Localization of harmonics generated in nonlinear shallow water waves

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The propagation of nonlinear shallow water waves over a random seabed is studied. A bathymetry which fluctuates randomly from a constant mean adds multiple scattering to resonant interactions and harmonic generation. By the method of multiple scales, nonlinear evolution equations for the harmonic amplitudes are derived. Effects of multiple scattering are shown to be represented by certain linear damping terms with complex coefficients related to the correlation function of the seabed disorder. For any finite number of harmonics, an equation governing the total wave energy is derived. By numerical solution of the amplitude equations, the effects of spatial attenuation (localization) on harmonic generation are studied.

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I. INTRODUCTION

Several characteristics of long waves in shallow water are of general interest to wave physics in many different contexts. The interplay between nonlinearity and dispersion has, on one hand, led to impressive advances in soliton dynamics and the inverse scattering theory [1]. On the other hand, independent discoveries in wave-wave interactions ushered the new age of oceanography [2,3] and nonlinear optics [4]. In particular, the mechanism of harmonic generation, first found in optics, is known to have a close cousin in shallow-water waves [5,6].

Anderson localization, originated in the study of transport in disordered quantum systems [7], is still an expanding topic in wave propagation in random media [8,9]. Many mathematical studies on nonlinear waves in random media have also appeared. In particular, Devillard and Souillard [10] have studied the one-dimensional nonlinear Schrödinger equation with a random potential. Extensions of this work for incident solitons and other types of random potentials have been advanced by many others (see e.g., Refs. [11–15]). For extensive reviews, see Refs. [16,17]. Relevant to long waves in shallow water, a theory for the Korteweg–de Vries equation [18] with a weak random potential has also been studied by Garnier [19]. In these mathematical models, a common feature is that the final differential equation has one or more stochastic coefficients.

In the past few decades, great efforts have been devoted in oceanography to wave prediction. For deep seas, focus has been directed to wind forcing, nonlinear energy transfer between different wavelengths and frequencies, and dissipation by wave breaking. For shallow seas, it is important to account in addition for dissipation from the seabed due to friction, as well as the effects of depth variation. Existing treatments of the latter aspect have largely been limited to deterministic modeling of refraction and/or scattering. Since some complex bathymetries can be best described as a random function of space, it is of practical value to see how multiple scattering by random bathymetry can cause spatial attenuation, i.e., localization by radiation damping. Only a few papers on the linearized aspects have appeared in the literature [20–25]. For nonlinear long waves in shallow water, the only known theories are of Howe [26] and Rosales and Papanicolaou [27]. For intermediate depth, the perturbation method of multiple scales, also known as the theory of homogenization in some contexts, has been shown to be an effective tool for analyzing weakly nonlinear waves in a weakly disordered medium of large spatial extent. The basic ideas were first explained for the simple case of a taut string embedded in a nonlinearly elastic surrounding, whose elastic properties contain a random component [28]. It has been shown that the wave envelope is governed by a cubic Schrödinger equation modified by a linear term with a complex damping coefficient, which is related to the statistical correlation function of the random perturbations. Effects of localization on the evolution of soliton envelopes and side-band instability have been examined. Similar analysis has been reported for small-amplitude water waves of intermediate wavelength over a seabed with weak disorder in depth. One- and two-dimensional nonlinear Schrödinger equations with a linear damping term have been derived for the envelope of a narrow-banded wave train. In one dimension, complex diffraction is found after a bisoliton passes over a finite strip of random seabed [29]. When the random bathymetry is two dimensional and confined in an elongated area of large width and length, the envelope of a uniform wave train is found to turn to a number of dark solitons in the shadow [30].

In this paper, we study long waves in shallow water over a random seabed in order to see how harmonic generation by nonlinearity is counteracted by localization. We shall begin with the Boussinesq equations [31] which account for weak nonlinearity and dispersion to the leading order. Evolution equations for all harmonic amplitudes will be derived. The effects of multiple scattering due to weakly random irregularities on the seabed will be shown to give rise to linear damping terms whose coefficients are found analytically for a prescribed correlation function. An equation for the evolution of the wave energy will be derived for any finite number of harmonics and used to verify numerical results. Physical implications will be examined through numerical solutions of these evolution equations.

II. BOUSSINESQ APPROXIMATION FOR LONG WATER WAVES

Consider one-dimensional long waves in shallow water. Using primes to distinguish quantities with physical dimensions, let $h'(x',t')$ denote the local depth beneath the still water level, $\eta'(x',t')$ the free surface displacement above, and $u'(x',t')$ the depth-averaged horizontal velocity of the water. It is well known that, to the leading order of nonlinearity and dispersion, the laws of mass and momentum conservation are approximated by

$$\frac{\partial \eta'}{\partial t'}+\frac{\partial}{\partial x'}[(h'+\eta')u']=0, \tag{2.1}$$

$$\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+g\frac{\partial \eta'}{\partial x'}=\frac{h'}{2}\frac{\partial^2}{\partial x'^2}\left(h'\frac{\partial u'}{\partial t'}\right)-\frac{h'^2}{6}\frac{\partial^3 u'}{\partial x'^2\partial t'}. \tag{2.2}$$

The accuracy of these equations can be made explicit by employing the following dimensionless variables without primes:

$$x=Kx', \quad t=t'K\sqrt{gH}, \quad \eta=\frac{\eta'}{a},$$

$$h=\frac{h'}{H}, \quad u=\frac{u'}{a\sqrt{g/H}}, \tag{2.3}$$

where K, a, and H are, respectively, the typical wave number, wave amplitude, and mean depth. Equations (2.1) and (2.2) can then be normalized in the following form:

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}[(h+\epsilon\eta)u]=0, \tag{2.4}$$

$$\frac{\partial u}{\partial t}+\epsilon u\frac{\partial u}{\partial x}+\frac{\partial \eta}{\partial x}=\frac{\mu^2 h}{2}\frac{\partial^2}{\partial x^2}\left(h\frac{\partial u}{\partial t}\right)-\frac{\mu^2 h^2}{6}\frac{\partial^3 u}{\partial x^2\partial t}, \tag{2.5}$$

where

$$\epsilon=\frac{a}{H}\ll 1, \quad \mu=KH\ll 1. \tag{2.6}$$

These equations are accurate to the leading order in ϵ and μ^2, which are small but independent parameters characterizing, respectively, nonlinearity and dispersion. Since terms of higher order in both effects are excluded, the accuracy of this Boussinesq approximation is limited to $O(\epsilon)=O(\mu^2)\ll 1$.

