Counting multiple solutions in glassy random matrix models

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This is a first step in counting the number of multiple solutions in certain glassy random matrix models introduced by N. Deo [Phys. Rev. E **65**, 056115 (2002)]. We are able to do this by reducing the problem of counting the multiple solutions to that of a moment problem. More precisely, we count the number of different moments when we introduce an asymmetry (tapping) in the random matrix model and then take it to vanish. It is shown here that the number of moments grows exponentially with respect to N, the size of the matrix. As these models map onto models of structural glasses in the high temperature phase (liquid), this may have interesting implications for the supercooled liquid phase in these spin glass models. Further, it is shown that the nature of the asymmetry (tapping) is crucial in finding the multiple solutions. This also clarifies some of the puzzles raised by E. Brézin and N. Deo [Phys. Rev. E **59**, 3901 (1999)].

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I. INTRODUCTION

Random matrix models can be used very effectively as simple mathematical toy models, where many new ideas in physics, biology, and economics can be tested analytically, Ref. [1–3]. Here, we try to understand the idea of tapping and counting, well studied in the context of granular media, in the glassy random matrix model introduced in Ref. [4]. There, it was demonstrated that the matrix models with gaps in their eigenvalue distribution had multiple solutions and were related to the high temperature phase of certain p-spin glass models, Ref. [5]. We approach the problem in much the same spirit as done for spin systems in Ref. [6]. This is a first step in understanding what happens when we tap the model, i.e., introduce a perturbation and remove it. This enables us to count the number of different configurations. Studies to understand the fluctuation-dissipation relations and the relations between the dynamical and Edwards temperature in the dynamical matrix models await further work. This study will also help us understand some of the puzzles that we raised in Ref. [7]. One of the puzzles in these models is that the long range correlators found in Ref. [8] by the mean field calculations differ from that found in Ref. [9,7] using the orthogonal polynomial methods. A resolution of this has been suggested in Ref. [10] where it is claimed that the difference arises due to the discreteness of the number of eigenvalues for double-well models with equal depths. Here, we try to understand these results using the method of moments.

Most of the studies and applications of matrix models correspond to eigenvalue distributions on a single cut in the complex plane where the eigenvalue density is nonzero, Ref. [1]. Here, we study a one-Hermitian matrix model with a more complicated eigenvalue structure. These have found applications in two-dimensional quantum gravity, string theory, disordered condensed matter systems, superconductors (with complex vector potential and with impurities), and glasses. Here, we study these models with applications to glasses in mind, as discussed in Ref. [4]. To illustrate some of the generic properties, we study a one-Hermitian matrix model with two cuts for the eigenvalue density. One of the important differences observed in these models is that these have multiple solutions, which show up in certain correlation functions. Here, we count the number of multiple solutions and explore the possibility that these multiple solutions arise by taking different paths in phase space (each path may correspond to a different metastable glassy state). It is important to establish the correspondence between the multiple solutions and the metastable glassy states. The barrier heights corresponding to these various solutions are also future goals.

I will discuss here the matrix model with double-well potential, the M^4 model (the Gaussian Penner model where similar things happen will be pursued elsewhere). A tapping is introduced, which corresponds to coupling the matrix model to an external source. The limit of taking the external sources to vanish gives different values for the moments in these models. This may result in different values for the partition function and, hence, the free energy. Taking different tappings corresponds to exploring the full space of configurations. Here, we present the first steps in counting the number of different configurations and find it to be exponentially large.

After this work was completed, we find that in a different context the results of exponentially large number of minima have been reported in a renormalizable matrix potential with S_N using a different method given by Soljacic and Wilczek, Ref. [11].

