PHYSICAL REVIEW E **68**, 025302(R) (2003)

Unstable periodic solutions embedded in a shell model turbulence

Sei Kato* and Michio Yamada†

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1, Meguro-ku, Komaba, Tokyo 153-8914, Japan

(Received 27 December 2002; published 12 August 2003)

An approach to intermittency of a shell model turbulence is proposed from the viewpoint of dynamical systems. We detected unstable solutions of the Gledzer-Ohkitani-Yamada shell model and studied their relation to turbulence statistics. One of the solutions has an unstable periodic orbit (UPO), which shows an intermittency where the scaling exponents of the structure function have a nonlinear dependence on its order, quite similar to that of turbulence solution at the same parameter values. The attractor in the phase space is found to be well approximated by a continuous set of solutions generated from the UPO through a one-parameter phase transformation, which implies that the intermittency of the shell model turbulence is described by this UPO.

DOI: 10.1103/PhysRevE.68.025302 PACS number(s): 47.27.Gs, 05.45.Jn, 05.45.Pq, 47.52.+j

Universal scaling property of the three-dimensional Navier-Stokes turbulence is one of the fundamental problems in understanding fluid turbulence. Since Landau's famous note on the Kolmogorov universality theory in 1941 (K41 scaling), many phenomenological models have been proposed for the intermittency, and rather good numerical agreements have been obtained between these models and experiments, on the scaling exponents of the structure functions. However, these models have remained phenomenological, and do not give an insight into a dynamical origin of the intermittency.

Recently, in the case of a plane Couette turbulence, Kawahara and Kida [1] showed that some mean properties including the mean velocity profile as well as the time development of coherent structures can be described by one of the unstable periodic orbits (UPOs) embedded in the turbulence orbits. Their success gives us an expectation that UPOs in fully developed turbulence, if found, would give basic information of statistical properties of turbulence. However, at the present stage of numerical facilities, it is still impossible to detect UPOs due to the extremely large number of degrees of freedom and the strong instability of the Navier-Stokes turbulence.

Instead, in this paper, we consider a shell model turbulence as a small model of Navier-Stokes turbulence, and study its unstable solutions and their statistics, showing that, as far as numerical pursuit of the UPO is successful, the intermittency in the shell model is described by one of the UPOs and that basic properties of the turbulence statistics are explained from this single UPO.

The shell model was first introduced by Gledzer [2] to describe two-dimensional turbulence by its steady solutions. Later Yamada and Ohkitani [3] complexified the model to describe three-dimensional turbulence by its chaotic solutions. In this model the wave number is one dimensional and discretized by octaves as $k_j=2^{j-4}$ $(j=1,2,\ldots,N)$, with which a complex velocity u_j variable is associated. The number of the shells, N, is usually taken to be 10–30. The dynamical equation for the complex shell velocity u_j is

$$\left(\frac{d}{dt}+\nu k_j^2\right)u_j=i[a_j u_{j+1}u_{j+2}+b_j u_{j-1}u_{j+1}+c_j u_{j-1}u_{j-2}]^* + f\delta_{j,1}, \tag{1}$$

where * denotes the complex conjugate, and the coupling constants are taken as $a_j=k_j$, $b_j=-\delta k_{j-1}$, $c_j=(1-\delta)k_{j-2}$, and $b_1=c_1=c_2=a_{N-1}=a_N=b_N=0$ to conserve the energy $E=\Sigma_j|u_j|^2$ when the viscosity ν and the external forcing f vanish. We are interested in the case of $\delta=1/2$, where the helicity $H=\Sigma_j(-1)^j k_j|u_j|^2$ is also conserved as in Navier-Stokes turbulence [4], but here we leave δ as a variable parameter for later use in numerical pursuit of UPO. In this study the external forcing is added to the first shell to avoid an unessential difficulty caused by energy transfer to lower wave numbers.

Many numerical evidences have been found for the K41 scaling of the energy spectrum in the inertial range of the shell model [3,5]. As for the higher order structure functions, a deviation from the K41 scaling, i.e., intermittency, has also been found similarly to the Navier-Stokes turbulence [6–8]. Also the probability density function (PDF) of the velocity variable u_j, which corresponds to the velocity difference in the Navier-Stokes case, has been found to have more deviation from the Gaussian distribution function at higher wave numbers [9].