We assume that the sea depth deviates only slightly from a constant mean value, by an amount somewhat larger than the typical wave amplitude. In dimensionless terms, h fluctuates from the constant 1 by $\sqrt{\epsilon}b(x)$, i.e.,

$$h(x)=1-\sqrt{\epsilon}b(x), \tag{2.7}$$

where b is a stationary random function of x with zero mean, $\langle b(x)\rangle=0$. Equation (2.4) becomes

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}[(1-\sqrt{\epsilon}b+\epsilon\eta)u]=0. \tag{2.8}$$

By retaining terms only up to $O(\epsilon)$ and $O(\mu^2)$, Eq. (2.5) reduces to

$$\frac{\partial u}{\partial t}+\epsilon u\frac{\partial u}{\partial x}+\frac{\partial \eta}{\partial x}=\frac{\mu^2}{3}\frac{\partial^3 u}{\partial x^2\partial t}, \tag{2.9}$$

which can be used to combine with Eq. (2.8) to yield

$$\frac{\partial^2\eta}{\partial t^2}-\frac{\partial^2\eta}{\partial x^2}=-\sqrt{\epsilon}\left(b\frac{\partial\eta}{\partial x}\right)+\frac{\epsilon}{2}\left(\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial t^2}+\frac{\partial^2\eta^2}{\partial t^2}\right)+\frac{\mu^2}{3}\frac{\partial^4\eta}{\partial x^4}. \tag{2.10}$$

We next seek approximations for the propagation of a wave train which is a simple harmonic at some station to the left of the region of disorder.

III. ASYMPTOTIC EXPANSIONS

As is readily shown in Eq. (2.10), without bed roughness and nonlinearity and since $\mu^2\ll 1$, the dispersion relation of a progressive wave $(\eta,u)\propto e^{\pm i(kx-\omega t)}$ is nearly a straight line. With weak nonlinearity the lowest few harmonics $e^{\pm im(kx-\omega t)}$, with $m=1,2,3,\ldots$, can resonate each other via quadratic interaction. Interactions with and among higher harmonics are weak because of the increasing phase mismatch.

For a plane seabed ($b\equiv 0$), Mei and Ünluata [5] have shown that both nonlinearity and dispersion become important after a physical distance of $O(1/\epsilon K)$. For a randomly rough seabed, it can also be estimated that the localization distance is inversely proportional to the root-mean-square height of the random perturbations, i.e., $O(1/\epsilon K)$ under the assumption of Eq. (2.7). We therefore introduce the multiple scale coordinates x and $X=\epsilon x$, and expand η and u in ascending powers of $\sqrt{\epsilon}$:

$$\eta=\eta_0+\epsilon^{1/2}\eta_1+\epsilon\eta_2+\cdots,$$

$$u=u_0+\epsilon^{1/2}u_1+\epsilon u_2+\cdots, \tag{3.1}$$

where each unknown function depends on t and on the fast and slow variables in space x and $X=\epsilon x$. For successive orders, perturbation equations are found from Eq. (2.10):

$$\frac{\partial^2\eta_0}{\partial t^2}-\frac{\partial^2\eta_0}{\partial x^2}=0, \tag{3.2}$$

$$\frac{\partial^2\eta_1}{\partial t^2}-\frac{\partial^2\eta_1}{\partial x^2}=-\frac{\partial}{\partial x}\left(b\frac{\partial\eta_0}{\partial x}\right), \tag{3.3}$$

$$\frac{\partial^2\eta_2}{\partial t^2}-\frac{\partial^2\eta_2}{\partial x^2}=-\frac{\partial}{\partial x}\left(b\frac{\partial\eta_1}{\partial x}\right)+2\frac{\partial^2\eta_0}{\partial X\partial x}+\frac{3}{2}\frac{\partial^2\eta_0^2}{\partial t^2}+\frac{\nu}{3}\frac{\partial^4\eta_0}{\partial x^4}, \tag{3.4}$$

where

$$\nu=\mu^2/\epsilon=O(1) \tag{3.5}$$

measures the relative importance of dispersion vs nonlinearity and is the reciprocal of Ursell's parameter. The right-hand side of Eq. (3.4) has been simplified by using the leading order approximation.

We now solve the perturbation problems sequentially.

A. Equation and solution at $O(\epsilon^0)$

Consider the evolution of a train of progressive waves whose harmonics have the amplitudes A_m, the frequencies $\omega_m=m\omega$, and the wave numbers k_m, with $m=1,2,\ldots$:

$$\eta_0=\frac{1}{2}\sum_{m=-\infty}^{\infty} A_m(X)e^{i\theta_m},\quad u_0=\frac{1}{2}\sum_{m=-\infty}^{\infty} B_m(X)e^{i\theta_m}, \tag{3.6}$$

where θ_m denotes the wave phase,

$$\theta_m=k_m x-\omega_m t, \tag{3.7}$$

with $k_{-m}=-k_m$ and $\omega_{-m}=-\omega_m$, and

$$A_{-m}=A_m^* \quad \text{and} \quad B_{-m}=B_m^*. \tag{3.8}$$

Here A^* denotes the complex conjugate of A. In order that the normalized mean depth is unity, we set $A_0=0$. It follows from Eq. (3.2) that

$$\omega=k,\quad \omega_m=k_m(=mk), \tag{3.9}$$

which are the first-order dispersion relations. The following normalized wave numbers and frequencies are implied: $k_m=k_m'/K$ and $\omega_m=\omega_m'/K\sqrt{gH}$.