II. NOTATIONS AND CONVENTIONS

Let M be a Hermitian matrix. The partition function to be considered is $Z=\int dM e^{-NtrV(M)}$, where $M=N\times N$, a Hermitian matrix. The Haar measure $dM=\prod_{i=1}^{N}dM_{ii}\prod_{i<j}dM_{ij}^{(1)}dM_{ij}^{(2)}$ with $M_{ij}=M_{ij}^{(1)}+iM_{ij}^{(2)}$ and N^2 independent variables. $V(M)$ is a polynomial in M: $V(M)=g_1M+(g_2/2)M^2+(g_3/3)M^3+(g_4/4)M^4+\cdots$. The partition function is invariant under the change of variable $M'=UMU^{\dagger}$, where U is a unitary matrix. We can use this invariance and go to the diagonal basis, i.e., $D'=UMU^{\dagger}$ such that D' is the matrix diagonal to M with eigenvalues

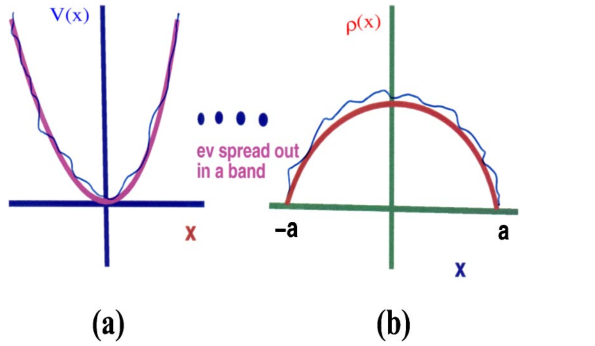

FIG. 1. (a) The confining potential. (b) The density of eigenvalues.

$\lambda_1, \lambda_2, \ldots, \lambda_N$. Then, the partition function becomes $Z = C\int_{-\infty}^{\infty}\Pi_{i=1}^{N} d\lambda_i \Delta(\lambda)^2 e^{-N\Sigma_{i=1}^{N} V(\lambda_i)}$, where $\Delta(\lambda) = \Pi_{i<j}|\lambda_i - \lambda_j|$ is the Vandermonde determinant. The integration over the group U with the appropriate measure is trivial and is just the constant C. By exponentiating the determinant as a "trace log," we arrive at the Dyson gas or Coulomb gas picture. The partition function is simply $Z = C\int_{-\infty}^{\infty}\Pi_{i=1}^{N} d\lambda_i e^{-S(\lambda)}$ with $S(\lambda) = N\Sigma_{i=1}^{N} V(\lambda_i) - 2\Sigma_{i,j,i\neq j}\ln|\lambda_i - \lambda_j|$.

This is just a system of N particles with coordinates λ_i on the real line, confined by a potential and repelling each other with a logarithmic repulsion. The spectrum or the density of eigenvalues $\rho(x) = (1/N)\Sigma_{i=1}^{N}\delta(x-\lambda_i)$ is in the large N limit or doing the saddle point analysis just the Wigner semicircle for a quadratic potential. The physical picture is that the eigenvalues try to be at the bottom of the well. But it costs energy to sit on top of each other because of logarithmic repulsion, so they spread. ρ has a support on a finite line segment. This continues to be true whether the potential is quadratic or a more general polynomial, and only depends on there being a single well though the shape of the Wigner semicircle is correspondingly modified. For the quadratic potential, the density is $\rho(x) = (1/\pi)\sqrt{(x-a)(b-x)}$ where $[a,b]$ are the ends of the cuts. See Figs. 1.

On changing the potential more drastically by having two humps or wells, the simplest example being a potential $V(M) = -(\mu/2)M^2 + (g/4)M^4$, the density can get disconnected support. The precise expressions for the density of eigenvalues are as follows:

$$\rho(x) = \frac{g}{\pi} x\sqrt{(x^2-a^2)(b^2-x^2)}, \quad a<x<b$$
$$= 0, \quad -b<x<-a, \tag{2.1}$$

where $a^2 = (1/g)[|\mu| - 2\sqrt{g}]$ and $b^2 = (1/g)[|\mu| + 2\sqrt{g}]$ with $|\mu| > 2\sqrt{g}$, which is the condition that the wells are sufficiently deep. The eigenvalues sit in the symmetric bands centered around each well. Thus, ρ has support on two line segments. As $|\mu|$ approaches $2\sqrt{g}$, $a \to 0$ and the two bands merge at the origin. Then the density is

$$\rho(x) = \frac{gx^2}{\pi}\sqrt{x^2 - \frac{2\mu}{g}}, \quad -\sqrt{\frac{2|\mu|}{g}} < x < \sqrt{\frac{2|\mu|}{g}},$$
$$= 0, \quad \text{otherwise.} \tag{2.2}$$

The phase diagram and density of eigenvalues for the M^4 potential are shown in Fig. 2.

