Vicious walks with a wall, noncolliding meanders, and chiral and Bogoliubov–de Gennes random matrices

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Spatially and temporally inhomogeneous evolution of one-dimensional vicious walkers with wall restriction is studied. We show that its continuum version is equivalent with a noncolliding system of stochastic processes called Brownian meanders. Here the Brownian meander is a temporally inhomogeneous process introduced by Yor as a transform of the Bessel process that is the motion of radial coordinate of the three-dimensional Brownian motion represented in spherical coordinates. It is proved that the spatial distribution of vicious walkers with a wall at the origin can be described by the eigenvalue statistics of Gaussian ensembles of Bogoliubov–de Gennes Hamiltonians of the mean-field theory of superconductivity, which have a particle-hole symmetry. We report that a time evolution of the present stochastic process is fully characterized by the change of symmetry classes from type *C* to type *C*I in the nonstandard classes of random matrix theory of Altland and Zirnbauer. The relation between the noncolliding systems of the generalized meanders of Yor, which are associated with the even-dimensional Bessel processes, and the chiral random matrix theory is also clarified.

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I. INTRODUCTION

Unbalance between short-ranged properties of interactions among elements and long-ranged cooperative effects realized in macroscopic levels is a significant feature of systems far from equilibrium. Even in one dimension the contact process, a model of infection of a contagious disease, exhibits a continuous phase transition at a critical value λ_c of the infection rate λ, and in $\lambda>\lambda_c$ infected and healthy individuals establish coexistence without detailed balance [1,2]. Boundary conditions locally imposed at the two edges in one-dimensional lattice play an essential role in determining the bulk properties in the asymmetric simple exclusion process, which can be regarded as a model of traffic flows in highways [3–5]. The purpose of the present paper is to propose one theoretical treatment of such emergence of long-range effects in simple (i.e., short-ranged) stochastic models using vicious-walker models originally introduced by Fisher for wetting and melting transitions [6]. The key point is the symmetry of higher-dimensional space, in which the nonequilibrium many-body system is embedded.

Consider N identical and independent simple and symmetric random walks with initial positions $x_1<x_2<\cdots<x_N$, where x_j are assumed to be even integers. One of the fundamental quantities in the vicious-walk problem [7] is the probability $\mathcal{N}_N(t,\mathbf{x})$, $\mathbf{x}=(x_1,\ldots,x_N)$, that all walkers retain the ordering of their positions up to time t; $x_1(s)<x_2(s)<\cdots<x_N(s)$ for all $0\leq s\leq t$. In other words, it is the probability that they never collide with each other for a time period t. If two of them collide, then both are annihilated since all walkers are vicious persons. It should be noted that this *noncolliding condition* seems to be very local and incidental, since any walker can enjoy free walking, while the relative distances from the nearest-neighbor walkers are greater than two units of lattice spacing. Fisher [6] and Huse and Fisher [8] derived the asymptotic form $\mathcal{N}_N(t,\mathbf{x})\sim t^{-\psi_N}$ in large t for finite $\mathbf{x}$ and determined the exponent as

$$\psi_N=\frac{1}{2}\binom{N}{2}=\frac{1}{4}N(N-1). \tag{1}$$

An interesting and important fact is that ψ_N is nonlinear in N expressing the long-ranged effect among vicious walkers, which is a result of accumulation of contact repulsive interactions between nearest-neighbor walkers during the time interval $[0,t]$. Moreover, Eq. (1) implies that the system possesses symmetry with respect to permutations of the walker positions. This hidden symmetry was clarified as follows: Huse and Fisher [6,8] mapped the enumeration problem of walks of the N particles from a set of positions $x_1<\cdots<x_N$ to $y_1<\cdots<y_N$ in time period t onto the diffusion problem of a single particle in the N-dimensional space with a set of wall restrictions (the phase space) from a position $\mathbf{x}=(x_1,\ldots,x_N)$ to $\mathbf{y}=(y_1,\ldots,y_N)$ in time t. Assume that $p(t,y|x)$ denotes the transition probability density of a one-dimensional Brownian motion from x to y in time t, that is, $p(t,y|x)=e^{-(y-x)^2/2t}/\sqrt{2\pi t}$, then by exploiting the method of images they derived the N-body Green function of vicious walkers in the determinantal form [9]

$$f(t,\mathbf{y}|\mathbf{x})=\det_{1\leq j,k\leq N}[p(t,y_k|x_j)]. \tag{2}$$

As discussed in Refs. [6,8] and explicitly shown in Ref. [10], Eq. (2) is found to be factorized into a product of the sym-

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metric part (the Schur function multiplied by Gaussian kernels) and the antisymmetric part (the product of differences of variables), and $\mathcal{N}_N(t,\mathbf{x})$ is obtained as an integration of Eq. (2) over $\mathbf{y}$ with restriction $y_1 < y_2 < \cdots < y_N$.

Based on the above knowledge of the function $\mathcal{N}_N(t,\mathbf{x})$, let us next consider the evolution of vicious walkers in time t, conditioned that they retain their ordering (i.e., noncolliding condition) up to a given finite time T. Katori and Tanemura [10] showed that this stochastic process was inhomogeneous both in space and time, and a transition in the particle distribution was observed as time t goes on from 0 to T. This transition is characterized by a symmetry change, which can be described neither in the real one-dimensional space nor in the N-dimensional phase space, but in the space of $N\times N$ Hermitian matrices. That is, the problem was exactly mapped to the statistics of N real eigenvalues of $N\times N$ Hermitian random matrices in a time-dependent Gaussian ensemble. Due to the Hermitian condition on $N\times N$ matrices with complex elements $H_{jk}=H^{\mathrm{R}}_{jk}+iH^{\mathrm{I}}_{jk}, i=\sqrt{-1}, 1\le j,k\le N$, $N(N+1)/2$ variables in set $\mathcal{R}=\{H^{\mathrm{R}}_{jk}:1\le j\le k\le N\}$ and $N(N-1)/2$ variables in set $\mathcal{I}=\{H^{\mathrm{I}}_{jk}:1\le j<k\le N\}$ are chosen as independent variables. These N^2 variables in total are assumed to be independently distributed following the Gaussian distributions with zero means. The variances of the variables in $\mathcal{R}$ and $\mathcal{I}$ are proportional to σ_{R} and σ_{I}, respectively, both of which are functions of t. As the time t approaches the final time T, the variance σ_{I} decreases to zero and a transition from the ensemble of complex Hermitian matrices (the Gaussian unitary ensemble, GUE) to that of real symmetric matrices (the Gaussian orthogonal ensemble, GOE) occurs. By integrating over the $N'=N^2-N$ variables other than the N eigenvalues, a transition of the eigenvalue statistics from the GUE class to the GOE class is formulated [11], and it is indeed realized as the time evolution of positions of vicious walkers [10,12,13].