B. Equation and solution at $O(\epsilon^{1/2})$

The forcing terms in Eq. (3.3) can be expanded and separated into time harmonics,

$$-\frac{\partial}{\partial x}\left(b\frac{\partial\eta_0}{\partial x}\right)=-\sum_{m=-\infty}^{\infty}F_m e^{-i\omega_m t}, \tag{3.10}$$

where the coefficients F_m are random functions of x,

$$F_m=\frac{1}{2}ik_m A_m(X)\frac{d}{dx}[b(x)e^{ik_m x}] \tag{3.11}$$

and $F_0=0$. The solution of Eq. (3.10) can be written in the form

$$\eta_1=\sum_{m=-\infty}^{\infty}\eta_1^{(m)}e^{-i\omega_m t},\quad \eta_1^{(0)}=0. \tag{3.12}$$

For every $\eta_1^{(m)}$, where $m\neq 0$, the governing equation is

$$\frac{d^2\eta_1^{(m)}}{dx^2}+k_m^2\eta_1^{(m)}=F_m(x). \tag{3.13}$$

By using the Green function

$$G_m(|x-x'|)=\frac{e^{ik_m|x-x'|}}{2ik_m}, \tag{3.14}$$

the solution for η_1 is easily found to be

$$\eta_1=\sum_{m=-\infty}^{\infty}e^{-i\omega_m t}\int_{-\infty}^{\infty}ik_m G_m(|x-x'|)\times\frac{A_m(X)}{2}\frac{d}{dx'}[b(x')e^{ik_m x'}]dx', \tag{3.15}$$

which behaves as outgoing waves at infinities. Using angular brackets to denote the ensemble average, we see readily that $\langle\eta_1\rangle=0$.

C. Problem at $O(\epsilon)$ and amplitude evolution equations

Let us take the ensemble average of Eq. (3.4),

$$\frac{\partial^2\langle\eta_2\rangle}{\partial t^2}-\frac{\partial^2\langle\eta_2\rangle}{\partial x^2}=-\frac{\partial}{\partial x}\left\langle b\frac{\partial\eta_1}{\partial x}\right\rangle+2\frac{\partial^2\eta_0}{\partial X\partial x}+\frac{3}{2}\frac{\partial^2\eta_0^2}{\partial t^2}+\frac{\nu}{3}\frac{\partial^4\eta_0}{\partial x^4}. \tag{3.16}$$

The forcing terms on the right-hand side are calculated below.

Using the known solution for $\eta_1^{(m)}$, the first forcing term on the right-hand side can be decomposed into

$$\left\langle b\frac{\partial\eta_1^{(m)}}{\partial x}\right\rangle=-\int_{-\infty}^{\infty}k_m^2\,\mathrm{sgn}(x-x')G_m(|x-x'|)\frac{A_m}{2}\times\frac{d}{dx'}[\langle b(x)b(x')\rangle e^{ik_m x'}dx']. \tag{3.17}$$

We now add the assumption that b is a stationary random function of x on the fast scale so that

$$\langle b(x)b(x')\rangle=\sigma^2\gamma(|x-x'|), \tag{3.18}$$

where $\sigma(X)$ is the root-mean-square height of the roughness and $\gamma(x-x')$ is the autocorrelation function of the bed roughness. It then follows that

$$\begin{aligned}\left\langle b\frac{\partial\eta_1^{(m)}}{\partial x}\right\rangle&=ik_m\frac{A_m}{4}e^{ik_m x}\int_{-\infty}^{\infty}dx'\,\mathrm{sgn}(x-x')e^{ik_m(|x-x'|)}\times\frac{d}{dx'}[\gamma(x-x')e^{ik_m(x'-x)}]\\&=-ik_m A_m\frac{\sigma^2}{4}e^{ik_m x}\int_{-\infty}^{\infty}d\xi\,\mathrm{sgn}(\xi)e^{ik_m|\xi|}\times\frac{d}{d\xi}[\gamma(\xi)e^{-ik_m\xi}].\end{aligned} \tag{3.19}$$

Note that Eq. (3.19) is zero for $m=0$ and the result for $m<0$ is equal to the complex conjugate of the result for $m>0$, hence

$$-\frac{\partial}{\partial x}\left[\sum_{m=-\infty}^{\infty} e^{-i\omega_m t}\left\langle b\frac{\partial \eta_1^{(m)}}{\partial x}\right\rangle\right]=\sum_{m=1}^{\infty} ik_m A_m(X)\beta_m e^{i\theta_m} + \text{c.c.}, \tag{3.20}$$

where the coefficient $\beta_m=\text{Re}\beta_m+i\,\text{Im}\beta_m$ is complex and defined by

$$\beta_m=\frac{\sigma^2}{4}ik_m\int_{-\infty}^{\infty}\text{sgn}(\xi)\left(\frac{d\gamma}{d\xi}-ik_m\gamma\right)e^{ik_m(|\xi|-\xi)}d\xi. \tag{3.21}$$

