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Large particle number limit in rain

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The way we conceptualize rain is fundamental in many branches of science since it provides the basis not only for rain modeling notably in meteorology and hydrology, but also for interpreting rain data (from gauges and radars). In order to empirically address this question, we use stereophotographic data to measure the positions and volumes of raindrops from $\sim 10\ \mathrm{m}^3$ regions containing 5000–15 000 of these drops. By determining the drop statistics in spheres of increasing size, we conduct a basic continuum mechanics thought experiment. We show that—presumably due to turbulence—there is no microscale-macroscale separation. We find that the large particle number (N) limit in rain is not a homogeneous continuum, but rather it is nonclassical, strongly inhomogeneous, and approaching a multifractal discontinuum.

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Ordinary bulk matter and rain are both particulate, yet everyday experience involves them in huge numbers; the corresponding macroscopic descriptions are continuous. Classical continuum mechanics (notably fluid mechanics) is justified whenever the ‘‘continuum hypothesis’’ is valid, i.e., whenever there exists a clear separation of scales between the microscale and the macroscale. In standard textbooks [1,2], this is illustrated by a thought experiment: imagine a sphere of radius r filled with air. When the sphere is very small—comparable to the mean intermolecular distance ($\approx 10^{-8}$ m)—there will be large fluctuations in properties such as mean density, or velocity, depending on whether zero, one, or a few particles happen to be in the sphere. However, as the sphere is made progressively larger, the number of particles increases until the relative fluctuations become very small. At scales several orders of magnitude larger (≈ 1 mm), the mean starts to vary again, this time because of turbulent variations in the macroscopic density, temperature, pressure, velocity, etc. The existence of this wide range of scales where the properties are independent of the size of the sphere justifies the continuum hypothesis, and allows us to define the macroscopic quantities by averaging over spheres much larger than the microscale while simultaneously much smaller than the macroscale.

The situation in rain and precipitation seems so closely analogous that precipitation physics has been founded on the—usually unstated and until now untested—assumption that drop sizes, rain rates, liquid water contents, and other macroscopic quantities also satisfy the continuum hypothesis (see the classic works [3,4]). This constitutes the classical approach. But is it really justified?

Precipitation and wind are clearly strongly and nonlinearly coupled; the latter is highly turbulent down to millimetre scales, whereas even in strong rain the mean interparticle distance is of the order of 10 cm. At these scales, the wind is multifractal—the result of cascade processes concentrating energy fluxes into smaller and smaller regions of space (see, e.g., the review in Ref. [5]). We should thus expect the precipitation to also exhibit a hierarchical clustering pattern down to a small scale, where rain ‘‘decouples’’ from the turbulence, i.e., the precipitation should tend to a multifractal rather than to a classical (homogeneous) large N limit. Indeed, there is no obvious theoretical reason why the rain rate field, which is the basic quantity of interest at larger scales, should be regular with respect to the volume measure, i.e., should have a pointwise density with respect to the latter [6]. There already exists evidence for the multifractality of rain, see, e.g., Refs. [7–17] and see the review [18].

Recent examples of the classical approach are Refs. [19–26]. These authors found systematic deviations from pure Poisson statistics (i.e., classical continuum) on both time series and spatial experiments, but in spite of these systematic inhomogeneities, the tendency has been to introduce *ad hoc* correction models, such as ‘‘doubly stochastic Poisson process,’’ ‘‘Poisson mixture,’’ and ‘‘compound Poisson processes,’’ the effect of which is to minimize the significance of the departures, and which in principle—if enough arbitrary parameters are introduced—can accommodate virtually any statistical behavior. To date, very few small scale studies have attempted to systematically consider the statistics as functions of scale. At the drop scale, an early exception was a study by Ref. [27] that used large pieces of chemically coated blotting paper and claimed evidence for fractal clus-