The above results suggest the possibility that vicious-walker-type problems with N walkers are generally mapped to some solvable problems in the spaces with appropriately higher dimensions $N+N'$, in which only the symmetries of the spaces should be considered and the interactions (restrictions) among the original walkers are resulted from integrating over the auxiliary N' variables. The symmetries of the higher-dimensional spaces govern the macroscopic behaviors of the systems. The interacting particle systems far from equilibrium will be exactly solved, if we are able to find relevant symmetries, which are generally hidden in the original descriptions of the systems.

Now we propose two kinds of problems of vicious walks, which will be solved in the present paper in order to demonstrate the above mentioned scheme for nonequilibrium systems. We assume that all walkers are located in the positive region of position as $0<x_1<x_2<\cdots<x_N$ and put an absorbing wall at the origin $\mathbf{0}$ (see Fig. 1). The problems are (i) to determine the probability $\hat{\mathcal{N}}_N(t,\mathbf{x})$ that all the walkers retain the ordering of their positions (noncolliding condition) by keeping apart from the wall up to time t and (ii) to find out the time-dependent matrix model, whose eigenvalue statistics realize this stochastic process of vicious walkers with a wall, and clarify the hidden symmetry transition in the time evolution.

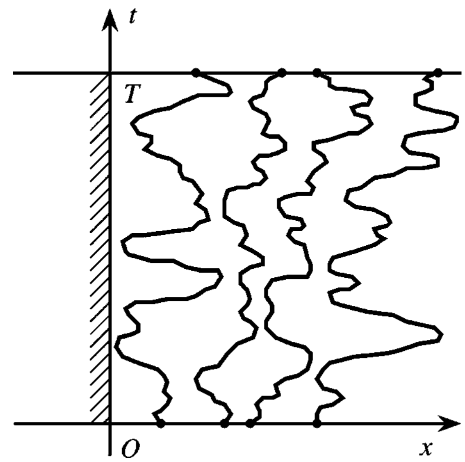

FIG. 1. Vicious walkers with a wall.

The former problem (i) was already solved by Krattenthaler *et al.* [14], and the exponent governing the asymptotic form in large t, $\hat{\mathcal{N}}_N(t,\mathbf{x})\sim t^{-\hat{\psi}_N}$, was determined as

$$\hat{\psi}_N=\frac{N^2}{2}. \tag{3}$$

So in this paper, we will start from their result and take the continuum limit of the model to solve the latter problem (ii). In Sec. II, we construct a system of noncolliding Brownian motions with a wall as a diffusion scaling limit of the corresponding vicious-walker model. An important result is that the N-body Green function of the obtained system is also in the determinantal form (2) for $0\le x_1<\cdots<x_N, 0\le y_1<\cdots<y_N$, if we replace $p(t,y|x)$ by

$$\hat{p}(t,y|x)=\frac{1}{\sqrt{2\pi t}}\{e^{-(y-x)^2/2t}-e^{-(y+x)^2/2t}\}. \tag{4}$$

This function becomes zero as $y\to 0$, representing the effect of the absorbing wall at $\mathbf{0}$. There are two distinct ways to derive Eq. (4), given as follows. (a) Consider a Brownian motion starting from the origin $\mathbf{0}$ in a space with dimensions $d\ge 2$. We adopt the d-dimensional spherical coordinate $\mathbf{r}=(r,\theta_1,\ldots,\theta_{d-1})$ to represent the motion. In particular, we can trace the radial coordinate (the modulus of the Brownian motion) $r=r(t)$ as shown in Fig. 2(a) for $d=3$. Since the transition probability density of r is generally described using the Bessel function, such a stochastic process of r is called the Bessel process [15–17]. If we multiply the transition probability density of the three-dimensional Bessel process by x/y, then Eq. (4) is obtained. (b) For a real path of the Brownian motion from $(x,0)$ to (y,t) in a spatiotemporal plane, we consider an imaginary path from $(-x,0)$ to (y,t), where $x,y>0$ [see Fig. 2(b)]. As an analogy of electrostatic problem, we subtract the transition probability density of the imaginary paths from that of the real paths to obtain Eq. (4) (the method of images).

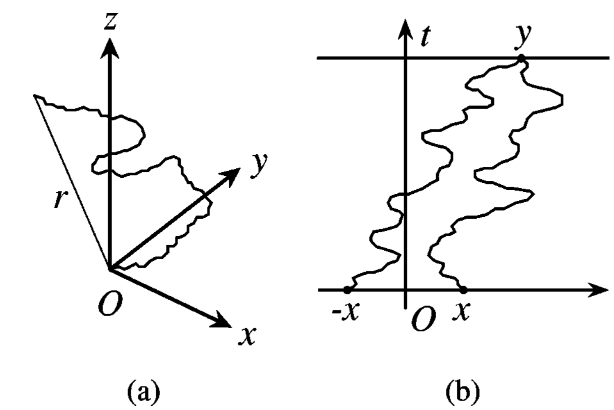

FIG. 2. (a) Three-dimensional Bessel process. (b) Method of images.

Yor studied a temporally inhomogeneous process called the *Brownian meander*, which is obtained as a transform of the three-dimensional Bessel process used in the above derivation (a) of Eq. (4). He also introduced the *d-dimensional generalized meanders* as the transform of the *d*-dimensional Bessel processes [17]. In Sec. III, we will give a general theory of noncolliding N walkers constructed as a conditioned system of *d*-dimensional generalized meanders. As a special case, it provides a proof that the noncolliding system of Brownian particles in the presence of a wall is equivalent with the noncolliding system of the Brownian meanders. This is the complete generalization, to many-particle systems with an arbitrary number of particles, N, of the fact that the single-particle Green function (4) of a Brownian motion with wall restriction at the origin is proportional to the transition probability density of a single three-dimensional Bessel process. A key point of our proof for this result is the proper transform from the N-body Green function (2) to the transition probability density by multiplying an appropriate ratio of $\hat{\mathcal{N}}_N$'s [see Eqs. (8) and (20) below]. Moreover, our general argument gives that, if we consider the problem (ii) for the *d*-dimensional Bessel processes and generalized meanders in the case of *even d*, it is solved using the Gaussian matrix theory with *chiral* symmetries, which is relevant for the physics of QCD at low energies [18–21]. Since Brownian meander is made from the *three-dimensional* Bessel process as mentioned above, it is concluded that the present problem with Eq. (4) is *not* related with chiral symmetry of matrices and that matrix models in the different symmetry classes should be considered.