Even in the shell model, it is difficult to detect the UPOs numerically in its chaotic stage corresponding to the fully developed turbulence because of their strong instability. Let us then utilize the property of the shell model—that when δ gets apart from 1/2, the chaotic solution becomes nonchaotic [10]. We first obtain limit cycles for $\delta<1/2$, and then trace the limit cycle along the following two paths in the parameter space (ν,δ): (a) $(1.5\times10^{-3},0.16)\rightarrow(1.5\times10^{-3},0.50)$ and (b) $(1.95\times10^{-3},0.4973)\rightarrow(1.95\times10^{-3},0.50)\rightarrow(1.5\times10^{-3},0.50)$, where the destination $(1.5\times10^{-3},0.50)$ is the chaotic state in which the intermittent fully developed turbu-

*Present address: IBM Research, Tokyo Research Laboratory, 1623-14, Shimotsuruma, Yamato-shi, Kanagawa 242-8502, Japan. Email address: seikato@jp.ibm.com

†Present address: Research Institute for Mathematical Sciences, Kyoto University, Oiwake-cho, Kitashirakawa, Sakyo-ku, Kyoto 606-8502, Japan. Email address: yamada@kurims.kyoto-u.ac.jp

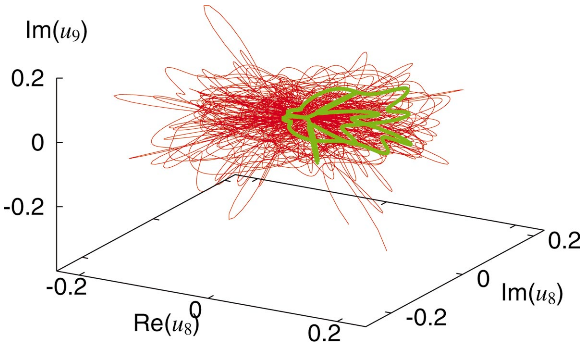

FIG. 1. (Color) Time evolution of the intermittency solution (green line) and the turbulence solution (red line) in the space of $(\mathrm{Re}(u_8),\mathrm{Im}(u_8),\mathrm{Im}(u_9))$.

lence is realized. For the path (a), we employ the Newton-Raphson method, which has been applied to identify steady solutions of the shell model [11]. For tracing the path (b), we use the Mees method [12,13], an extension of the Newton-Raphson method, by which we solve $2N+1$ equations for $u_j(0)$ and the period T.

We found two kinds of unstable solutions for $(\nu,\delta) = (1.5\times10^{-3},0.50)$ and $N=12$. One is an unstable steady solution, which is found, by the path (a), with the error $\Delta_j = |\dot{u}_j|$ being about 10^{-18} for all j. The energy spectrum of this steady solution shows $-5/3$ power law in the inertial range and the sum of the phases of sequential three velocities u_j, u_{j+1}, and u_{j+2} is equal to $3\pi/2 \pmod{2\pi}$, which is the ideal value for the energy cascade process maximizing the energy flux towards higher wave numbers, although the steady solutions give no energy cascade. In the case of infinite number of shells without viscosity and external forcing, the model equation (1) allows two steady solutions,

$$u_j = C_j^{(1)} k_j^{-1/3}, \quad u_j = C_j^{(2)} k_j^{-1/3+\log_2(\delta-1)/3}, \tag{2}$$

where $C_j^{(1)}$ and $C_j^{(2)}$ are complex constants periodic with respect to j with period 3 [10]. The slope of the energy spectrum suggests that the unstable steady solution obtained in our calculation may approach the former steady solution of Eq. (2) as $\nu\to0$, while in the case of $0<\delta<1$ in our calculation, no solution was found which may correspond to the latter solution. This solution corresponds to the standard model solution of the unstable fixed point found by Kockelkoren *et al.* of the shell map [14].

