p+w_0^A]}-\epsilon\frac{h_\alpha(w_0^A)\sin\bar{\theta}_0^A\sin[(k_0/k_f)\bar{\theta}_0^A]}{[v_p+w_0^A]^2}$$
$$+\epsilon\frac{h'_\alpha(w_0^A)\cos\bar{\theta}_0^A\cos[(k_0/k_f)\bar{\theta}_0^A]}{[v_p+w_0^A]}\Bigg)\Bigg]-\int_0^t d\tau\sum_n T_{nk_0}(\tau)\Bigg(\sin\frac{nk_0}{k_f}(\bar{\theta}_\tau^A+\omega\tau)$$
$$+nk_0\frac{q_\alpha a}{k_f^2m_\alpha}\frac{\sin(\bar{\theta}_\tau^A+2\omega\tau)\cos[(nk_0/k_f)(\bar{\theta}_\tau^A+\omega\tau)]}{[v_p+w_\tau^A]^2}\Bigg)\Bigg[\frac{dF_\alpha}{dv}(w_\tau^A)+\epsilon\frac{dh_\alpha}{dv}(w_\tau^A)\cos\frac{k_0}{k_f}(\bar{\theta}_\tau^A+\omega\tau)$$
$$+\frac{q_\alpha a}{k_f m_\alpha}\Bigg(\frac{F''_\alpha(\omega_\tau^A)\cos(\bar{\theta}_\tau^A+2\omega\tau)}{[v_p+w_\tau^A]}-\epsilon\frac{h'_\alpha(w_\tau^A)\sin(\bar{\theta}_\tau^A+2\omega\tau)\sin[(k_0/k_f)(\bar{\theta}_\tau^A+\omega\tau)]}{[v_p+w_\tau^A]^2}$$
$$+\epsilon\frac{h''_\alpha(w_\tau^A)\cos(\bar{\theta}_\tau^A+2\omega\tau)\cos[(k_0/k_f)(\bar{\theta}_\tau^A+\omega\tau)]}{[v_p+w_\tau^A]}\Bigg)\Bigg]\Bigg\}$$
$$+(\text{corresponding terms from the other half phase plane}), \tag{E2}$$

where $w_\tau^A(\bar{\theta},\bar{J},t)\equiv v_p+(k_f/m_\alpha)\bar{J}_\tau^A(\bar{\theta},\bar{J},t)$ and

$$\bar{\theta}_\tau^A(\bar{\theta},\bar{J},t)=-2\,\mathrm{am}\left[\frac{t-\tau}{\kappa\tau_b}-F\left(\frac{\bar{\theta}}{2}\middle|m\right),m\right], \tag{E3}$$

$$\bar{J}_\tau^A(\bar{\theta},\bar{J},t)=\frac{2m_\alpha}{k_f^2}\frac{1}{\tau_b\kappa}\mathrm{dn}\left[\frac{t-\tau}{\kappa\tau_b}-F\left(\frac{\bar{\theta}}{2}\middle|m\right),m\right]. \tag{E4}$$

The arguments of the trigonometric functions in Eq. (E2) must be corrected to include a shift of π in $\bar{\theta}$ and $\bar{\theta}_\tau^A$ when $q_\alpha>0$. These corrections will be introduced in the asymptotic expansions of the various terms in Eq. (E2) in powers of $\Delta\epsilon$. We now substitute the perturbative expansion for T, Eq. (31), into Eq. (E2) and solve the resulting hierarchy equations for the $T^{(k)}(x,t)$. We illustrate the simplest situation, when the time-asymptotic field is zero.