The latter derivation (b) of Eq. (4) gave us a hint to find out the true symmetries, which govern the distribution of vicious walkers with a wall; *particle-hole symmetry*. The particle-hole symmetry is important in the BCS theory of superconductivity. In particular, its microscopic mean-field treatment ignores any local interactions among particles and holes, but considers this symmetry with the so-called Bogoliubov–de Gennes (BdG for short) Hamiltonian. A random matrix theory of the BdG-type Hamiltonians was introduced and developed by Altland and Zirnbauer in order to describe the energy-level statistics and transport properties in a metallic quantum dot in contact with a superconductor in a magnetic field [22,23]. They studied the Gaussian ensembles of the BdG Hamiltonian in the form

$$\mathcal{H}=U^{\dagger}\begin{pmatrix}\omega & 0\\ 0 & -\omega\end{pmatrix}U, \quad \omega=\mathrm{diag}(\omega_1,\ldots,\omega_N),$$

where U is an appropriate unitary matrix. As we will give a summary of their results in Sec. IV, Altland and Zirnbauer discovered four new classes of eigenvalue statistics in addition to the previously known three classic Wigner-Dyson classes (two of them are GUE and GOE, whose relation with the vicious walks without a wall was reported in Ref. [10]) and three chiral symmetry classes (two of them will be argued in Sec. III associated with the noncolliding systems of generalized meanders constructed from the even-dimensional Bessel processes). In the four nonstandard symmetry classes, denoted by C, CI, D and DIII in Cartan's notation [23], the classes C and CI are relevant in the present vicious-walk problem. They have the probability density functions of nonnegative eigenvalues in the form

$$p^{\mathrm{BdG}}_{\alpha,\beta}(\omega;\sigma^2)\propto e^{-|\omega|^2/2\sigma^2}\prod_{1\le j<k\le N}|\omega_k-\omega_j|^{\beta}\prod_{\ell=1}^{N}|\omega_\ell|^{\alpha}, \tag{5}$$

where $|\omega|^2=\Sigma_{j=1}^{N}\omega_j^2$ and the indices α and β are specified as

$$\alpha=2,\quad \beta=2 \quad \text{for class } C,$$

$$\alpha=1,\quad \beta=1 \quad \text{for class } C\mathrm{I}.$$

We will show in Sec. IV that the transition of distribution of vicious-walker positions with a wall is described by the symmetry change from class C to class CI of the BdG Hamiltonians. This fact was already reported by Nagao [24], but in this paper complementary results will be given. In an earlier paper [13], the transition from GUE to GOE realized in the vicious walks without a wall was characterized by the graphical expansions with time-dependent coefficients for the moments of walkers. In Sec. V, we will introduce the Möbius graph expansions for the moments of the vicious walkers with a wall. Moreover, using exact results of dynamical correlations by Nagao [24], closed formulas for the moments will be given. Such graphical expansions will be reported in detail in the present paper for the nonstandard symmetry classes C and CI of Altland and Zirnbauer. Concluding remarks are given in Sec. VI.

II. VICIOUS WALK WITH A WALL AND ITS DIFFUSION SCALING LIMIT

A. Determinantal formula

First, we consider the N independent, simple and symmetric random walks on an integer lattice $\mathbf{Z}=\{\ldots,-2,-1,0,1,2,\ldots\}$ starting from the sites $\{x_j\}$, $j=1,2,\ldots,N$, and denote the position of the jth walker at time $n=0,1,2,\ldots$ by $x_j(n)$. Assume that the initial positions are all distinct non-negative even integers and ordered as 0

$\leqslant x_1 < x_2 < \cdots < x_N$. Then we impose the noncolliding condition up to a given time $m \geqslant 0$,

$$x_1(n) < x_2(n) < \cdots < x_N(n), \quad n = 1,2,\ldots,m.$$

Such conditional walks are called *vicious walks up to time m* [6]. Here we impose further restriction on the walks as

$$x_j(n) \geqslant 0, \quad j = 1,2,\ldots,N, \quad n = 1,2,\ldots,m.$$

In other words, there is a wall at the origin and all the walkers are conditioned *never to collide with each other or to collide with the wall during the time interval* $0 \leqslant n \leqslant m$.

Let $\hat{N}_N(m,\mathbf{y}|\mathbf{x})$, $\mathbf{x}=(x_1,\ldots,x_N)$, $\mathbf{y}=(y_1,\ldots,y_N)$, be the total number of the vicious walks with wall restriction, in which the N walkers start from the positions x_j, $j=1,2,\cdots,N$, and arrive at the positions y_j, $j=1,2,\cdots,N$, at time m. Krattenthaler *et al.* gave the determinantal formula to this number as [14]

$$\hat{N}_N(m,\mathbf{y}|\mathbf{x}) = \det_{1\leqslant j,k\leqslant N}\left[\binom{m}{\frac{m+x_j-y_k}{2}} - \binom{m}{\frac{m+x_j+y_k+2}{2}}\right].$$

Suppose that all random walks start from given initial positions $\mathbf{x}$. Since the total number of walks is 2^{mN}, the probability that they are vicious walks with wall restriction and end up with positions $\mathbf{y}$ is $\hat{N}_N(m,\mathbf{y}|\mathbf{x})/2^{mN}$.