The other unstable solution is an unstable periodic solution, which was detected by the path (b) when we started from the limit cycle at $(\nu,\delta)=(1.95\times10^{-3},0.4973)$. The period of this solution is $T=179.98$, and the relative error $\Delta_j^{rel} = |[u_j(T)-u_j(0)]/u_j(0)|$ is about 10^{-5} for all j. As described below, this solution shows the intermittency, and we refer to this solution as the intermittency solution. The largest eddy turnover time $T_L = 2\pi/(\sqrt{E}k_1)$ is equal to 65.6 and the Kolmogorov time scale is 1.20. We note that the period of the solution is around twice the largest eddy turnover time, consistent with the fact that the periodic solution consists of similar two parts, as seen in Fig. 1 where the orbit of this solution in the phase space is shown, by taking $\mathrm{Re}(u_8)$, $\mathrm{Im}(u_8)$, and $\mathrm{Im}(u_9)$ as coordinates, together with the orbit of a chaotic solution (which we will call turbulence solution hereafter) at the same values of parameters. We see that the intermittency solution is embedded in the turbulence solution. The time development of the energy transfer function in low wave number region shows two peaks in one period, while that at the highest wave number shows eight peaks, indicating that the intermittency solution has a ramifying cascade process of energy, as assumed in the Navier-Stokes turbulence. In Fig. 2, we show the PDF of $\mathrm{Im}(u_8)$ together with that of the turbulence solution. We see that the PDF of the intermittency solution outlines the shape of the PDF of the turbulence solution. Accordingly, the statistical properties of the turbulence solution are expected to be well described by this intermittency solution.

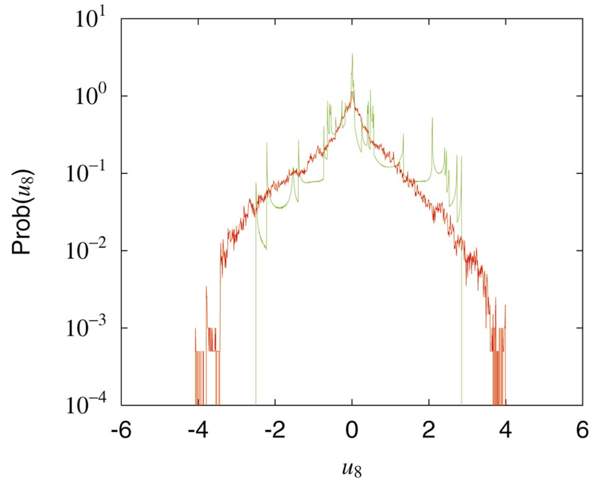

FIG. 2. (Color) Log-linear plots of PDFs for the intermittency solution (green line) and turbulence solution (red line).

To see it more quantitatively, we calculated the scaling exponent ζ_p of the pth order structure function $S_p(k_j) = \langle |u_j|^p \rangle$, where the brackets stand for the time average. The viscosity for this solution is not small enough to produce a wide inertial range necessary especially for the normal fitting method for large values of p. For this reason, we make use of the fitting method of the least squares with the extended self-similarity (ESS) [15,16]

$$S_p(k_j) = S_3(k_j)^{\zeta_p} \tag{3}$$

for the evaluation of ζ_p in the wave number range of $1\leqslant j \leqslant 5$, where the time-averaged energy flux

$$\Pi_j = \left\langle -\frac{d}{dt}\sum_{i=1}^{j} |u_j|^2 \right\rangle$$
$$= \langle -k_j\, \mathrm{Im}(u_j u_{j+1} u_{j+2} - \tfrac{1}{4} u_{j-1} u_j u_{j+1}) \rangle \tag{4}$$

of the intermittency solution takes a constant value $\Pi_j \sim \epsilon = \langle 2\nu\Sigma_j k_j^2 |u_j|^2 \rangle$. Figure 3 shows the scaling exponents obtained in this way for the intermittency solution and for the turbulence solution. We see that the intermittency solution shows a multiscaling property, and the scaling exponents agree well with those obtained from the turbulence solution.

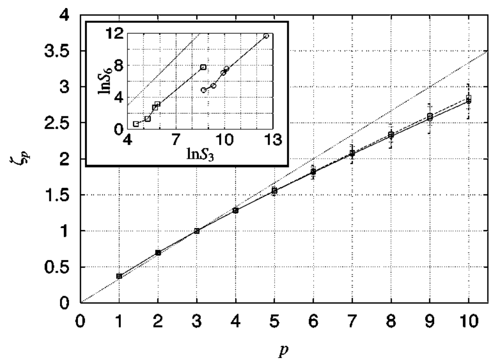

FIG. 3. Scaling exponent ζ_p of the pth order structure function obtained by ESS fitting for the intermittency solution (□) and turbulence solution (○). The error bars show the standard deviations obtained from the method of the least squares. The dotted line corresponds to K41 ($\zeta_p = p/3$). In the inset, we show the log-log plot of the sixth order structure function against the third order structure function in the range of $1 \leqslant j \leqslant 5$. The symbols are the same as for the scaling exponents. The slopes are 2.0 for K41, 1.82 for the intermittency solution, and 1.81 for the turbulence solution.