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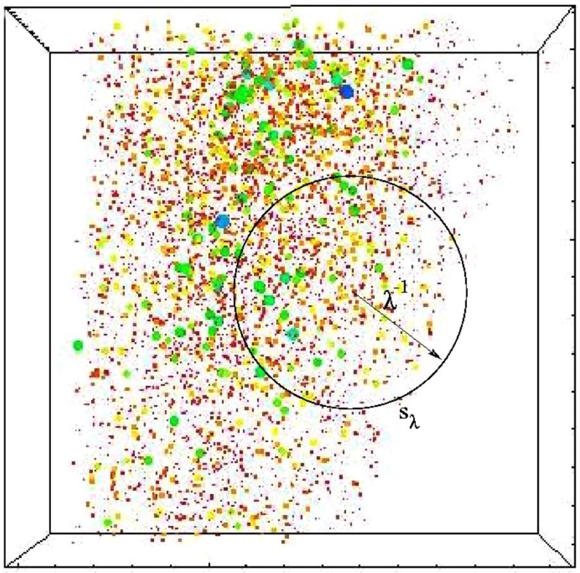

FIG. 1. (Color online) A reconstruction of one of the $f295$ triplets showing only the 15 000 drops in the ROI. This is a side view; the top is to the right and the bottom to the left. The cameras are on the lower side and the flashes on the upper side. Only the relative sizes of the drops are accurate.

tering of individual drop positions and liquid water. A later study of a space-time rain section was performed with a lidar [28]. In support of these early results, there is a study by Lavergnat and Golé [29] who found that the histograms of the arrival times of rain drops follow a power law (Pareto) behavior, which is a scaling distribution also implying hierarchical clustering of arrivals. These three studies confirmed the existence of scaling in various small scale rain statistics. Other relevant small scale scaling studies are those of cloud liquid water; Refs. [27,30] have shown that the statistics are indeed multifractal down to at least 10 m.

The experimental details of our most recent stereophotographic approach—the HYDROP experiment—have only just appeared [31] (see also Ref. [32]). The experiment involves three large format cameras to perform stereophotography of a ≈ 10 m^3 region (≈ 1000 times larger than those used previously) with 5000–20 000 drops "frozen" by two 50 μs, 1 KJ flashlamps. Even with 60 mm negatives, the photographic resolution limited us to detecting drops $\geqslant 0.2$ mm in a region ≈ 2 m across.

The negatives were scanned using a special scanner with 6-μm and 12 bit resolution. Since there could be as many as 100 000 drops on each image only a fraction of which was inside the sharply focused, well lit "region of interest" (ROI), a sophisticated algorithm first geometrically rectified the negatives and then matched them with at least two negatives determining the position to ± 3 mm (laterally), ± 3 cm (depth). The volumes were determined either from the diameter (large drops) or backscatter intensity (accuracies were ± 0.25 mm for drops >1 mm and $\pm 50\%$ <1 mm); 90% of these drops were matched.

During the three-year experimental period, a total of nearly 450 triplets were acquired. However, due to the difficulty of making reconstructions, only two storms were initially analyzed [31,32] precluding clear scientific conclusions. We have now performed 18 reconstructions (e.g., Fig. 1) from five storms obtaining convincing multifractal results in four out of the five storms (see Table I; see Ref. [33] for the full details).

Is the variability in Fig. 1 due to the chance fluctuations of a basically homogeneous, classical process, or to a systematic scaling, multifractal process? To answer this, we placed larger and larger spheres at random over the ROI and calculated the corresponding (normalized) ηth power densities at resolution λ [34] ($\langle\rangle$ denotes an ensemble average):

$$(\rho^{(\eta)})_\lambda \propto \sum_{V_i \in S_\lambda} V_i^\eta \quad \text{and} \quad \langle(\rho^{(\eta)})_\lambda\rangle = 1. \tag{1}$$

Here, V_i is the volume of the ith drop in the sphere S_λ at scale ratio $\lambda = L/r$, r is the radius of the sphere, and L is a large outer scale (the scale of the experimental region). The rain drops are observed at a much finer resolution, where each of them is resolved. Therefore, $(\rho^{(\eta)})\lambda$ is a coarse-grained observation of the rain drops. Increasing η gives increasing weight to the larger drops allowing one to explore the effects of the drop size distribution on the statistics; various η values yield the different fields: $(\rho^{(0)})\lambda = n_\lambda$, number density; $(\rho^{(1)})\lambda = V_\lambda$ liquid water content (LWC); $(\rho^{(7/6)}) = R_\lambda$, nominal rain rate (i.e., using a theoretical fall speed); $(\rho^{(2)})\lambda = Z_\lambda$, radar reflectivity factor, etc.