By following the procedure in Ref. [31], the nonlinear forcing term in Eq. (3.16) can be shown to be

$$\frac{3}{2}\frac{\partial^2\eta_0^2}{\partial t^2}=\sum_{m=1}^{\infty}-\frac{3}{8}\omega_m^2 e^{i\theta_m}\left[\sum_{l=1}^{\infty}2A_l^*A_{m+l}+\sum_{l=1}^{[m/2]}\alpha_l A_l A_{m-l}\right] + \text{c.c.}, \tag{3.22}$$

where $[m/2]$ is the integer part of $m/2$, and α_l is a coefficient equal to 1 for $l=[m/2]$ and equal to 2 otherwise. The last forcing term is

$$\frac{\nu}{3}\frac{\partial^4\eta_0}{\partial x^4}=\sum_{m=1}^{\infty}\frac{\nu}{6}k_m^4 A_m e^{i\theta_m}+\text{c.c.} \tag{3.23}$$

In summary Eq. (3.16) can be rewritten as

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\langle\eta_2\rangle=\sum_{m=1}^{\infty}ik_m A_m\beta_m e^{i\theta_m}+\sum_{m=1}^{\infty}ik_m\frac{dA_m}{dX}e^{i\theta_m} -\sum_{m=1}^{\infty}\frac{3}{8}\omega_m^2 e^{i\theta_m}\left[\sum_{l=1}^{\infty}2A_l^*A_{m+l} +\sum_{l=1}^{[m/2]}\alpha_l A_l A_{m-l}\right]+\sum_{m=1}^{\infty}\frac{\nu}{6}k_m^4 A_m e^{i\theta_m} +\text{c.c.} \tag{3.24}$$

To ensure solvability of the preceding equation, secular terms proportional to $\exp(i\theta_m)$ must be removed. With the help of the dispersion relation $\omega_m=k_m$, we get

$$\frac{dA_m}{dX}+\beta_m A_m-i\frac{\nu}{6}k_m^3A_m+\frac{3}{8}i\omega_m\left[\sum_{l=1}^{\infty}2A_l^*A_{m+l} +\sum_{l=1}^{[m/2]}\alpha_l A_l A_{m-l}\right]=0,\quad m=1,2,\ldots,\infty. \tag{3.25}$$

This result constitutes an infinite number of nonlinearly coupled equations governing the slow spatial evolution of harmonic amplitudes, and extends the theory of Ref. [6] for harmonic generation over a smooth seabed (see also Ref. [31]). The linear terms with complex coefficients β_m represent the effects of multiple scattering by disorder. It will be shown in the following section that $(\text{Re}\beta_m)^{-1}$ is positive and represents the length scale of spatial attenuation, i.e., localization.

To proceed further, one must first prescribe the correlation function $\gamma(|x-x'|)$, truncate the series at some finite but large n, and solve the truncated system by numerical means. The truncated differential system is

$$\frac{dA_m}{dX}+\beta_m A_m-i\frac{\nu}{6}k_m^3A_m+\frac{3}{8}i\omega_m\left[\sum_{l=1}^{n-m}2A_l^*A_{m+l} +\sum_{l=1}^{[m/2]}\alpha_l A_l A_{m-l}\right]=0,\quad m=1,2,\ldots,n. \tag{3.26}$$

Note that the infinite l series must be truncated at $l=n-m$ because the $A_l^*A_{m+l}$ terms are summed over l such that $m+l\leqslant n$, only the first n harmonics are taken into account.

We remark that the two-harmonic system $(n=2)$ without randomness is the basis of the second-harmonic-generation theory in nonlinear optics [4], and has been shown in Ref. [5] to give good predictions for laboratory observations of shallow water waves over a plane seabed [32].

IV. THE COEFFICIENTS β_m

For illustration we assume the correlation function to be Gaussian:

$$\gamma(\xi)=\exp\left(-\frac{\xi^2}{2l^2}\right), \tag{4.1}$$

where $l=Kl'$ is the ratio of the correlation distance to the characteristic wavelength.

To calculate β_m, let us note first that

$$\left\langle b(x)\frac{db}{dx'}\right\rangle=-\left\langle b(x')\frac{db}{dx}\right\rangle=-\frac{d\gamma}{d\xi}. \tag{4.2}$$

The following integrals can be evaluated as

$$\int_{-\infty}^{\infty}\text{sgn}(\xi)\gamma(\xi)e^{ik_m(|\xi|-\xi)}d\xi =-\int_{-\infty}^{0}e^{-(\xi^2/2l^2)}e^{-2ik_m\xi}d\xi+\int_0^{\infty}e^{-(\xi^2/2l^2)}d\xi =l\sqrt{2}\left(\frac{\sqrt{\pi}}{2}-\int_{-\infty}^{0}e^{-u^2-2i\sqrt{2}k_m lu}du\right) \tag{4.3}$$

and

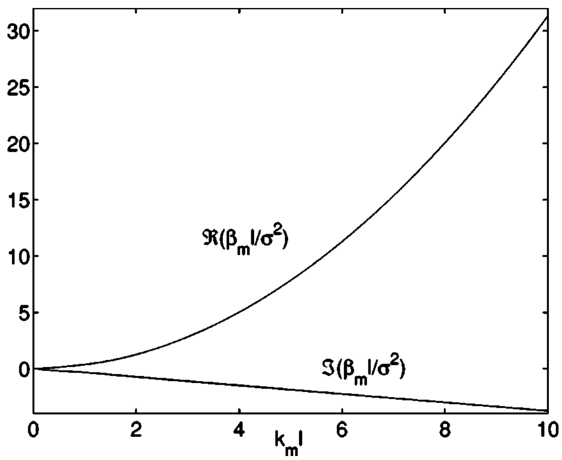

FIG. 1. Real and imaginary parts of $\beta_m l/\sigma^2$.