The simplest way to determine $\rho(z)$ explicitly is to use the generating function $F(z) = \langle 1/N \,\mathrm{Tr}\, 1/(z-M)\rangle$ and its saddle point or Schwinger-Dyson equation also known in the mathematics literature as the Riemann-Hilbert problem $F(z) = \frac{1}{2}[V'(z) + \sqrt{\Delta}(z)]$ with $\Delta(z) = V'(z)^2 - 4b(z)$ and $b(z) = gz^2 + \mu + g\langle 1/N \,\mathrm{Tr}\, M^2\rangle$ (see Ref. [12]). The density $\rho(x)$ is then determined by the formula $\rho(z) = -(1/2\pi)\mathrm{Im}\sqrt{\Delta(z)}$. In what follows, the matrix model is tapped (that is, a small perturbation is added, which breaks

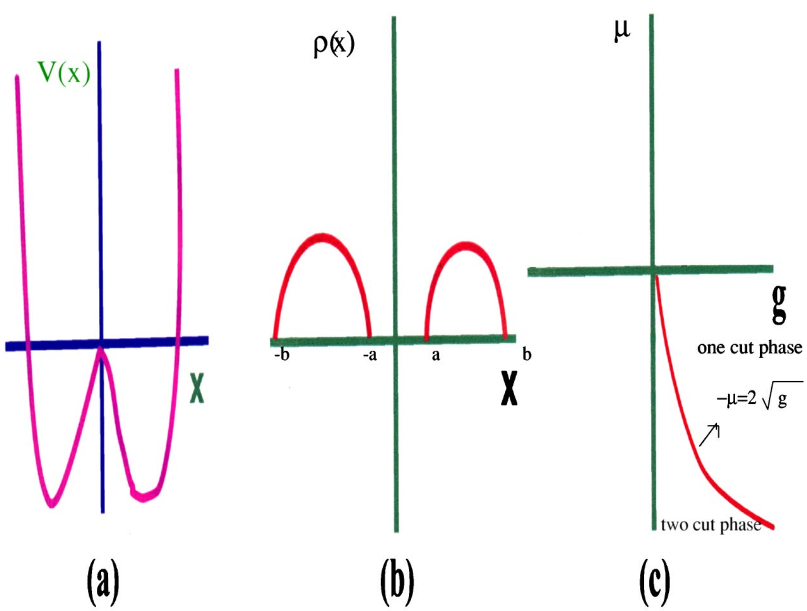

FIG. 2. (a) The double-well potential. (b) Density of eigenvalues. (c) The phase diagram.

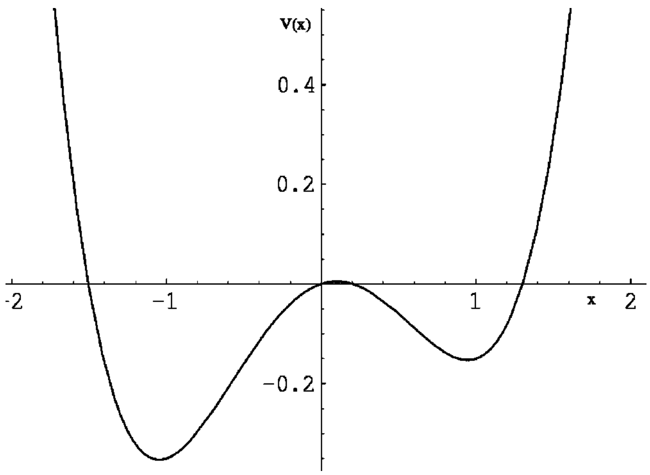

FIG. 3. The asymmetric potential $\tilde{V}(x)$.