This means that the basic statistical properties of the turbulence are well approximated by this intermittency solution which may be considered to be a skeleton of the turbulence.

This agreement of the scaling exponents may seem curious, as the intermittency solution shares only a small part of the region in the phase space occupied by the turbulence solution (Fig. 1). The nonforced shell model is known to have a phase symmetry corresponding to the spatial translational symmetry of the Navier-Stokes equation [8,17]: If $\{u_j(t)\}$ is a solution of the shell model, then $\{u_j^{(\theta,\phi)}(t)\}$, which is defined as

$$\begin{aligned} u_j^{(\theta,\phi)} &= e^{i\theta} u_j, \\ u_{j+1}^{(\theta,\phi)} &= e^{i(\phi-\theta)} u_j, \\ u_{j+2}^{(\theta,\phi)} &= e^{-i\phi} u_j, \end{aligned} \tag{5}$$

where θ and ϕ are arbitrary real constants and $j \equiv 1 \pmod 3$ is also a solution of the shell model. In our case, we added a constant external forcing term to the first shell, and the phase symmetry is reduced to the following [see Eq. (5)]:

$$\begin{aligned} u_j^{(\phi)} &= u_j, \\ u_{j+1}^{(\phi)} &= e^{i\phi} u_j, \\ u_{j+2}^{(\phi)} &= e^{-i\phi} u_j. \end{aligned} \tag{6}$$

In Fig. 4 we show the orbits of u_7, u_8, and u_9 of the intermittency solution and of the turbulence solution. Again, we see that while the orbits of u_7 share a similar area, the orbits of u_8 and u_9 of the intermittency solution take only a small part of the region of the turbulence solution. However, the phase transformation (6) means that the rotated orbit of u_8 (u_9) by an angle ϕ ($-\phi$) is also an orbit of a solution that has exactly the same statistical properties as the intermittency solution. As ϕ is an arbitrary constant, the orbit of u_8 (u_9) thus generates a continuous set of the intermittency solutions, which this time cover most of the region (attractor) of the turbulence solution in the phase space. In other words, the attractor is well approximated by the continuous set generated from a single intermittency solution, and this explains why the statistical properties of the single solution agree well with those of the turbulence solution.

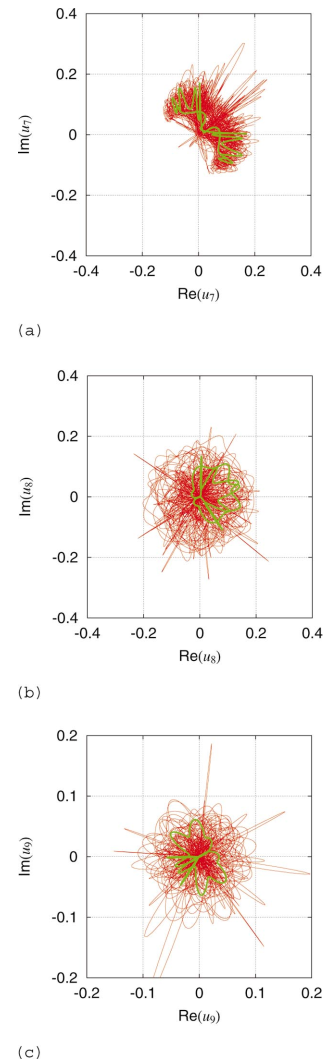

FIG. 4. (Color) Time evolution of the shell velocities (a) u_7, (b) u_8, and (c) u_9 of the intermittency solution in the complex plane. In each figure, the green line represents the intermittency solution and the red line represents the turbulence solution.