B. Diffusion scaling limit

In order to take the continuum limit of the vicious walks to derive the system of noncolliding Brownian motions, we introduce a function $\phi_L(x) = 2[Lx/2]$ for $L>0$, $x\in\mathbf{R}$ (the set of all real numbers), where $[z]$ denotes the largest integer not greater than z, and let $\phi_L(\mathbf{x}) = (\phi_L(x_1),\ldots,\phi_L(x_N))$. By Stirling's formula, we can take the diffusion scaling limit as

$$\lim_{L\to\infty}\left(\frac{L}{2}\right)^N 2^{-N\phi_{L^2}(t)}\hat{N}_N[\phi_{L^2}(t),\phi_L(\mathbf{y})|\phi_L(\mathbf{x})] = \hat{f}_N(t,\mathbf{y}|\mathbf{x})$$

for $\mathbf{x},\mathbf{y}$ with $0\leqslant x_1<\cdots<x_N$, $0\leqslant y_1<\cdots<y_N$, where

$$\hat{f}_N(t,\mathbf{y}|\mathbf{x}) = \det_{1\leqslant j,k\leqslant N}[\hat{p}(t,y_k|x_j)] \tag{6}$$

with Eq. (4). Let $T>0$ and consider a system of N Brownian motions conditioned never to collide with each other or to collide with the wall at $x=0$ in $[0,T]$. Set

$$\hat{\mathcal{N}}_N(t,\mathbf{x}) = \int_{0\leqslant y_1<\cdots<y_N} d\mathbf{y}\,\hat{f}_N(t,\mathbf{y}|\mathbf{x}), \tag{7}$$

where $d\mathbf{y} = \prod_{j=1}^N dy_j$. Then the transition probability density from the state $0\leqslant x_1<\cdots<x_N$ at time s to the state $0\leqslant y_1<\cdots<y_N$ at time $t(\geqslant s)$ of such a system is given by

$$\hat{g}_{N,T}(s,\mathbf{x};t,\mathbf{y}) = \frac{\hat{f}_N(t-s,\mathbf{y}|\mathbf{x})\hat{\mathcal{N}}_N(T-t,\mathbf{y})}{\hat{\mathcal{N}}_N(T-s,\mathbf{x})}, \tag{8}$$

since the numerator is the probability that we have the noncolliding (with each other and with a wall) Brownian paths from $\mathbf{x}$ at time s to $\mathbf{y}$ at time t *and* these paths keep noncolliding from time t up to time T as well, and the denominator is the probability that the Brownian paths are noncolliding all during time interval $[s,T]$.

It is useful to rewrite Eq. (6) as

$$\begin{aligned}\hat{f}_N(t,\mathbf{y}|\mathbf{x}) = {} & (2\pi t)^{-N/2}\mathrm{sp}_{\xi(\mathbf{y})}(e^{x_1/t},\ldots,e^{x_N/t}) \\ & \times e^{-(|\mathbf{x}|^2+|\mathbf{y}|^2)/2t}\prod_{j=1}^N (e^{x_j/t}-e^{-x_j/t}) \\ & \times \prod_{1\leqslant j<k\leqslant N}\{(e^{x_k/t}-e^{x_j/t})(e^{(x_j+x_k)/t}-1)\} \\ & \times\left\{\prod_{j=1}^N e^{x_j/t}\right\}^{-N+1},\end{aligned} \tag{9}$$

where $\xi(\mathbf{y}) = (\xi_1(\mathbf{y}),\ldots,\xi_N(\mathbf{y}))$ with $\xi_j(\mathbf{y}) = y_{N-j+1}-(N-j+1)$, $j=1,2,\ldots,N$,

$$\mathrm{sp}_\xi(z_1,\ldots,z_N) = \frac{\det(z_i^{\xi_j+N-j+1} - z_j^{-(\xi_j+N-j+1)})}{\det(z_i^{N-j+1}-z_i^{-(N-j+1)})},$$

and $|\mathbf{x}|^2 = \Sigma_{j=1}^N x_j^2$. Note that $\mathrm{sp}_\lambda(z_1,\ldots,z_N)$ is the character of the irreducible representation corresponding to a partition λ of the symplectic Lie algebra (see, for example, Lectures 6 and 24 in Ref. [25]). Since we know the formula

$$\mathrm{sp}_\xi(1,\ldots,1) = \prod_{1\leqslant i<j\leqslant N}\frac{\ell_j^2-\ell_i^2}{m_j^2-m_i^2}\prod_{j=1}^N\frac{\ell_j}{m_j},$$

with $\ell_j = \xi_j+N-j+1$, $m_j = N-j+1$ [25], and the integral ((17.6.6) on p. 354 in Ref. [26])

$$\begin{aligned}\int d\mathbf{x}e^{-|\mathbf{x}|^2/2}\prod_{1\leqslant i<j\leqslant N}|x_j^2-x_i^2|^{2\gamma}\prod_{j=1}^N|x_j|^{2a-1} \\ = 2^{aN+\gamma N(N-1)}\prod_{j=1}^N\frac{\Gamma(1+j\gamma)\Gamma[a+\gamma(j-1)]}{\Gamma(1+\gamma)},\end{aligned}$$

we have the asymptotic

$$\hat{\mathcal{N}}_N(t,\mathbf{x}) = \frac{1}{\tilde{c}_N}\hat{h}_N(\mathbf{x}/\sqrt{t})[1+\mathcal{O}(\mathbf{x}/\sqrt{t})] \tag{10}$$

for $|\mathbf{x}|/\sqrt{t}\to 0$, where $\tilde{c}_N = (\pi/2)^{N/2}\Pi_{j=1}^N\Gamma(2j)/\Gamma(j)$ with the Gamma function $\Gamma(z)$ and

$$\hat{h}_N(\mathbf{x}) = \prod_{1\leqslant j<k\leqslant N}(x_k^2-x_j^2)\prod_{\ell=1}^N x_\ell. \tag{11}$$

Substituting Eqs. (9) and (10) into Eq. (8), we find

$$\hat{g}_{N,T}(0,\mathbf{0};t,\mathbf{y}) = \hat{c}_N T^{N^2/2} t^{-N(2N+1)/2} e^{-|\mathbf{y}|^2/2t} \hat{h}_N(\mathbf{y}) \hat{\mathcal{N}}_N(T-t,\mathbf{y}), \tag{12}$$

where $\hat{c}_N = 1/\prod_{j=1}^{N}\Gamma(j)$. The transition probability densities (8) and (12) define the *N noncolliding Brownian motions with wall restriction* in time interval $(0,T]$ [27].

It should be noted that $\hat{\mathcal{N}}_N(t,\mathbf{x})$ is the noncolliding probability of Brownian motions with wall restriction and Eq. (10) gives the power-law behavior $\hat{\mathcal{N}}_N(t,\mathbf{x}) \sim t^{-\hat{\psi}_N}$ in large t for finite $\mathbf{x}$ with the critical exponent (3).