$$\int_{-\infty}^{\infty}\operatorname{sgn}(\xi)\frac{d\gamma}{d\xi}e^{ik_m(|\xi|-\xi)}d\xi$$

$$=\frac{1}{l^2}\left(\int_{-\infty}^{0}\xi e^{-(\xi^2/2l^2)}e^{-2ik_m\xi}d\xi-\int_{0}^{\infty}\xi e^{-(\xi^2/2l^2)}d\xi\right)$$

$$=-2-2ik_m l\sqrt{2}\int_{-\infty}^{0}e^{-u^2-2i\sqrt{2}k_m l u}du. \tag{4.4}$$

These results can be used to give

$$\frac{\beta_m l}{\sigma^2}=k_m^2 l^2\frac{\sqrt{2\pi}}{8}(1+e^{-2k_m^2l^2})-i\frac{k_m l}{2}\left(1-\frac{k_m l}{\sqrt{2}}\times e^{-2k_m^2l^2}\int_0^{\sqrt{2}k_m l}e^{u^2}du\right). \tag{4.5}$$

Clearly $\mathrm{Re}\beta_m>0$ for all m, implying that all harmonics are localized. For each harmonic, the product $\mathrm{Re}\beta_m l$ is the ratio of the correlation length to the localization length. For sufficiently high harmonics, $\mathrm{Re}\beta_m l$ is essentially proportional to $k_m^2l^2=m^2k^2l^2$. Thus higher harmonics are more localized, i.e., attenuated in a shorter distance. Since $l=Kl'$, for a fixed l', shorter waves are more strongly localized. A similar result has been reported before in Ref. [29]. Figure 1 shows the dependence of the real and imaginary parts of $\beta_m l/\sigma^2$ on $k_m l$.

V. EVOLUTION OF WAVE ENERGY

To help understand the physics of harmonic generation and localization, and to provide a means to assess computational accuracy, we shall prove the following general relation on the first-order wave energy:

$$\frac{d}{dX}\sum_{m=1}^{n}|A_m|^2=-2\sum_{m=1}^{n}\mathrm{Re}(\beta_m)|A_m|^2, \tag{5.1}$$

where n is any integer representing the highest harmonic in the truncated differential system. Physically, due to second-order multiple scattering by disorder, the total wave energy of all leading-order harmonics decreases with propagation distance. Without disorder $\mathrm{Re}\beta_m=0$, the total leading-order energy is conserved. Although Bryant [6] first derived the evolution equations for an infinite number of harmonics generated in shallow-water waves over a smooth bed, his nth-order differential system was not properly truncated. In consequence, he did not succeed in proving Eq. (5.1).

We shall prove the general relation by the method of induction. For one harmonic $(n=1)$, Eq. (5.1) can be shown readily by multiplying the governing equation by A^*,

$$\frac{dA}{dX}+\left(\beta-i\frac{\nu}{6}k^3\right)A=0, \tag{5.2}$$

and adding the result to its complex conjugate. For $n=2$, the governing differential equations are

$$\frac{dA_1}{dX}+\left(\beta_1-i\frac{\nu}{6}k_1^3\right)A_1+\frac{3}{4}ik_1A_1^*A_2=0, \tag{5.3}$$

$$\frac{dA_2}{dX}+\left(\beta_2-i\frac{\nu}{6}k_2^3\right)A_2+\frac{3}{8}ik_2A_1^2=0. \tag{5.4}$$

Multiplying Eq. (5.3) by A_1^* and adding the resulting equation to its complex conjugate yields

$$\frac{dA_1A_1^*}{dX}+2\,\mathrm{Re}(\beta_1)A_1A_1^*-\frac{3}{4}k_1\mathrm{Im}(A_1^{*2}A_2)=0. \tag{5.5}$$

Similarly for the second harmonic,

$$\frac{dA_2A_2^*}{dX}+2\,\mathrm{Re}(\beta_2)A_2A_2^*-\frac{3}{8}k_2\mathrm{Im}(A_1^2A_2^*)=0. \tag{5.6}$$

Adding Eqs. (5.5) and (5.6), the following is obtained

$$\sum_{m=1}^{2}\left(\frac{d|A_m|^2}{dX}+2\,\mathrm{Re}\beta_m|A_m|^2\right)=\frac{3}{4}k_1\mathrm{Im}(A_1^{*2}A_2)+\frac{3}{8}2k_1\mathrm{Im}(A_1^2A_2^*). \tag{5.7}$$

Since $k_2=2k_1$, the right-hand side of Eq. (5.7) is zero; Eq. (5.1) is proven. In the limit of a smooth bottom, $\beta_m=0$, $m=1,2$, the total energy of the two harmonics is constant in X, although energy can be interchanged between the first and second harmonics. This result for the two harmonics is known as the Manley-Rowe relation in parametric electronics.

For any integer $n>2$, we first multiply Eq. (3.26) by A_m^*, add the result to its complex conjugate, and then perform the summation in m from 1 to n to get

$$\sum_{m=1}^{n}\left(\frac{d}{dX}|A_m|^2+2\,\mathrm{Re}(\beta_m)|A_m|^2\right)=\frac{3}{8}H_n, \tag{5.8}$$

where

$$H_n=\sum_{m=1}^{n} k_m \mathrm{Im}\left[\sum_{l=1}^{n-m} A_l^* A_{m+l} A_m^* + \sum_{l=1}^{[m/2]} \alpha_l A_l A_{m-l} A_m^*\right]. \tag{5.9}$$

The task is to prove that, if H_n vanishes for any n, H_{n+1} must vanish also. Note that we can rewrite

$$\begin{aligned} H_{n+1} &= \sum_{m=1}^{n+1} k_m \mathrm{Im}\left[\sum_{l=1}^{n+1-m} A_l^* A_{m+l} A_m^* + \sum_{l=1}^{[m/2]} \alpha_l A_l A_{m-l} A_m^*\right] \\ &= H_n + (n+1) k_1 \mathrm{Im}\left[\sum_{l=1}^{[(n+1)/2]} \alpha_l A_l A_{n+1-l} A_{n+1}^*\right] \\ &\quad + \sum_{m=1}^{n} m k_1 \mathrm{Im}[2 A_{n+1-m}^* A_{n+1} A_m^*] \end{aligned} \tag{5.10}$$

with $\alpha_l=1$ for $l=(n+1)/2$. It is necessary to examine separately odd and even values of n.