1. Transient expansion along $a=0$

Along the basic branch $a=0$, the equation for $\tilde{T}^{(0)}(p)$ can be obtained by setting $a=0$ in Eq. (E2) and noting from Eq. (E3) that $\bar{\theta}_\tau^0(\bar{\theta},\bar{J},t)=\bar{\theta}-(k_f^2/m_\alpha)\bar{J}(t-\tau)$ and $\bar{J}_\tau^0(\bar{\theta},\bar{J},t)=\bar{J}$, where the superscripts correspond to $A\equiv 0$. We find

$$\tilde{T}_k^{(0)}(p)=\frac{4}{k}\sum_\alpha \frac{q_\alpha}{m_\alpha}\int_0^\infty dt\, e^{-pt}\int_{-k_f\pi}^{+k_f\pi} d\overline{\theta}\int^+ d\overline{J}\cos\frac{k}{k_f}(\overline{\theta}+\omega t)\left[F_\alpha\left(v_p+\frac{k_f}{m_\alpha}\overline{J}\right)+\epsilon_0 h_\alpha\left(v_p+\frac{k_f}{m_\alpha}\overline{J}\right)\cos\frac{k_0}{k_f}\left(\overline{\theta}-\frac{k_f^2}{m_\alpha}Jt\right)\right]$$
$$-\frac{4}{k}\sum_\alpha \frac{q_\alpha^2}{m_\alpha^2}\int_0^\infty dt\, e^{-pt}\int_{-k_f\pi}^{+k_f\pi} d\overline{\theta}\int^+ d\overline{J}\cos\frac{k}{k_f}(\overline{\theta}+\omega t)\int_0^t d\tau\sum_n T_{nk_0}^{(0)}(\tau)\sin\frac{nk_0}{k_f}\left[\overline{\theta}-\frac{k_f^2}{m_\alpha}J(t-\tau)+\omega\tau\right]$$
$$\times\left\{\frac{dF_\alpha}{dv}\left(v_p+\frac{k_f}{m_\alpha}\overline{J}\right)+\epsilon_0\frac{dh_\alpha}{dv}\left(v_p+\frac{k_f}{m_\alpha}\overline{J}\right)\cos\frac{k_0}{k_f}\left[\overline{\theta}-\frac{k_f^2}{m_\alpha}J(t-\tau)+\omega\tau\right]\right\}$$
$$+(\text{corresponding terms from the other half phase plane}). \tag{E5}$$

The projector $I-P_a$ in Eq. (E2) becomes unnecessary in Eq. (E5) because all the time-asymptotic parts vanish for $a=0$. Also, no modifications are necessary in Eq. (E5) for $q_\alpha>0$, since the straight line trajectories $\overline{\theta}_\tau^0$ and $\overline{J}_\tau^0$ do not depend on the sign of q_α. Carrying out the $\overline{\theta}$ integrations yields

$$\tilde{T}_k^{(0)}(p)=\delta_{k,k_0}\epsilon_0\frac{4\pi k_f}{k}\sum_\alpha\frac{q_\alpha}{m_\alpha}\int_0^\infty dt\,e^{-pt}\int^+ d\overline{J}\cos k\left(v_pt+\frac{k_f}{m_\alpha}\overline{J}t\right)h_\alpha\left(v_p+\frac{k_f}{m_\alpha}\overline{J}\right)-\frac{4\pi k_f}{k}\sum_\alpha\frac{q_\alpha^2}{m_\alpha^2}\int_0^\infty dt\,e^{-pt}$$
$$\times\int^+ d\overline{J}\int_0^t d\tau\left\{T_k^{(0)}(\tau)\frac{dF_\alpha}{dv}\left(v_p+\frac{k_f}{m_\alpha}\overline{J}\right)+\frac{\epsilon_0}{2}[T_{k+k_0}^{(0)}(\tau)+T_{k-k_0}^{(0)}(\tau)]h_\alpha\left(v_p+\frac{k_f}{m_\alpha}\overline{J}\right)\right\}\sin k\left[\frac{k_f}{m_\alpha}J(t-\tau)+v_p(t-\tau)\right]$$
$$+(\text{corresponding terms from the other half phase plane}). \tag{E6}$$