The statistics of the classical large N limit is straightforward: the probability per unit volume of finding a particle is constant; the number in any volume is a Poisson random variable; there exists a scale-independent particle size distribution (the first two moments denoted by $\langle\nu\rangle$, $\langle\nu^2\rangle$). This

TABLE I. A summary of some of the relevant characteristics of the five storms. The event ID corresponds to the film no.; $f207$, $f204$ were from the same storm, for each storm, all events were within 20 min of each other. For the convergence scale l_c, $q=2$ is chosen since it is critical for the classical central limit theorem.

Event ID	Number of triplets	ROI bounds along Z (m)	Number of drops in each ROI	l_c for $\eta=1$, $q=2$	C_1 ($\eta=1$)	α ($\eta=1$)
$f142$	3	[4–6.5]	10 000	>2 m		
$f145$	3	[4.88–7.22]	6500	24 cm	0.13	1.6
$f207$–$f204$	7	[4.5–7]	22 500	20 m	0.24	1.3
$f229$	2	[4.7–7.8]	15 000	40 cm	0.10	1.6
$f295$	3	[4.7–7.8]	15 000	23 cm	0.09	1.6

implies that for large regions ($\lambda \to 1$), for any finite variance distribution, the LWC converges to a Gaussian:

$$\Pr(\rho_\lambda) \propto \exp\left[\frac{(\rho_\lambda - \langle n\rangle\langle v\rangle)^2 volS_\lambda}{2\langle n\rangle\langle v^2\rangle}\right]. \quad (2)$$

Here, "Pr" indicates "probability" and $\langle n\rangle$ is the average number density. The continuum limit is obtained by taking S_λ large so that the Gaussian tends to a "sure" Dirac delta function: $\Pr(\rho_\lambda) \to \delta(\rho_\lambda - \langle n\rangle\langle v\rangle)$ (i.e., $\rho_\lambda \approx \langle n\rangle\langle v\rangle$). On the contrary, in the multifractal large N limit, we have

$$\Pr((\rho^{(\eta)})_\lambda \geqslant \lambda^\gamma) \approx \lambda^{-c(\gamma,\eta)}, \quad (3)$$

where $c(\gamma,\eta)$ is the codimension function [35], γ is the corresponding order of singularity, and "$\approx$" means equality to within slowly varying factors. As the averaging sphere gets larger (smaller λ), the values $(\rho^{(\eta)})_\lambda \approx \lambda^\gamma$ do indeed get smoothed (the distribution is less and less "spread"), but this occurs in a power law way. While the effect is not so great here, $L=2$ m, and the scale of convergence (l_c) to a multifractal regime is about 40 cm ($\lambda \approx 5$), if we consider the global rain process, we may have $L=10^4$ km ($\lambda \approx 10^8$; see Refs. [36,37]), so that the effects of the clustering at all scales can be very large. Considering the statistical moments, we have

$$\langle (\rho^{(\eta)})_\lambda^q \rangle = \lambda^{K(q,\eta)}, \quad (4)$$

where K and c are related by a Legendre transformation. In estimating $(\rho^{(\eta)})_\lambda^q$, some spheres centered near the ROI boundary having volumes largely outside were rejected. This effect implies a small change in the effective sample as a function of radius; it is relatively more important for larger r and is partially corrected by the normalization used for $(\rho^{(\eta)})_\lambda$ [see Eq. (1)].

In Fig. 2, at the far right (small scale), the log moments for various q's are curved reflecting the lack of convergence. At $l_c \approx 40$ cm (corresponding to about 50 raindrops per sphere), the lines become straight indicating power law convergence. In Fig. 2(a) ($\eta=0$), high q emphasizes the spheres with particularly large N; these are therefore subject to the largest statistical fluctuations, they require more particles (larger r, smaller λ) to converge. Figure 2(b) ($\eta=1$) shows the equivalent graphs for V. As expected, the fluctuations are larger since the drop size distribution is now important; convergence to the multifractal limit is at $\approx$50–60 cm (Table I). In both cases, it is a simple matter to compare the actual results with the classical predictions: we simply take the actual drop volumes and randomize their coordinates so that the probability densities are rigorously spatially uniform. The results are shown in the smooth curves; in all cases, the latter display significantly smaller variations. As expected from the graphs, χ^2 goodness of fit tests show that the classical large N limit can be rejected with high degrees of certainty (in Fig. 1, at scale l_c, at a level 0.9999).