C. Transition from class *C* to class *C*I

From Eq. (10), the $T\to\infty$ limit of Eq. (8) is determined and simply given as

$$\hat{p}_N(0,\mathbf{x};t,\mathbf{y}) \equiv \lim_{T\to\infty} \hat{g}_{N,T}(0,\mathbf{x};t,\mathbf{y}) = \frac{\hat{h}_N(\mathbf{y})}{\hat{h}_N(\mathbf{x})} \hat{f}_N(t,\mathbf{y}|\mathbf{x}), \tag{13}$$

where we have set $s=0$ and used Eq. (11). Moreover, we can take the $\mathbf{x}\to\mathbf{0}$ limit of Eq. (13) to obtain

$$\hat{p}_N(0,\mathbf{0};t,\mathbf{y}) = \hat{c}'_N t^{-N(2N+1)/2} e^{-|\mathbf{y}|^2/2t} \hat{h}_N(\mathbf{y})^2, \tag{14}$$

where $\hat{c}'_N = (2/\pi)^{N/2}/\prod_{j=1}^{N}\Gamma(2j)$. That is, we have the identity

$$\hat{p}_N(0,\mathbf{0};t,\mathbf{y}) = N!\, p_{2,2}^{\mathrm{BdG}}(\mathbf{y};t)$$

for $0\le y_1<\cdots<y_N$, where $p_{2,2}^{\mathrm{BdG}}(\omega;\sigma^2)$ is the probability density function (5) of non-negative eigenvalues of the BdG Hamiltonian in class *C* ($\alpha=\beta=2$). On the other hand, if we set $t=T$ in Eq. (8), for $\hat{\mathcal{N}}_N(0,\mathbf{y})=1$, we have

$$\hat{g}_{N,T}(0,\mathbf{0};T,\mathbf{y}) = \hat{c}_N t^{-N(N+1)/2} e^{-|\mathbf{y}|^2/2t} \hat{h}_N(\mathbf{y}), \tag{15}$$

which implies the identity

$$\hat{g}_{N,T}(0,\mathbf{0};T,\mathbf{y}) = N!\, p_{1,1}^{\mathrm{BdG}}(\mathbf{y};t).$$

That is, at $t=T$, $\hat{g}_{N,T}(0,\mathbf{0};T,\mathbf{y})$ is identified with the probability density function (5) of non-negative eigenvalues of the BdG Hamiltonian in class *C*I ($\alpha=\beta=1$).

The above results mean the following facts. If we consider the *N* noncolliding Brownian motions with wall restriction up to a finite time $T>0$, in which all particles start from the origin, as the ratio $t/T\to 0$, the distribution of particle positions is asymptotically described by the eigenvalue statistics of the BdG Hamiltonian in class *C*. On the other hand, at the final time $t=T$, it can be identified with the eigenvalue statistics of the BdG Hamiltonian in class *C*I. There occurs, thus, a transition from the class *C* distribution to the class *C*I distribution as time t goes on from 0 to T in our stochastic process.

The essential difference between Eqs. (14) and (15) is found in the exponents of the factors $\hat{h}_N(\mathbf{y})^\beta$ such that $\beta=2$ in Eq. (14) and $\beta=1$ in Eq. (15). This factor expresses strong repulsive interactions among particles and between the wall and each particle, in which the larger exponent β gives stronger repulsion for short distances. At the very early stage of the process, $t/T\ll 1$, the repulsion may be strong, since the noncolliding condition will be imposed for a long time period up to time T in the future. As the time t goes on, the repulsion strength decreases as does the remaining time until T, and attains its minimum at $t=T$.

D. Stochastic differential equations

As explained in Ref. [10] in the case without wall restriction, the positions $\mathbf{x}(t)=(x_1(t),\ldots,x_N(t))$ of the *N* noncolliding Brownian motions solve the stochastic differential equations in a modified type of Dyson's Brownian motion model [26,28]. In the present case with wall restriction, we have

$$dx_j(t) = \hat{E}_j^T[\mathbf{x}(t)]dt + dB_j(t),$$

for $0\le x_1<\cdots<x_N, 0<t\le T$, where

$$\hat{E}_j^T(\mathbf{x}) = \frac{\partial}{\partial x_j}\ln \hat{\mathcal{N}}_N(T-t;\mathbf{x}),$$

and $\{B_j(t)\}_{j=1}^N$ are *N* independent standard Brownian motions

$$B_j(0)=0, \quad \langle B_j(t)\rangle = 0,$$

$$\langle (B_j(t)-B_j(s))(B_k(t)-B_k(s))\rangle = |t-s|\,\delta_{jk}$$

for any $t,s>0, j,k=1,2,\ldots,N$. In particular, in the limit $T\to\infty$, Eqs. (10) and (11) give the equations

$$dx_j(t) = dB_j(t) + \frac{1}{x_j(t)}dt + \sum_{1\le k\le N, k\neq j}\left\{\frac{1}{x_j(t)-x_k(t)} + \frac{1}{x_j(t)+x_k(t)}\right\}dt \tag{16}$$

for $1\le j\le N$ [27]. In the stochastic differential equation for the position of the jth particle, the drift terms $dt/x_j(t)$ and $dt/(x_j(t)-x_k(t))$ represent repulsive forces from the wall at the origin and that from the kth particle, respectively. In addition to these, there are terms of the form $dt/(x_j(t)+x_k(t))$, $1\le k\le N, k\neq j$, which can be interpreted as effective repulsive forces from the mirror images of other particles located at $-x_k(t)$, $1\le k\le N$, $k\neq j$.

III. NONCOLLIDING MEANDERS AND CHIRAL RANDOM MATRIX THEORY

A. Definitions of elementary processes

Consider a diffusion equation in dimension $d \geq 2$,

$$\frac{\partial}{\partial t} u(t,\mathbf{y}|\mathbf{x}) = \frac{1}{2} \sum_{j=1}^{d} \frac{\partial^2}{\partial y_j^2} u(t,\mathbf{y}|\mathbf{x})$$

with the initial condition $u(0,\mathbf{y}|\mathbf{x}) = \delta(\mathbf{x}-\mathbf{y})$. We use the spherical coordinates $\mathbf{x}=(x,\theta_1,\ldots,\theta_{d-1}),\mathbf{y}=(y,\varphi_1,\ldots,\varphi_{d-1})$ and integrate over all the angular variables to obtain a differential equation for the radial coordinate (the modulus)

$$\frac{\partial}{\partial t}\overline{u}(t,y|x) = \frac{1}{2}\left[\frac{\partial^2}{\partial y^2} + \frac{d-1}{y}\frac{\partial}{\partial y}\right]\overline{u}(t,y|x).$$

The unique solution of this equation satisfying the initial condition $\overline{u}(0,y|x)y^{d-1}dy = \delta(x-y)dy$ for $x>0$ is given as

$$\overline{u}(t,y|x) = \frac{1}{(xy)^{\nu}} \frac{1}{t} e^{-(x^2+y^2)/2t} I_{\nu}\left(\frac{xy}{t}\right),$$

where $\nu=(d-2)/2$ and $I_\nu(z)$ is the modified Bessel function,

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+\nu}}{n!\Gamma(\nu+n+1)}.$$