Transforming back to $v=v_p+(k_f/m_\alpha)\overline{J}$, calculating the Laplace transform, and carrying out an identical calculation in the other half phase plane gives

$$\tilde{T}_k^{(0)}(p)=\delta_{k,k_0}\epsilon_0\frac{4\pi}{k}\sum_\alpha q_\alpha\int_{\mathbb{R}}dv\,\frac{ph_\alpha(v)}{p^2+k^2v^2}-\frac{4\pi}{k}\tilde{T}_k^{(0)}(p)\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\int_{\mathbb{R}}dv\,\frac{kvF'_\alpha(v)}{p^2+k^2v^2}$$
$$-\frac{\epsilon_0}{2}\frac{4\pi}{k}[T_{k+k_0}^{(0)}(\tau)+T_{k-k_0}^{(0)}(\tau)]\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\int_{\mathbb{R}}dv\,\frac{kvh'_\alpha(v)}{p^2+k^2v^2}. \tag{E7}$$

Since F_α and h_α are even, this becomes

$$\tilde{T}_k^{(0)}(p)\left[1+\frac{4\pi}{k}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\int_{\mathbb{R}}dv\,\frac{F'_\alpha(v)}{kv-ip}\right]+\frac{\epsilon_0}{2}\frac{4\pi}{k}\times\sum_\alpha\frac{q_\alpha^2}{m_\alpha}[\tilde{T}_{k+k_0}^{(0)}(p)+\tilde{T}_{k-k_0}^{(0)}(p)]\int_{\mathbb{R}}dv\,\frac{h'_\alpha(v)}{kv-ip}$$
$$=-\delta_{k,k_0}\epsilon_0\frac{4\pi i}{k}\sum_\alpha q_\alpha\int_{\mathbb{R}}dv\,\frac{h_\alpha(v)}{kv-ip} \tag{E8}$$

and Eq. (33) follows.

2. The threshold equation

Next we derive the threshold equation, Eq. (39), for a small critical initial amplitude. If $T^{(0)}$, Eq. (38), is substituted into Eq. (27), this threshold equation becomes explicit. From Eq. (38), $T_{nk_0}^{(0)}=O(\epsilon_0^n)$, and we shall need to keep only $T_{k_0}^{(0,1)}$ and $T_{2k_0}^{(0,2)}$, which generate, respectively, the coefficients $S_{1,j}^{(0,1)}$ and $S_{2,j}^{(0,2)}$ according to Eq. (D48) (with errors of order ϵ_0^3). By symmetry,

$$S_{1,j}^{(0,1)}=\frac{1}{2i}\left[\frac{N_{k_0}^+(-ij\omega)}{D_{k_0}^+(-ij\omega)}-\frac{N_{k_0}^+(ij\omega)}{D_{k_0}^+(ij\omega)}\right]=\mathrm{Im}\left[\frac{N_{k_0}^+(-ij\omega)}{D_{k_0}^+(-ij\omega)}\right], \tag{E9}$$

$$S_{2,j}^{(0,2)}=\frac{1}{2i}\left[\frac{N_{k_0}^+(-ij\omega)C_{2k_0}^+(-ij\omega)}{D_{k_0}^+(-ij\omega)D_{2k_0}^+(-ij\omega)}-\frac{N_{k_0}^+(ij\omega)C_{2k_0}^+(ij\omega)}{D_{k_0}^+(ij\omega)D_{2k_0}^+(ij\omega)}\right]$$
$$=\mathrm{Im}\left[\frac{N_{k_0}^+(-ij\omega)C_{2k_0}^+(-ij\omega)}{D_{k_0}^+(-ij\omega)D_{2k_0}^+(-ij\omega)}\right], \tag{E10}$$

where $N_k(p)$, $D_k(p)$, and $C_k(p)$ are defined in Eqs. (34), (35), and (36), and the superscript "+" indicates analytic continuation. These expressions, in conjunction with Eqs. (D55), yield $\rho_1(T_0)=\epsilon_0\sigma_1 S_{1,1}^{(0,1)}-4\epsilon_0^2 s_\alpha\sigma_2 S_{2,2}^{(0,2)}+O(\epsilon_0^3)$ and $\rho_2(T_0)=-4\epsilon_0 s_\alpha\sigma_2 S_{1,2}^{(0,1)}+O(\epsilon_0^2)$. Thus, to leading order,