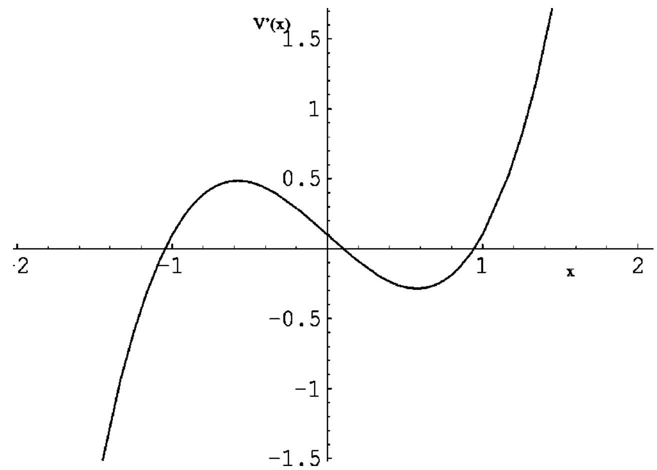

FIG. 4. Derivative of the asymmetric potential $\tilde{V}'(x)$.

the Z_2 symmetry) and the number of solutions corresponding to the different moments of the model is counted.

III. INTRODUCING ASYMMETRY (TAPPING)

Let us put a matrix source A, with an eigenvalue a_n, which will ultimately vanish in the partition function

$$Z_N(A)=\int dM e^{-N\,\mathrm{Tr}(V(M)-AM)}. \tag{3.1}$$

Using Harish-Chandra-Itzykson-Zuber formula

$$Z_N(A)=\int \prod_1^N d\lambda_i \frac{\Delta(\lambda)}{\Delta(a)} e^{-\sum_1^N [V(\lambda_i)-a_i\lambda_i]}, \tag{3.2}$$

where

$$\Delta(\lambda)=\mathrm{Det}\lambda_i^{j-1}. \tag{3.3}$$

Then, in terms of the moments, the partition function becomes

$$Z_N(A)=\frac{\mathrm{Det}(m_n(a_k))}{\Delta(a)} \tag{3.4}$$

with

$$m_n(a)=\int dx e^{-N[V(x)-ax]}x^n. \tag{3.5}$$

Let us consider $m_n(a)$ if N goes to infinity before $a\to 0$.

(a) First take a non-Z_2 symmetric $V(x)$ with two wells, see Fig. 3.

(i) The saddle point is the solution of $V'(x)=a$, see Fig. 4.

(ii) If a is positive, we have three solutions but the action is lowest at x_3.

(iii) x_3 is still the leading saddle-point solution for $a<0$.

Therefore, the behavior of $m_n(a)$ for small a is independent of the sign of a. This corresponds to the case studied in Ref. [8], where the difference between the depths of the asymmetric wells is large.

(b) However, if V is symmetric, for example, $V(x)=-1/2x^2+g/4x^4$, when $a\to 0$ the saddle points are

$$x_c=\pm\frac{1}{\sqrt{g}}+\frac{a}{2}+O(a^2), \tag{3.6}$$

($x\approx 0$ has a higher action) then

$$S(x_c)=\frac{1}{2g}\mp\frac{a}{\sqrt{g}}. \tag{3.7}$$

The integral m_n is dominated by

$$x=+\frac{1}{\sqrt{g}}+\frac{a}{2} \quad \text{for } a>0,$$

$$=-\frac{1}{\sqrt{g}}+\frac{a}{2} \quad \text{for } a<0. \tag{3.8}$$

The moments are thus given by

$$m_n=\frac{1}{g^{n/2}}e^{-N/2g}e^{+aN/\sqrt{g}}\sqrt{\frac{2\pi}{3N}} \quad \text{for } a>0,$$

$$=\left(\frac{-1}{\sqrt{g}}\right)^n e^{-N/2g}e^{-aN/\sqrt{g}}\sqrt{\frac{2\pi}{3N}} \quad \text{for } a<0. \tag{3.9}$$

For n even, the two results are the same; but for n odd, we get opposite signs. Note that the Z_2 symmetry would say that $m_n=0$ for n odd and $a\to 0$. The set of moments would be $2^{N/2}$ corresponding to the number of different possible moments (only the odd moments are different for different n).

(c) We have to check whether the nonuniformity of the limits $N\to\infty$, $a\to 0$ may be present if V is nonsymmetric but has two wells of equal depths.