The argument on the symmetry recovery given by Frisch [18] may be appropriate to be mentioned here. He claimed that the symmetries of the Navier-Stokes equation, which are broken in the process of transition to turbulence, recover "in a statistical sense" in the inviscid limit. In our study of the shell model, the turbulence solution recovers its phase symmetry by wandering around the area in the phase space covered by a set of rotated intermittency solutions, just as Frisch claimed. Also a remark should be made on the instability of the orbit. We calculated the Lyapunov exponents λ_j ($1 \leqslant j$

$\leqslant 2N$) along with the intermittency solution and with the turbulence solution, and substitute them into the Kaplan-Yorke formula for attractor dimension D_{KY},

$$D_{KY}=p-\frac{1}{\lambda_{p+1}}\sum_{j=1}^{p}\lambda_j \quad \left(p=\max\left\{J\left|\sum_{j=1}^{J}\lambda_j\geqslant 0\right.\right\}\right). \tag{7}$$

The obtained value for the intermittency solution is 6.28, which approximates 7.02 for the turbulence solution. Obviously the continuous set of the solutions generated by the intermittency solution through one-parameter phase transformation is two-dimensional in contrast with the Kaplan-Yorke dimension of the attractor, 7.02. This suggests that the attractor is thin in the orthogonal directions to the two-dimensional continuous set. This conjecture may be supported by the observation in Fig. 1, that the attractor is nearly flat in the horizontal direction while very thin in the vertical direction.

We should mention the burstlike structure found by Okkels [19], who investigated the behavior of local attractor of a real valued Gledzer-Ohkitani-Yamada model. He discussed the intermittency in relation to the stability of the manifold of $\dot{r}_n=0$ derived from Eq. (8) of Ref. [19]. It is interesting that in our case essential properties of the turbulence are involved in a single UPO, rather than a multidimensional manifold.

In conclusion, we have detected two kinds of unstable solutions in the shell model. We showed that the scaling exponents of the intermittency solution agree well with those of the turbulence solution. The attractor is approximated by a continuous set generated from a single intermittency solution through the phase transformation. Therefore, basic statistical properties of the intermittency of the shell model turbulence can be described by the intermittency of the unstable periodic orbit. Detailed structure of unstable periodic orbits is now under investigation and will be reported.

The authors would like to thank G. Kawahara and S. Kida for their fruitful comments. They are also grateful to H. Nakao for his continuous encouragement.

[1] G. Kawahara and S. Kida, J. Fluid Mech. **449**, 291 (2001).

[2] E.B. Gledzer, Dokl. Akad. Nauk SSR **209**, 1046 (1973) [Sov. Phys. Dokl. **18**, 216 (1973)].

[3] M. Yamada and K. Ohkitani, J. Phys. Soc. Jpn. **56**, 4210 (1987).

[4] L. Kadanoff, D. Lohse, and J. Wang, Phys. Fluids **7**, 617 (1995).

[5] K. Ohkitani and M. Yamada, Prog. Theor. Phys. **81**, 329 (1989).

[6] M.H. Jensen, G. Paladin, and A. Vulpiani, Phys. Rev. A **43**, 798 (1991).

[7] D. Prisarenko *et al.*, Phys. Fluids A **5**, 2533 (1993).

[8] R. Benzi, L. Biferale, and G. Parisi, Physica D **65**, 163 (1993).

[9] M. Yamada, S. Kida, and K. Ohkitani, in *Unstable and Turbulent Motion of Fluid*, edited by S. Kida (World Scientific, Singapore, 1993), p. 188.

[10] L. Biferale, A. Lambert, R. Lima, and G. Paladin, Physica D **80**, 105 (1995).

[11] N. Schörghofer, L. Kadanoff, and D. Lohse, Physica D **88**, 40 (1995).

[12] A.I. Mees, *Dynamics of Feedback Systems* (Wiley, New York, 1981).

[13] T.S. Parker and L.O. Chua, *Practical Numerical Algorithms for Chaotic Stystems* (Springer-Verlag, New York, 1989).

[14] J. Kockelkoren, F. Okkels, and M.H. Jensen, J. Stat. Phys. **93**, 833 (1998).

[15] R. Benzi, G.R.S. Ciliberto, and R. Tripiccione, Europhys. Lett. **24**, 275 (1993).

[16] R. Benzi *et al.*, Phys. Rev. E **48**, R29 (1993).

[17] O. Gat, I. Procaccia, and R. Zeitak, Phys. Rev. E **51**, 1148 (1995).

[18] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, 1995).

[19] F. Okkels, Phys. Rev. E **63**, 056214 (2001).