$$\Gamma(\epsilon_0,T_0)=\epsilon_0\sigma_1 S_{1,1}^{(0,1)}\sum_\alpha\left[\frac{|q_\alpha|^5}{k_0^3m_\alpha^3}\right]^{1/2}\frac{dF_\alpha}{dv}(v_p)-4\epsilon_0^2\left\{\sigma_2 S_{2,2}^{(0,2)}\sum_\alpha s_\alpha\left[\frac{|q_\alpha|^5}{k_0^3m_\alpha^3}\right]^{1/2}\frac{dF_\alpha}{dv}(v_p)+\frac{\sigma_2}{2}S_{1,2}^{(0,1)}\sum_\alpha s_\alpha\left[\frac{|q_\alpha|^5}{k_0^3m_\alpha^3}\right]^{1/2}\frac{dh_\alpha}{dv}(v_p)\right\}+O(\epsilon_0^3) \tag{E11}$$

and Eq. (27) becomes Eq. (39).

3. Time-asymptotic field amplitude near the threshold

Finally, we present the details that lead from the general formula for the time-asymptotic amplitude, Eq. (32), to the value of a near the threshold, Eq. (56). From Eq. (32), to first order in both a and $\Delta\epsilon$, $a=\mu\Delta\epsilon$ where

$$\mu=-\frac{k_0\Sigma_\alpha s_\alpha[|q_\alpha|^3/m_\alpha]^{1/2}h_\alpha(v_p)}{\Sigma_\alpha[|q_\alpha|^5/m_\alpha^3]^{1/2}F_\alpha''(v_p)}-\frac{\Sigma_\alpha[|q_\alpha^5|/k_0^3m_\alpha^3]^{1/2}[\frac{1}{2}\epsilon_0^{(1)}\rho_2(T^{(0,1)}+T^{(1,0)})h_\alpha'(v_p)+\rho_1(T^{(1,0)})F_\alpha'(v_p)]}{\sigma_1\Sigma_\alpha[|q_\alpha|^5/k_0^5m_\alpha^3]^{1/2}F_\alpha''(v_p)}$$
$$+\frac{\chi_2\chi_3\epsilon_0^{(1)}+\chi_2\Sigma_\alpha[|q_\alpha|^5/k_0^5m_\alpha^3]^{1/2}\{(\sigma_1/2v_p)F_\alpha'(v_p)-\epsilon_0^{(1)}(|q_\alpha|/m_\alpha)[2\rho_3(T^{(0,1)})F_\alpha'''(v_p)-(1/4v_p)\lambda_2(T^{(0,1)})F_\alpha''(v_p)]\}}{[\sigma_1\Sigma_\alpha[|q_\alpha^5|/k_0^5m_\alpha^3]^{1/2}F_\alpha''(v_p)]^2}. \tag{E12}$$

From the definitions of ρ_i and λ_i, Eqs. (D55) and (D57), simple integrations lead to

$$\rho_1(T^{(1,0)})=\sigma_1 S_{1,1}^{(1,0)},$$
$$\rho_2(T^{(1,0)}+T^{(0,1)})=-4\sigma_2 s_\alpha(S_{1,2}^{(1,0)}+S_{1,2}^{(0,1)}),$$
$$\rho_3(T^{(0,1)})=\frac{1}{2}\sigma_2 S_{1,1}^{(0,1)},\quad \lambda_2(T^{(0,1)})=\sigma_1 S_{1,3}^{(0,1)}, \tag{E13}$$