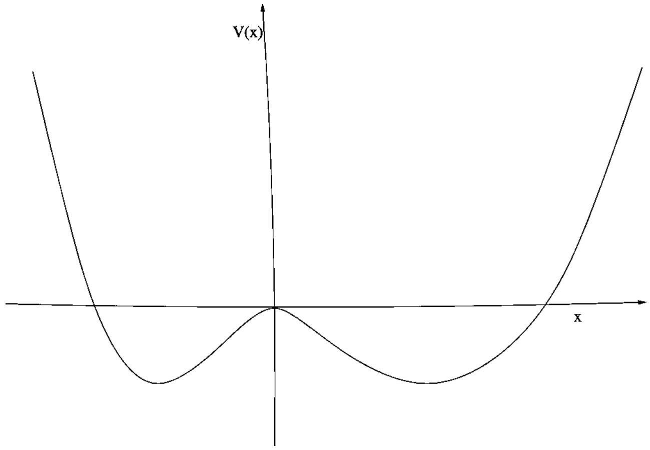

FIG. 5. The asymmetric potential $V(x)$ with two wells of equal depths.

The same series of arguments follows through for the asymmetric potential with two wells of equal depths as for the purely symmetric potential. Hence, there would be multiple solutions of the same multiplicity $2^{N/2}$ in the moments for this problem as well. This is the same situation considered in Ref. [10] (though here only one of the $2^{N/2}$, the symmetric solution, as is referred to in Ref. [12] was considered) and we arrive at the same symmetric answer as in Ref. [10] where they make the unequal wells equal (asymmetry tending to zero limit) (Fig. 5).

IV. FIRST STEPS IN COUNTING MULTIPLE SOLUTIONS

Let us reformulate the problem in a slightly different way to enable counting and bring out some different results in a form easily comparable to formulas in Ref. [13]. We consider the measure

$$Z^{-1}\exp(-N\,\mathrm{tr}\,V(M)+N\,\mathrm{tr}\,MA)d^{N^2}M, \tag{4.1}$$

where V is an arbitrary polynomial and $A=\mathrm{diag}(a_0,\ldots,a_{N-1})$ can be assumed diagonal.

One diagonalizes M; if $M=\Omega\Lambda\Omega^{\dagger}$ where $\Lambda=\mathrm{diag}(\lambda_0,\ldots,\lambda_{N-1})$, the integral over Ω is the usual Itzykson-Zuber integral on the unitary group and one finds

$$\rho_N(\lambda_0,\lambda_1,\ldots,\lambda_{N-1})=Z^{-1}\Delta(\lambda_i)\frac{\det(\exp N\lambda_j a_l)}{\Delta(a_l)}\times\exp\left(-N\sum_{i=0}^{N-1}V(\lambda_i)\right). \tag{4.2}$$

Replacing powers of λ in the Van der Monde with the orthogonal polynomials $P_k(\lambda)$ of the measure $\exp[-NV(\lambda)]d\lambda$. The partition function Z can then be expressed as

$$Z=\frac{N!}{\Delta(a_l)}\int\prod_{i=0}^{N-1}d\lambda_i\det[P_k(\lambda_i)]\exp N\sum_{i=0}^{N-1}[-V(\lambda_i)+a_i\lambda_i]$$
$$=\frac{N!}{\Delta(a_l)}\det\left(\int d\lambda P_k(\lambda)\exp N[-V(\lambda)+a_l\lambda]\right). \tag{4.3}$$

Hence, ρ_N becomes

$$\rho_N(\lambda_0,\lambda_1,\ldots,\lambda_{N-1})=\frac{1}{N!}\frac{\det(P_k(\lambda_i))_{i,k=0,\ldots,N-1}\det(\exp Na_l\lambda_j)_{j,l=0,\ldots,N-1}}{\det\left(\int d\lambda P_k(\lambda)\exp N[-V(\lambda)+a_l\lambda]\right)_{k,l=0,\ldots,N-1}}\exp\left(-N\sum_{i=0}^{N-1}V(\lambda_i)\right). \tag{4.4}$$