A. n=odd

Let $n=2p-1$ in Eq. (5.10), where p is a positive integer. Recall that $\alpha_l=1$ only for $l=p$, and $\alpha_l=2$ otherwise. Assuming $H_n=H_{2p-1}=0$, the remaining part of Eq. (5.10) is

$$\begin{aligned} H_{2p} = k_1 \mathrm{Im}\Bigg[& 2p A_{2p}^* \sum_{l=1}^{p-1} 2 A_l A_{2p-l} + 2p A_{2p}^* A_p A_p \\ & + 2 A_{2p} \sum_{m=1}^{2p-1} m A_{2p-m}^* A_m^*\Bigg]. \end{aligned} \tag{5.11}$$

Expanding the sum over m,

$$\begin{aligned} \sum_{m=1}^{2p-1} m A_{2p-m}^* A_m^* &= A_1^* A_{2p-1}^* + 2 A_2^* A_{2p-2}^* + \cdots + p A_p^* A_p^* \\ &\quad + \cdots + (2p-2) A_{2p-2}^* A_2^* + (2p-1) A_{2p-1}^* A_1^* \\ &= 2p \sum_{m=1}^{p-1} A_m^* A_{2p-m}^* + p A_p^* A_p^*, \end{aligned} \tag{5.12}$$

we get

$$\begin{aligned} H_{2p} = k_1 \mathrm{Im}\Bigg[& 2p A_{2p}^* \sum_{l=1}^{p-1} 2 A_l A_{2p-l} + 2p A_{2p}^* A_p A_p \\ & + 2p A_{2p}\left(2\sum_{m=1}^{p-1} A_m^* A_{2p-m}^* + A_p^* A_p^*\right)\Bigg] = 0, \end{aligned} \tag{5.13}$$

since the quantity inside the brackets is real.

B. n=even

Let $n=2p$ in Eq. (5.10), where p is an integer. Now $\alpha_l=2$ for all l as $(2p+1)/2$ is not an integer. With $H_{2p}=0$, the remaining part of Eq. (5.10) is

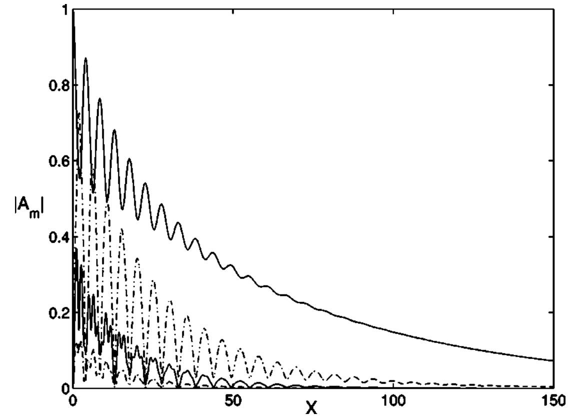

FIG. 2. Evolution of the first four harmonics ($m=1,2,3,4$ from top down) over a semi-infinite region of disorder. $\sigma=0.2$, $\nu=1$, and $l=k_1=1$.

$$\begin{aligned} H_{2p+1} = k_1 \mathrm{Im}\Bigg[& (2p+1) A_{2p+1}^* \sum_{l=1}^{p} 2 A_l A_{2p+1-l} \\ & + 2 A_{2p+1} \sum_{m=1}^{2p} m A_{2p+1-m}^* A_m^*\Bigg]. \end{aligned} \tag{5.14}$$

Expanding the sum over m,

$$\begin{aligned} \sum_{m=1}^{2p} m A_{2p+1-m}^* A_m^* &= A_1^* A_{2p}^* + 2 A_2^* A_{2p-1}^* + \cdots + (2p-1) A_{2p-1}^* A_2^* \\ &\quad + (2p) A_{2p}^* A_1^* \\ &= (2p+1) \sum_{m=1}^{p} A_m^* A_{2p+1-m}^*, \end{aligned} \tag{5.15}$$

we get

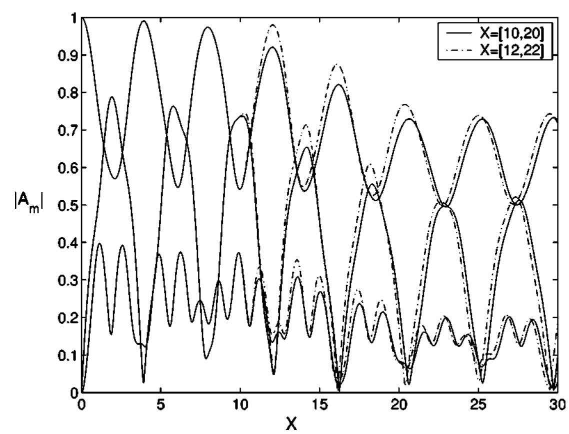

FIG. 3. Incident wave phase effect on the first three harmonics ($m=1,2,3$ from top down). The width of the disordered region is 10. Solid lines: disorder is in $X=[10,20]$; dash-dotted line: disorder is in $X=[12,22]$. $\sigma=0.2$, $\nu=1$, and $l=k_1=1$.