The slopes of the straight lines in Fig. 2 yield $K(q)$ which can itself be characterized by fitting it to the "universal" form [35]: $K(q)=[C_1/(\alpha-1)](q^\alpha - q)$. $0 \leqslant \alpha \leqslant 2$ is the Levy index which characterizes the curvature of $K(q)$, while $C_1 = K'(1)$ characterizes the sparseness of the mean. For the five storms ($\eta=1$), we found $\alpha=1.5\pm0.2$ and $C_1 \approx 0.14$

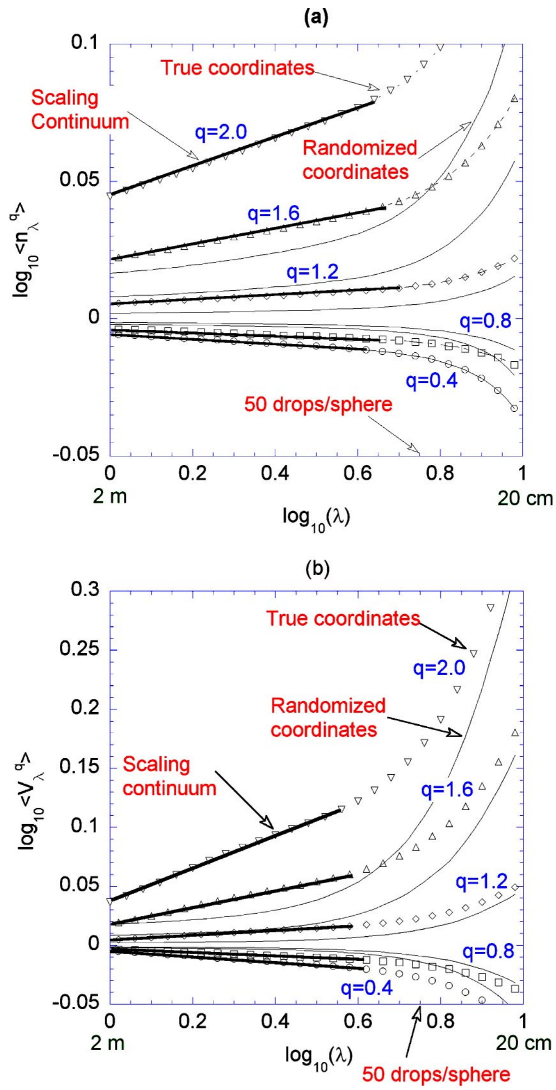

FIG. 2. (Color online) (a) The number density ($\eta=0$) for the storm $f295$ (three reconstructions). The figure shows a few representative moments showing the convergence (curved points at the small scales), and then the convergence to multiscaling (the straight lines). The predictions for the classical continuum are the smooth curves. The ensemble average is estimated by summing over 5000 spheres per reconstruction, all reconstructions. (b) Same as Fig. 2(a) but for the LWC ($\eta=1$).

±0.06 (Table I), which are close to the values found for radar reflectivities of rain and from rain gauges ($\alpha=1.4$, 1.35, $C_1=0.12$, 0.16, respectively [38]). Only for one storm ($f142$), did the multifractal convergence fail to occur within the observed scale range. This storm had large drops and weakest winds [as measured by an adjacent vertically pointing ultrahigh frequency (UHF) radar] weakest winds. Presumably, in this case, the inertia of the drops lead to a relative decoupling with the turbulence so that $l_c > 2$ m (see [32] for further discussion on this point).

Although particulate, rain is usually theorized as a mathematical field with a value at each point that corresponds to a density with respect to the volume measure. Precipitation physics and rain estimates are based on the implicit assumption of a microscale-macroscale separation allowing local means to be defined and the phenomena to be treated classically. The turbulent nature of rain makes this implausible; experiments routinely find variability in rain down to the smallest observed scales. By simply and directly applying the classical continuum thought experiment on raindrops in a 10 m^3 volume, we showed that the large N limit is multifractal; the values strongly depend on the averaging volumes. Elsewhere [38], we have shown how this can be exploited to give more realistic LWC estimates from radar. Future work is needed to include the coupling between the wind and the drops in order to determine the implications for the rain rate.

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