This formula has a simple structure. On introducing the functions $F_k(\lambda)=h_k^{(-1/2)}P_k(\lambda)\exp[-N/2V(\lambda)]$ and $G_l(\lambda)=\exp(Na_l\lambda-N/2V(\lambda))$, we have

$$\rho_N(\lambda_0,\lambda_1,\ldots,\lambda_{N-1})=\frac{1}{N!}\frac{\det(F_k(\lambda_i))_{i,k=0,\ldots,N-1}\det(G_l(\lambda_j))_{j,l=0\ldots N-1}}{\det\left(\int d\lambda F_k(\lambda)G_l(\lambda)\right)_{k,l=0,\ldots,N-1}}. \tag{4.5}$$

The matrix $[\int d\lambda G_l(\lambda)F_k(\lambda)]_{l,k=0,\ldots,N-1}$ has an inverse α_{kl}. Putting the three determinants together, we get the following:

$$\rho_N(\lambda_0,\lambda_1,\ldots,\lambda_{N-1})=\frac{1}{N!}\det(K(\lambda_i,\lambda_j))_{i,j=0,\ldots,N-1}, \tag{4.6}$$

where

$$K(\lambda,\mu)=\sum_{k,l=0}^{N-1}F_k(\lambda)\alpha_{kl}G_l(\mu). \tag{4.7}$$

The kernel satisfies the following property:

$$[K*K](\lambda,\rho)=K(\lambda,\rho) \tag{4.8}$$

($[K*K](\lambda,p)=\int d\mu K(\lambda,\mu)K(\mu,p)$). Thus, we obtain the determinant formulas

$$\rho_n(\lambda_0,\lambda_1,\ldots,\lambda_{n-1})=\frac{(N-n)!}{N!}\det(K(\lambda_i,\lambda_j))_{i,j=0,\ldots,n-1} \tag{4.9}$$

for any $n\leqslant N$. The kernel K has the form

$$K(\lambda,\mu)=\sum_{k=0}^{N-1} F_k(\lambda)\hat{F}_k(\mu) \tag{4.10}$$

with $\hat{F}_k(\mu)=\Sigma_l\alpha_{kl}G_l(\mu)$, but $\hat{F}_k\neq F_k$. Thus, K is not symmetric. In order to get further properties for K, we consider the integral

$$\begin{aligned}I&=\int d\lambda[G_l(\lambda)F_k(\lambda)]_{l,k=0,\ldots,N-1}\\&=\int d\lambda\,\frac{P_k(\lambda)}{\sqrt{h_k}}\exp(N[-V(\lambda)+a_l\lambda])\\&=\frac{1}{\sqrt{h_k}}\int d\lambda\sum_{i=0}^{k} C_i\lambda^i\exp(N[-V(\lambda)+a_l\lambda])\\&=\frac{1}{\sqrt{h_k}}\sum_{i=0}^{k} C_i\int d\lambda\lambda^i\exp(N[-V(\lambda)+a_l\lambda])\\&=\frac{1}{\sqrt{h_k}}\sum_{i=0}^{k} C_i m_i,\end{aligned} \tag{4.11}$$

m_i are the moments. For symmetric potential $V(\lambda)$, the above expression becomes (using the expression for the moments found in the preceding section)

$$\begin{aligned}I&=\int d\lambda[G_l(\lambda)F_k(\lambda)]_{l,k=0,\ldots,N-1}\\&=\frac{1}{\sqrt{h_k}}\sum_{i=0}^{k} C_i g^{i/2}e^{-N/2g+a_lN/\sqrt{g}}\sqrt{\frac{2\pi}{3N}}=\alpha_{kl}^{-1}\quad\text{for } a_l>0\\&=\frac{1}{\sqrt{h_k}}\sum_{i=0}^{k} C_i\left(-\frac{1}{\sqrt{g}}\right)^i e^{-N/2g-a_lN/\sqrt{g}}\sqrt{\frac{2\pi}{3N}}=\alpha_{kl}'^{-1}\\&\quad\text{for } a_l<0.\end{aligned} \tag{4.12}$$

Summarizing

$$I=\begin{cases}\alpha_{kl}^{-1}, & a_l>0,\\ \alpha_{kl}'^{-1}, & a_l<0.\end{cases}$$

Recall that $x_c=\pm 1/\sqrt{g}+a/2$, thus only for $\pm 1/\sqrt{g}\geqslant a/2$ the above result holds, i.e., the integral equation (4.12) has two values depending on whether $a_l>0$ or $a_l<0$. Whereas for $\pm 1/\sqrt{g}\leqslant a/2$, the usual single-well result, as given in Ref. [13], is found.