If we set $p^{(\nu)}(t,y|x) = \overline{u}(t,y|x)y^{d-1}$, then it is normalized as $\int_0^\infty p^{(\nu)}(t,y|x)dy=1$ for any $x>0$. We define $p^{(\nu)}(t,y|0)$ by the $x\to 0$ limit of $p^{(\nu)}(t,y|x)$. Then we have

$$p^{(\nu)}(t,y|x) = \frac{y^{\nu+1}}{x^{\nu}} \frac{1}{t} e^{-(x^2+y^2)/2t} I_{\nu}\left(\frac{xy}{t}\right), \quad x>0, y\geq 0,$$

$$p^{(\nu)}(t,y|0) = \frac{y^{2\nu+1}}{2^{\nu}\Gamma(\nu+1)t^{\nu+1}} e^{-y^2/2t}, \quad y \geq 0. \tag{17}$$

The $d=2(\nu+1)$-dimensional Bessel process is defined so that its transition probability density is given by Eq. (17) [15–17].

For $0\leq u \leq T, w\geq 0$, $\kappa \in [0,2(\nu+1))$, we consider

$$h_T^{(\nu,\kappa)}(u,w) = \int_0^\infty dz\ p^{(\nu)}(T-u,z|w)z^{-\kappa}. \tag{18}$$

That is, we multiply a weight $z^{-\kappa}$ at the final time T, so that, as the arrival position z is nearer to the origin, the path is more enhanced. Then the transition probability density from x at time s to y at time t with such bias at time T, $0\leq s<t \leq T$, is given by

$$p_T^{(\nu,\kappa)}(s,x;t,y) = \frac{1}{h_T^{(\nu,\kappa)}(s,x)} p^{(\nu)}(t-s,y|x) h_T^{(\nu,\kappa)}(t,y) \tag{19}$$

for $x,y\geq 0$. This bias makes the process defined by Eq. (19) temporally inhomogeneous, and Yor called it the *generalized meander* indexed (ν,κ). In particular, when $\nu=1/2$ and $\kappa=1$, the process is called the *Brownian meander* [17]. The transformation from Eq. (17) to $p_T^{(\nu,\kappa)}(s,x;t,y)$ by Eq. (19) is a generalization of the h transform of Doob [29]. Note that if we replace T by u in Eq. (18), then $h_u^{(\nu,\kappa)}(u,w) = w^{-\kappa}$, since $p^{(\nu)}(0,z|w) = \delta(z-w)$. In the case $\nu=1/2$, $\kappa=1$, and $s=0$, Eq. (19) becomes, by this replacement, $p^{(1/2)}(t,y|x)x/y$, which is equal to Eq. (4) for $I_{1/2}(z) = \sqrt{2/(\pi z)}\sinh z$. This is the derivation (a) of Eq. (4) mentioned in Sec. I.

B. Noncolliding systems and rectangular random matrices

Now we consider a system of N generalized meanders conditioned that they never collide with each other for a time interval $(0,T], T>0$. Using the determinantal formula in Eq. (2) and following the same way as Eqs. (8) and (19), the transition probability density is given by

$$g_{N,T}^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = \frac{f_{N,T}^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y})\mathcal{N}_{N,T}^{(\nu,\kappa)}(T-t,\mathbf{y})}{\mathcal{N}_{N,T}^{(\nu,\kappa)}(T-s,\mathbf{x})} \tag{20}$$

for $0\leq s<t\leq T, 0\leq x_1<\cdots<x_N, 0\leq y_1<\cdots<y_N$, where

$$f_{N,T}^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = \det_{1\leq j,k\leq N}[p_T^{(\nu,\kappa)}(s,x_j;t,y_k)], \tag{21}$$

and

$$\mathcal{N}_{N,T}^{(\nu,\kappa)}(T-t,\mathbf{x}) = \int_{0\leq y_1<\cdots<y_N} d\mathbf{y} f_{N,T}^{(\nu,\kappa)}(T-t,\mathbf{x},T,\mathbf{y}).$$

Since $f_{N,T}^{(\nu,0)}(s,\mathbf{x};t,\mathbf{y})$ is temporally homogeneous and independent of T, we will write it as $f_N^{(\nu)}(t-s,\mathbf{y}|\mathbf{x})$. Moreover, note that

$$f_{N,T}^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = \frac{1}{h_T^{(\nu,\kappa)}(s,\mathbf{x})} f_N^{(\nu)}(t-s,\mathbf{y}|\mathbf{x}) h_T^{(\nu,\kappa)}(t,\mathbf{y}), \tag{22}$$

where $h_T^{(\nu,\kappa)}(t,\mathbf{x}) = \prod_{j=1}^N h_T^{(\nu,\kappa)}(t,x_j)$, and that $h_T^{(\nu,\kappa)}(T,\mathbf{x}) = \prod_{j=1}^N x_j^{-\kappa}$. Then Eq. (20) can be written as

$$g_{N,T}^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = \frac{1}{\widetilde{\mathcal{N}}_N^{(\nu,\kappa)}(T-s,\mathbf{x})} \times f_N^{(\nu)}(t-s,\mathbf{y}|\mathbf{x}) \widetilde{\mathcal{N}}_N^{(\nu,\kappa)}(T-t,\mathbf{y}) \tag{23}$$

with

$$\widetilde{\mathcal{N}}_N^{(\nu,\kappa)}(t,\mathbf{x}) = \int_{0\leq y_1<\cdots<y_N} d\mathbf{y} f_N^{(\nu)}(t,\mathbf{y}|\mathbf{x}) \prod_{j=1}^N y_j^{-\kappa}. \tag{24}$$

The important point is that we can confirm that Eq. (23) with $\nu=1/2$ and $\kappa=1$ is equal to the transition probability density (8) of the noncolliding Brownian motions with a wall. In other words, we found the equivalence between *the noncol-*

liding Brownian motions with a wall and *the noncolliding Brownian meanders*. If we set $N=1$ in Eq. (24) with $\nu=1/2,\kappa=1$, we have $\widetilde{\mathcal{N}}_1^{(1/2,1)}(t,x)=1/x$. Then Eq. (23) with $s=0$ is reduced to the equality $\hat{p}(t,y|x)=p^{(1/2)}(t,y|x)x/y$, which is the statement given as the derivation (a) of Eq. (4) in Sec. I.