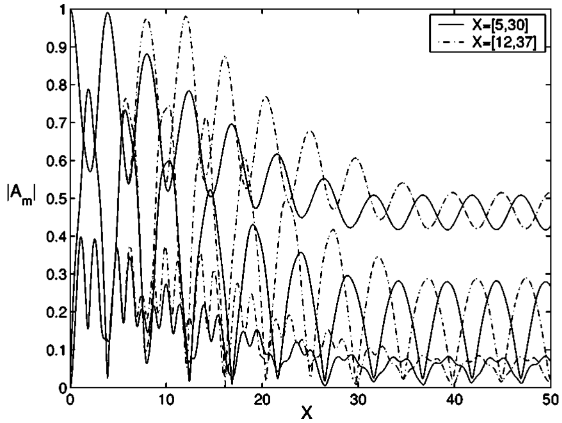

FIG. 4. Effects of incident wave phase on the first three harmonics ($m=1,2,3$ from top down). The width of the disordered region is 25. Solid lines: disorder is in $X=[5,30]$, dash-dotted line: disorder is in $X=[12,37]$. $\sigma=0.2$, $\nu=1$, and $l=k_1=1$.

$$H_{2p+1}=k_1 \mathrm{Im}\left[(2p+1)A^*_{2p+1}\sum_{l=1}^{p} 2A_l A_{2p+1-l}+\text{c.c.}\right]=0. \tag{5.16}$$

Thus $H_{n+1}=0$ if $H_n=0$. Since $H_1=H_2=0$, it follows by induction that $H_3=H_4=H_5=\cdots=H_n=0$ for any integer *n*. The energy relation (5.1) is proven.

VI. NUMERICAL RESULTS OF HARMONIC EVOLUTION AND LOCALIZATION

For $n=2$, an analytical solution is possible only for $\beta_1=\beta_2$, which is not the case here unless the bottom is perfectly smooth so that $\beta_1=\beta_2=0$. In this section, we report the numerical results based on finite-difference solution of the differential system truncated at $n=6$. Accuracy is assured since the energy relation is satisfied in all cases with errors in the range of 10^{-9}–10^{-12}. The results also differ very little from those for $n=10$.

The typical features can be displayed for a semi-infinite region of disorder with $\sigma=0, X<0; \sigma=0.2, X>0$. Incident waves with only the first harmonic arrive from $X<0$. For the parameters $k_1=1$, $l=1$, and $\nu=\mu^2/\epsilon=1$, Fig. 2 shows the evolution of the first four harmonics. Second and higher harmonics are seen to grow at the expense of the first, but they all attenuate with distance; the higher harmonics die out sooner. Once the second and higher harmonics are sufficiently diminished, energy exchange with the first harmonic becomes insignificant; the latter then attenuates monotonically with distance.

Next we examine the effects of several parameters in turn: the position and size of the region of randomness, the root-mean-square height σ, the ratio of correlation length to wavelength *l*, and the ratio of dispersion to nonlinearity $\mu^2/\epsilon=\nu$. Let the incident wave first go through harmonic generation over a smooth bed, then pass over a disordered region of finite extent.

To see the possible effects of the wave phase, we compare in Fig. 3 two cases where the randomness has the same σ

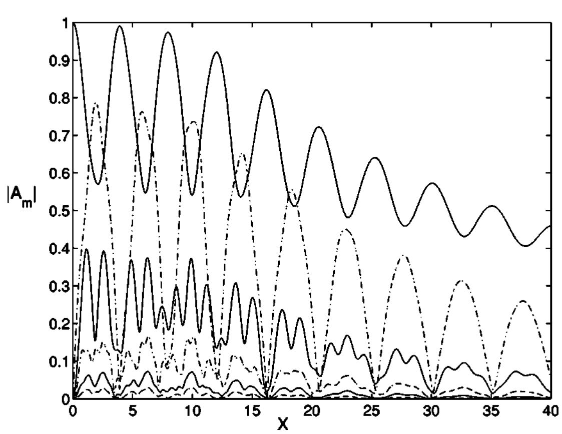

FIG. 5. Localization of harmonics over a semi-infinite region ($X>10$) of weak disorder ($\sigma=0.2$) ($m=1,2,\ldots,6$ from top down). Other parameters are $l=1$, $\nu=1$, and $k_1=1$. Solid lines: odd harmonics (1,3,5); dash-dotted lines: even harmonics (2,4,6).

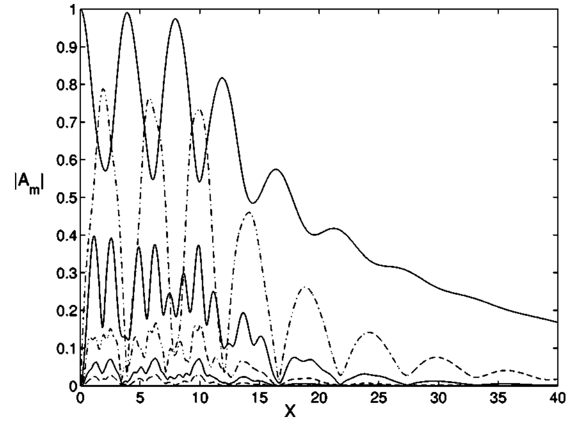

FIG. 6. Localization of harmonics over a semi-infinite region ($X>10$) of stronger disorder ($\sigma=0.35$) ($m=1,2,\ldots,6$). Other parameters are $l=1$, $\nu=1$, and $k_1=1$. Solid lines: odd harmonics (1,3,5); dash-dotted lines: even harmonics (2,4,6).