From the equation for $K(\lambda,\mu)$, i.e., Eq. (4.10) which depends on the integral equation (4.12) through a sum, it may be possible that there are 2^N solutions for certain kernels, this would correspond to an exponentially large number of solutions depending on the path or different combinations of a_l taken. For $\rho_N(\lambda_0,\ldots,\lambda_{N-1})$ and Z, i.e., Eqs. (4.3) and (4.4) which are related to I through a determinant, it is risky to consider the large N behavior of I before computing $\det_{[N\times N]}I$. Counting at the level of $K(\lambda,\mu)$, $\rho_N(\lambda_0,\ldots,\lambda_{N-1})$, Z, and the free energy still remains an open problem and needs a nonperturbative treatment (as shown in Ref. [12]). This will be pursued in a future work.

V. AN EXPLICIT CALCULATION OF THE INTEGRAL EQUATION (4.12) FOR THE DOUBLE-WELL PROBLEM

For the double-well matrix model the orthogonal polynomials, are not known polynomials but we do know the form for the polynomial at large N, i.e., when $(N-n)\approx O(1)$. The polynomials are given by

$$\psi_n(\lambda)=\frac{1}{\sqrt{f}}\left[\cos(N\zeta-(N-n)\phi+\chi+(-1)^n\eta)(\lambda)+O\left(\frac{1}{N}\right)\right], \tag{5.1}$$

where f, ζ, ϕ, χ, and η are functions of λ and are given by

$$\begin{aligned}f(\lambda)&=\frac{\pi}{2\lambda}\frac{(b^2-a^2)}{2}\sin 2\phi(\lambda),\\ \zeta'(\lambda)&=-\pi\rho(\lambda),\\ \cos 2\phi(\lambda)&=\frac{\lambda^2-\frac{(a^2+b^2)}{2}}{\frac{(b^2-a^2)}{2}},\\ \cos 2\eta(\lambda)&=b\,\frac{\cos\phi(\lambda)}{\lambda},\\ \sin 2\eta(\lambda)&=a\,\frac{\sin\phi(\lambda)}{\lambda},\\ \chi(\lambda)&=\frac{1}{2}\phi(\lambda)-\frac{\pi}{4}.\end{aligned} \tag{5.2}$$

Let us consider the Eq. (4.12) with the above asymptotic ansatz for ϕ_k for large k, then

$$I=\int P_k(\lambda)e^{-N[V(\lambda)-\tilde{a}\lambda]},$$

$$P_k(\lambda)e^{-(N/2)V(\lambda)}=\sqrt{h_k}\psi_k(\lambda),$$

$$I=\sqrt{h_k}\mathrm{Re}\int \frac{d\lambda}{\sqrt{f(\lambda)}}e^{i[N\zeta-(N-k)\phi+\chi+(-1)^k\eta]}e^{-N[1/2(-1/2\mu\lambda^2+g/4\lambda^4)-\tilde{a}\lambda]}$$

$$=\sqrt{h_k}\mathrm{Re}\int d\lambda \exp\left(N\left[\frac{1}{2N}\ln f(\lambda)+i\zeta+i\gamma_{N,k}\frac{\phi}{N}+i(-1)^k\frac{\eta}{N}+\frac{1}{4}\mu\lambda^2-g/8\lambda^4+\tilde{a}\lambda-\frac{\pi}{4N}\right]\right), \tag{5.3}$$

where $\gamma_{N,k}$ is given by $-(N-k)+\frac{1}{2}$. In the saddle point approximation, the exponent $S(\lambda)$ is to be minimized. The action