We then consider the limit $|\mathbf{x}|\to 0$ to define the noncolliding generalized meanders, all starting from the origin $\mathbf{0}$ at the initial time $s=0$ for non-negative integers ν (i.e., even d). Consider an arbitrary $N_1\times N_2$ complex matrices with $N_1\geqslant N_2$. We denote by $\mathcal{M}(N_1,N_2;\mathbf{C})$ the space of all such matrices. It is known [30] that any matrix $A\in\mathcal{M}(N_1,N_2;\mathbf{C})$ can be expressed by

$$A=U^{\dagger}\Lambda V, \tag{25}$$

where U and V are unitary matrices with sizes N_1 and N_2, respectively, and Λ is the $N_1\times N_2$ matrix in the form

$$\Lambda=\begin{pmatrix}\hat{\Lambda}\\ \mathbf{0}\end{pmatrix} \quad \text{with} \quad \hat{\Lambda}=\mathrm{diag}(a_1,a_2,\ldots,a_{N_2}), \tag{26}$$

and where $a_j\geqslant 0, 1\leqslant j\leqslant N_2$. The matrices (U,V) parametrize the coset space $U(N_1)\times U(N_2)/[U(1)]^{N_2}$, where $[U(1)]^{N_2}$ is the diagonal subgroup of $U(N_2)$, and thus the (U,Λ,V) can be regarded as "spherical coordinates" in the space $\mathcal{M}(N_1,N_2;\mathbf{C})$. It should be noted that $\{a_1,\ldots,a_{N_2}\}$ are not eigenvalues of A; they will be referred to as "radial coordinates." The following integral formula proved in Ref. [20] is useful. Let $d\mu(U,V)$ be the Haar measure of $U(N_1)\times U(N_2)/[U(1)]^{N_2}$. For $A,B\in\mathcal{M}(N_1,N_2;\mathbf{C})$, set $A=U_A^{\dagger}\Lambda_A V_A$, $B=U_B^{\dagger}\Lambda_B V_B$, where $U_A,U_B\in U(N_1)$, $V_A,V_B\in U(N_2)/[U(1)]^{N_2}$,

$$\Lambda_A=\begin{pmatrix}\hat{\Lambda}_A\\ \mathbf{0}\end{pmatrix}, \quad \Lambda_B=\begin{pmatrix}\hat{\Lambda}_B\\ \mathbf{0}\end{pmatrix},$$

with $\hat{\Lambda}_A=\mathrm{diag}(a_1,\ldots,a_{N_2})$, $\hat{\Lambda}_B=\mathrm{diag}(b_1,\ldots,b_{N_2})$, $a_j\geqslant 0, b_j\geqslant 0, 1\leqslant j\leqslant N_2$. Then for an arbitrary constant σ,

$$\int d\mu(U_A,V_A)\exp\left(-\frac{1}{2\sigma^2}\mathrm{tr}\{(A-B)^{\dagger}(A-B)\}\right) \propto \frac{\det_{1\leqslant j,k\leqslant N_2}\left[\exp\left(-\frac{a_j^2+b_k^2}{2\sigma^2}\right)I_{N_1-N_2}\left(\frac{a_jb_k}{\sigma^2}\right)\right]}{\prod_{j=1}^{N_2}(a_jb_j)^{N_1-N_2}\prod_{1\leqslant j<k\leqslant N_2}(a_j^2-a_k^2)(b_j^2-b_k^2)}.$$

Note that this integral formula can be regarded as a version of the Harish-Chandra (Itzykson-Zuber) formula [31–33].

Using this integral formula, Eq. (21) with a non-negative integer ν and $\kappa=0$ is written as

$$f_N^{(\nu)}(t,\mathbf{y}|\mathbf{x})\propto\prod_{1\leqslant j<k\leqslant N}(x_j^2-x_k^2)\prod_{j=1}^{N}y_j^{2\nu+1} \times\prod_{1\leqslant j<k\leqslant N}(y_j^2-y_k^2)\int d\mu(U_Y,V_Y) \times\exp\left(-\frac{1}{2t}\mathrm{tr}\{(X-Y)^{\dagger}(X-Y)\}\right).$$

Since $\mathrm{tr}\{(X-Y)^{\dagger}(X-Y)\}\to\mathrm{tr}Y^{\dagger}Y=|\mathbf{y}|^2$ as $|\mathbf{x}|\to 0$, we have

$$\lim_{|\mathbf{x}|\to 0}\frac{f_N^{(\nu)}(t,\mathbf{y}|\mathbf{x})}{\prod_{1\leqslant j<k\leqslant N}(x_j^2-x_k^2)} \propto\prod_{j=1}^{N}y_j^{2\nu+1}\prod_{1\leqslant j<k\leqslant N}(y_j^2-y_k^2)e^{-|\mathbf{y}|^2/2t}. \tag{27}$$

The above argument proves the following result. Let ν be a non-negative integer and $0\leqslant\kappa<2(\nu+1)$. The limit $|\mathbf{x}|\to 0$ of $g_{N,T}^{(\nu,\kappa)}(0,\mathbf{x},t,\mathbf{y})$ is given by

$$g_{N,T}^{(\nu,\kappa)}(0,\mathbf{0};t,\mathbf{y})=c\ e^{-|\mathbf{y}|^2/2t}\prod_{j=1}^{N}y_j^{2\nu+1} \times\prod_{1\leqslant j<k\leqslant N}(y_j^2-y_k^2)\widetilde{\mathcal{N}}_N^{(\nu,\kappa)}(T-t,\mathbf{y}), \tag{28}$$

where c is a normalization constant determined by $\int_{0\leqslant y_1<\ldots<y_N}d\mathbf{y}g_{N,T}^{(\nu,\kappa)}(0,\mathbf{0};t,\mathbf{y})=1$. The N noncolliding generalized meanders *all starting from the origin* $\mathbf{0}$ *at time 0* are defined by the transition probability density (28).