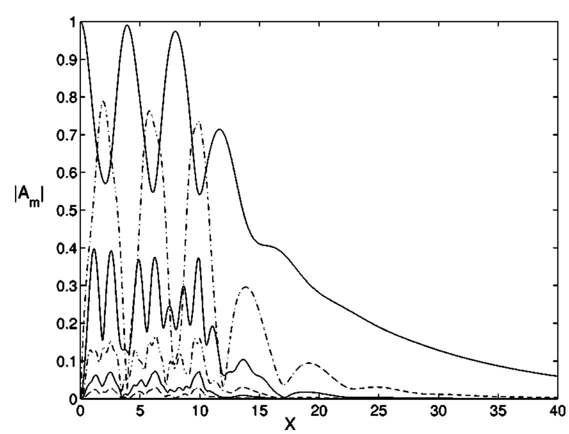

FIG. 7. Localization of harmonics over a semi-infinite region ($X>10$) of larger correlation length relative to the wavelength ($l=2$). Other parameters are $\sigma=0.35$, $\nu=1$, and $k_1=1$. Solid lines from top down: odd harmonics (1,3,5); dash-dotted line from top down: even harmonics (2,4,6).

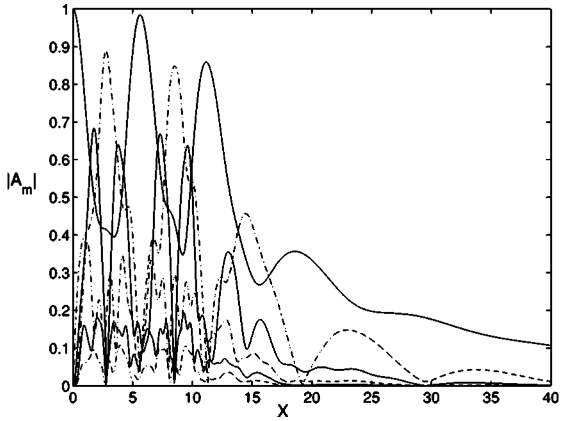

FIG. 8. Localization of harmonics over a semi-infinite region of disorder ($X>10$) for more nonlinear waves ($\nu=0.5$). Other parameters are $\sigma=0.35$, $l=1$, and $k_1=1$. Solid lines from top down: odd harmonics (1,3,5); dash-dotted line from top down: even harmonics (2,4,6).

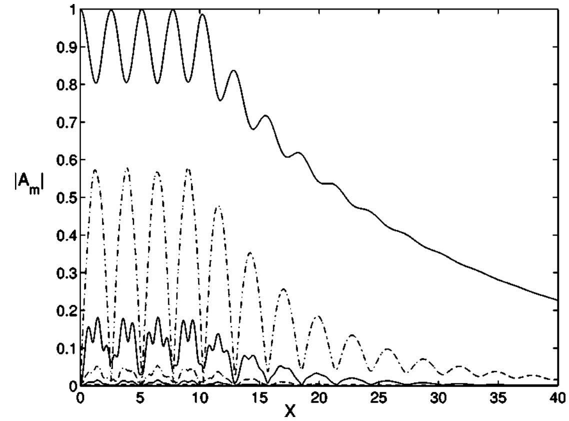

FIG. 9. Localization of harmonics over a semi-infinite region of disorder ($X>10$) for less nonlinear waves ($\nu=2$). Other parameters are $\sigma=0.35$, $l=1$, and $k_1=1$. Solid line: odd harmonics (1,3,5); dash-dotted line: even harmonics (2,4,6).

$=0.2$ and horizontal extent $=10$. In both cases the first harmonic amplitude is unity at $X=0$, while all higher harmonics are zero. However, in one, disorder begins at $X=10$ where the first harmonic is the smallest and the second harmonic the largest. In the other, disorder begins at $X=12$ where the first harmonic is the largest and the second harmonic the smallest. After passing the region of disorder, the reduced harmonics of the two wave trains are seen to differ in phase but not in amplitude.

Figure 4 compares two cases for a wider region of disorder equal to 25. For the same incident waves at $X=0$, the random bottom extends over $X=[5,30]$ in one and over $X=[12,37]$ in the other. After the region of randomness, harmonics of both waves are diminished. Again they differ in phase but not in amplitude. It may be concluded that the wave phase relative to the position of the randomness is of little consequence on localization.

As expected from the expressions of β_m, larger σ leads to faster localization. Quantitative confirmation can be seen in Figs. 5 and 6. In both cases, randomness starts at $X=10$ where the first harmonic is the smallest.

As pointed out before, greater l means shorter wavelength relative to the correlation length and stronger localization. This can be seen by comparing Figs. 6 and 7, both for a semi-infinite region of randomness $X>10$ with $\sigma=0.35$.

Finally, we examine the effects of nonlinearity and/or dispersion by varying the ratio $\mu^2/\epsilon=\nu$. The region of disorder is semi-infinite $X>10$. For the same region of randomness, $X>10$ and $\sigma=0.35$, we add Figs. 8 and 9 to compare them to Fig. 6, corresponding to $\nu=0.5$, 2, and 1, respectively.

It is clear that as ν increases (weaker nonlinearity and/or stronger dispersion), the second and higher harmonic amplitudes decrease and the modulation of all harmonics becomes more rapid. For the most nonlinear case with $\nu=0.5$, energy conversion from the first to the second and higher harmonics is very effective, yet they all are localized by radiation damping.

In conclusion, we have shown that two major advances of modern physics, harmonic generation and localization, are intertwined in a problem of coastal oceanography. Moreover, the asymptotic method of multiple scales, widely used for wave propagation in slowly varying or periodic media, is seen to be an effective tool for a weakly random medium as well. The same technique can likely be applied to other problems such as sound propagation in a shallow sea over a randomly rough seabed.

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