$$S(\lambda)=\frac{i\gamma_{N,k}\phi(\lambda)}{N}+i\zeta+i(-1)^k\frac{\eta(\lambda)}{N}+\frac{1}{4}\mu\lambda^2-g/8\lambda^4+\tilde{a}\lambda+\frac{1}{2N}\ln f(\lambda) \tag{5.4}$$

will have a first derivative which vanishes as shown below:

$$\frac{i\gamma_{N,k}\phi'(\lambda)}{N}+i\zeta'+i(-1)^k\frac{\eta'(\lambda)}{N}+\frac{1}{2}\mu\lambda-g/2\lambda^3+\tilde{a}+\frac{1}{2Nf(\lambda)}f'(\lambda)=S'(\lambda)=0,$$

$$\frac{i\gamma_{N,k}\phi'(\lambda)}{N}-i\pi\rho(\lambda)+i(-1)^k\frac{\eta'(\lambda)}{N}+\frac{1}{2}\mu\lambda-g/2\lambda^3+\tilde{a}+\frac{1}{2Nf(\lambda)}f'(\lambda)=0, \tag{5.5}$$

where we have used the relation for ζ in terms of ρ from Eq. (4.14). Solving for the density $\rho(\lambda)$, we get

$$\rho(\lambda)=\frac{i}{\pi}\left(-\frac{1}{2}\mu\lambda+g/2\lambda^3-\tilde{a}\right)+\frac{\gamma_{N,k}\phi'(\lambda)}{\pi N}-\frac{i}{2N\pi f(\lambda)}f'(\lambda)+(-1)^k\frac{\eta'(\lambda)}{\pi N}. \tag{5.6}$$

For the symmetric potential using the expression for $\rho(\lambda)$ and in the large N limit neglecting the last terms, for small a the equation for λ has solutions λ_{pm} and 0. Thus, in the saddle point approximation, the integral I for large k becomes

$$I_{\pm}=I_0P_k(\lambda_{\pm})e^{-N(V(\lambda_{\pm})-a\lambda_{\pm})}+\text{h.o.t.} \tag{5.7}$$

(the $\lambda\sim 0$ solution gives a higher action), where I_0 is a constant. Hence, we have shown in an explicit example for the symmetric double-well potential that the integral equation (4.12) for a large k in the saddle point approximation has two solutions, which solution is chosen depends on whether $a \geqslant$ or $\leqslant 0$. This result indicates the possibility that the kernel, partition function, free energy can have 2^N solutions depending on the path $\{a_l\}$ taken as these functions all depend on the integral I, Eq. (4.12). Thus, here evidence is presented that there exists an exponentially large number of solutions, i.e., $e^{N\ln 2}$, in the double-well matrix models depending upon the path taken in parameter space $\{a_l\}$. It will be interesting to explore the possibility that these exponentially large number of solutions correspond to the metastable solutions of the supercooled p-spin glass that these random matrix models map into.

VI. CONCLUSIONS

We have been able to map the problem of counting the number of multiple solutions found in Ref. [12] to a moment problem. The multiple solutions were discovered in the recurrence coefficients of the orthogonal polynomials in Ref. [12]. It was known that there are an infinite number of solutions. The counting problem is mapped onto counting the number of ways to get different moments. The set of moments grows exponentially as $2^{N/2}$. In order to show this, we have to introduce a small perturbation that breaks Z_2 symmetry into the moment integral and then take the small asymmetry parameter to zero (which we call tapping the matrix). As an added bonus, we are able to understand some of the puzzles and controversies that are found in Ref. [12] and studied in Refs. [9,7]. The counting at the level of the kernel, $\rho_N(\lambda_0,\ldots,\lambda_{N-1})$, Z, and the free energy still remains an open problem and needs a nonperturbative treatment. This will be pursued in a future work.

The number of moments in these random matrix models are exponentially rising with N. These matrix models are connected with the high temperature phase of structural-glasses as has been discussed in Refs. [4,5]. There could be interesting properties of the supercooled liquid phase, which may be explored analytically in these simple models. For example, it will be worthwhile to study how the metastable states of the liquid are related to the different paths of taking the small perturbation parameter, as introduced here, to zero. Future work on finding barrier heights is underway.

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