where σ_i, $i=1,2$, are given below Eq. (D38), $S_{1,j}^{(1,0)}$ are zero for $j\neq 1$ since the only nonzero Fourier component in $T^{(1,0)}$ corresponds to $k=k_0$, $S_{1,1}^{(1,0)}$ is obtained by inserting $T_{k_0}^{(1,0)}$ into Eq. (D48), and the $S_{1,j}^{(0,1)}$, $j=1,2,\ldots$, were already obtained in Eq. (E9). Here, however, $N_{k_0}^+/D_{k_0}^+$ in Eq. (E9) is replaced by the modified function $[N_{k_0}^+(p)/D_{k_0}^+(p)]-\tilde{A}_{k_0}^{(0)}(p)$. This has a significant effect, because $N_{k_0}^+/D_{k_0}^+$ has a singularity at $p=\pm ik_0v_p$, as $\epsilon_0^{(1)}\to 0$, which causes $S_{1,1}^{(0,1)}$ to be of order $1/\epsilon_0^{(1)}$. This can be seen from Eq. (E9) for $j=1$, since at $\omega=\pm k_0v_p$ the denominator reduces to a term proportional to $\Sigma_\alpha(q_\alpha^2/m_\alpha)F_\alpha'(v_p)=O(\epsilon_0^{(1)})$. Thus, the second term in the numerator of Eq. (40) is non-negligible at leading order, whereas the first term in the denominator is negligible at leading order due to Eq. (54) [because there is no singularity at $p=\pm 2ik_0v_p$ and $S_{2,2}^{(0,2)}=O(1)$]. However, the coefficient $S_{1,1}^{(1,0)}$ in Eqs. (E13) and therefore (E12) is $O(1)$ [and the term that contains it is $O(\epsilon_0^{(1)})$ due to Eq. (54)], because the singularities in $[N_{k_0}^+(p)/D_{k_0}^+(p)]-\tilde{A}_{k_0}^{(0)}(p)$ at $p=\pm ik_0v_p$ are removable, due to Eq. (50). Indeed, Taylor expanding $N_{k_0}^+$ and $D_{k_0}^+$ in this modified function about $\pm ik_0v_p$ shows that the singular terms cancel, at leading order in $\epsilon_0^{(1)}$. Taking the limit $\delta z\to 0$ in that expansion and exploiting the symmetries yields

$$S_{1,1}^{(1,0)}=\frac{a_0}{2k_0v_p}-\mathrm{Im}\left[\frac{i\Sigma_\alpha q_\alpha[\mathrm{P}\int_{\mathbb{R}}dv\,h_\alpha'(v)/(v-v_p)+i\pi h_\alpha'(v_p)]}{\Sigma_\alpha(q_\alpha^2/m_\alpha)[\mathrm{P}\int_{\mathbb{R}}dv\,F_\alpha''(v)/(v-v_p)+i\pi F_\alpha''(v_p)]}\right], \tag{E14}$$

where a_0 was given in Eq. (49). According to Eq. (E13), Eq. (E12) then becomes Eq. (56).

Equation (56) is accurate to order $\epsilon_0^{(1)}\Delta\epsilon$, even though we did not calculate the terms of order $O(\epsilon_0^{(1)}\Delta\epsilon)$ in the solution for the transient field, Eq. (55). This follows from Eq. (54); in principle, the solution for a, Eq. (32), contains a term of the form $\epsilon_0^{(1)}\Delta\epsilon\Sigma_\alpha[|q_\alpha^5|/k_0^3m_\alpha^3]^{1/2}\,\rho_1(T^{(1,1)})F_\alpha'(v_p)$, but this term is "pushed" to order $O(\epsilon_0^{(1)^2}\Delta\epsilon)$ by Eq. (54). In cases in which only Eq. (53) is satisfied (*and* $\Sigma_\alpha[|q_\alpha^5|/k_0^3m_\alpha^3]^{1/2}\,F_\alpha'(v_p)\neq 0$), one must add to Eq. (56) the contribution due to $T^{(1,1)}$, by carrying out the perturbation analysis of the transient equation through first order in both $\Delta\epsilon$ and $\epsilon_0^{(1)}$. This calculation, which is quite tedious, will be omitted here.

APPENDIX F: BUCHANAN-DORNING SOLUTIONS

In this appendix we verify that the two-wave BGK-like solutions discovered by Buchanan and Dorning [15] satisfy the nonlinear condition Eq. (50), and thus are a special case of the solutions developed in this paper. At $t=0$ the distribution function corresponding to the approximate invariants $\mathcal{E}_\alpha^{(\pm)}$ [see Eq. (15)] is

$$\begin{aligned}\mathcal{G}_\alpha&\left(\frac{m_\alpha}{2}(v\mp v_p)^2+\epsilon\frac{q_\alpha}{k_0}\cos k_0x+\epsilon\frac{q_\alpha}{k_0}\frac{v\mp v_p}{v\pm v_p}\cos k_0x\right)\\&=\mathcal{G}_\alpha\left(\frac{m_\alpha}{2}(v\mp v_p)^2+\epsilon\frac{q_\alpha}{k_0}\frac{2v}{v\pm v_p}\cos k_0x\right)\\&=\mathcal{G}_\alpha\left(\frac{m_\alpha}{2}(v\mp v_p)^2\right)+\epsilon\frac{q_\alpha}{k_0}\frac{2v}{v\pm v_p}\\&\quad\times\mathcal{G}'_\alpha\left(\frac{m_\alpha}{2}(v\mp v_p)^2\right)\cos k_0x+O(\epsilon^2),\end{aligned}\tag{F1}$$

where $\mathcal{G}_\alpha$ must satisfy certain criteria [15] which in fact ensure that Eqs. (45) and (46) are satisfied. Clearly, the initial condition in Eq. (F1) is of the form $F_\alpha(v)+\epsilon h_\alpha(v)\cos k_0x$ (at leading order in ϵ) with $F_\alpha(v)=\mathcal{G}_\alpha[(m_\alpha/2)(v\mp v_p)^2]$ and $h_\alpha(v)=(q_\alpha/k_0)[2v/(v\pm v_p)]\mathcal{G}'_\alpha[(m_\alpha/2)(v\mp v_p)^2]$. Differentiating F_α yields

$$\begin{aligned}F'_\alpha(v)&=m_\alpha(v\mp v_p)\mathcal{G}'_\alpha\left(\frac{m_\alpha}{2}(v\mp v_p)^2\right)\\&=\frac{k_0m_\alpha}{q_\alpha}\frac{(v\pm v_p)(v\mp v_p)}{2v}h_\alpha(v),\end{aligned}\tag{F2}$$

so that

$$\frac{k_0m_\alpha}{q_\alpha}h_\alpha(v)=\frac{2v}{v^2-v_p^2}F'_\alpha(v)=\frac{F'_\alpha(v)}{v+v_p}+\frac{F'_\alpha(v)}{v-v_p}.\tag{F3}$$

Dividing by $v\mp v_p$ and taking the principal value integral gives

$$\begin{aligned}\frac{k_0m_\alpha}{q_\alpha}\mathrm{P}\int\frac{h_\alpha(v)}{v\mp v_p}dv&=\mathrm{P}\int\frac{F'_\alpha(v)}{(v\mp v_p)^2}dv+\mathrm{P}\int\frac{F'_\alpha(v)}{(v+v_p)(v-v_p)}dv\\&=\mathrm{P}\int\frac{F''_\alpha(v)}{v\mp v_p}dv,\end{aligned}\tag{F4}$$

where the first term was integrated by parts, the second is zero because the integrand is odd, and

$$\begin{aligned}F''_\alpha(v)=&\,m_\alpha^2(v\mp v_p)^2\mathcal{G}''_\alpha\left(\frac{m_\alpha}{2}(v\mp v_p)^2\right)\\&+m_\alpha\mathcal{G}'_\alpha\left(\frac{m_\alpha}{2}(v\mp v_p)^2\right),\end{aligned}\tag{F5}$$

which at $v=\pm v_p$ gives

$$F''_\alpha(\pm v_p)=m_\alpha\mathcal{G}'_\alpha(0)=\frac{k_0m_\alpha}{q_\alpha}h_\alpha(\pm v_p).\tag{F6}$$

From Eqs. (F4) and (F6) it follows that Eq. (50) is satisfied. Then, substituting Eq. (F6) into Eq. (49) gives $a_0=1$, i.e., $a=\epsilon$, which is what we should expect since we know that these undamped "BGK-like" waves travel without changing amplitude.

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