C. Chiral Gaussian ensembles and transition of chiral symmetries

For the space $\mathcal{M}(N_1,N_2;\mathbf{C})$ of all $N_1\times N_2$ complex matrices, we introduce the integration measure $dv(A)=\prod_{j=1}^{N_1}\prod_{k=1}^{N_2}dA_{jk}^{\mathrm{R}}dA_{jk}^{\mathrm{I}}$ for $A=(A_{jk})\in\mathcal{M}(N_1,N_2;\mathbf{C})$ with $A_{jk}=A_{jk}^{\mathrm{R}}+iA_{jk}^{\mathrm{I}}, i=\sqrt{-1}$. The chiral GUE (chGUE for short) with variance σ^2 is the ensemble of matrices $A\in\mathcal{M}(N_1,N_2;\mathbf{C})$ with the probability measure

$$d\mu^{\mathrm{chGUE}}(A;\sigma^2)\propto\exp\left(-\frac{1}{2\sigma^2}\mathrm{tr}A^{\dagger}A\right)dv(A).$$

For $A\in\mathcal{M}(N_1,N_2;\mathbf{C})$ with the polar coordinates (25) and (26), we can show that the probability density function of the radial coordinates $\mathbf{a}=(a_1,\ldots,a_{N_2})$ of $A\in\mathcal{M}(N_1,N_2;\mathbf{C})$ in chGUE with variance σ^2 is given as

$$p^{\mathrm{chGUE}}(\mathbf{a};\sigma^2)\propto e^{-|\mathbf{a}|^2/2\sigma^2}\prod_{j=1}^{N_2} a_j^{2(N_1-N_2)+1} \times \prod_{1\le j<k\le N_2}(a_j^2-a_k^2)^2.$$

Next we set $\mathcal{M}(N_1,N_2;\mathbf{R})$ as the space of all $N_1\times N_2$ *real* matrices for $N_1\geqslant N_2$. The chiral GOE (chGOE) with variance σ^2 is the ensemble of matrices $B\in\mathcal{M}(N_1,N_2;\mathbf{R})$ with the probability measure

$$d\mu^{\mathrm{chGOE}}(B;\sigma^2)\propto \exp\left(-\frac{1}{2\sigma^2}\mathrm{tr}B^TB\right)dv'(B),$$

with $dv'(B)=\prod_{j=1}^{N_1}\prod_{k=1}^{N_2}dB_{jk}$. The probability density function of the radial coordinates $\mathbf{b}=(b_1,\ldots,b_{N_2})$ is given in the form

$$p^{\mathrm{chGOE}}(\mathbf{b};\sigma^2)\propto e^{-|\mathbf{b}|^2/2\sigma^2}\prod_{j=1}^{N_2} b_j^{N_1-N_2}\prod_{1\le j<k\le N_2}|b_j^2-b_k^2|.$$

Then we consider the distribution of the sum of two rectangular matrices $C=A+B$, in which A and B are chosen from chGUE and chGOE, respectively. The distribution function of C is the convolution of those of chGUE and chGOE. Consider the ensemble of matrices $C\in\mathcal{M}(N_1,N_2;\mathbf{C})$, in which the probability measure is given as

$$d\mu^{\mathrm{chGUE/GOE}}(C;\sigma_1^2,\sigma_2^2) = \int_{B\in\mathcal{M}(N_1,N_2;\mathbf{R})} d\mu^{\mathrm{chGUE}}(C-B;\sigma_1^2)d\mu^{\mathrm{chGOE}}(B;\sigma_2^2).$$

We denote the probability density function of the radial coordinates $\mathbf{c}=(c_1,\ldots,c_{N_2})$ of matrix C in this ensemble by $p^{\mathrm{chGUE/GOE}}(\mathbf{c};\sigma_1^2,\sigma_2^2;N_1,N_2)$.

Comparing the above definitions and Eq. (28), we can prove the following equality for non-negative integers ν:

$$g_{N,T}^{(\nu,\nu+1)}(0,\mathbf{0};t,\mathbf{y})=N!\,p^{\mathrm{chGUE/GOE}}\left(\mathbf{y};t\left(1-\frac{t}{T}\right),\frac{t^2}{T};N+\nu,N\right),$$

where $0\le y_1<\cdots<y_N$. It implies that, if $\nu=0,1,2,\ldots$, the time evolution of the noncolliding generalized meanders indexed $(\nu,\nu+1)$ is represented by the transition of the eigenvalue statistics from the chGUE class to the chGOE class.

IV. GAUSSIAN ENSEMBLES OF BOGOLIUBOV–DE GENNES RANDOM MATRICES

Since we have found that the noncolliding Brownian motion with a wall is equivalent with the noncolliding system of Brownian meanders with indices $\nu=1/2$ and $\kappa=1$, it does not belong to the chiral symmetry classes discussed in the preceding section. We have to consider the ensembles of Hermitian matrices in the form of the BdG Hamiltonian for the mean-field theory of superconductivity.

As shown in Sec. II C, the time evolution of spatial distribution of particles in the present system can be regarded as a transition of the eigenvalue statistics of the BdG random Hamiltonian in the class C to class CI. The former statistics are characterized by the exponent $\beta=2$ of the repulsive factor $\hat{h}_N(\mathbf{y})^\beta$ and the latter by $\beta=1$. It does not imply, however, that the functional form of the distribution is maintained in form (5) with $\alpha=\beta$ and only the exponent β changes continuously as the time passes. In this section, using a version of the Harish-Chandra (Itzykson-Zuber) integral formula over a unitary group, we show the fact that the time evolution of the present process is described by *two-matrix model* coupling random matrices, one of which is chosen from a Gaussian ensemble of the BdG Hamiltonian matrices of class C and the other of which is from that of class CI. There the time dependence of variances of these two ensembles is different from each other.

A. Hermitian and real symmetric matrices with particle-hole symmetry

We consider the space of the Hermitian matrices specified by the following:

$$\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{C})=\left\{\mathcal{H}=\begin{pmatrix} a & b\\ b^\dagger & -a^T\end{pmatrix}:\quad a \text{ is an } N\times N \text{ Hermitian matrix and } b \text{ is an } N\times N \text{ complex symmetric matrix}\right\}.$$

Since the dimension of the space of a is N^2 and that of b is $N(N+1)$, the dimension of $\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{C})$ is $N(2N+1)$. Define

$$\mathcal{C}=\begin{pmatrix} 0 & I_N\\ -I_N & 0\end{pmatrix}, \tag{29}$$

where I_N is the $N\times N$ unit matrix. Then $\mathcal{H}\in\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{C})$ has the following symmetry [22,23]:

$$\mathcal{H}=-\mathcal{C}\mathcal{H}^T\mathcal{C}^{-1}. \tag{30}$$

Assume that φ_j is the $2N$-dimensional eigenvector of $\mathcal{H}$ with an eigenvalue ω_j; $\mathcal{H}\varphi_j=\omega_j\varphi_j$. Then by Eq. (30), $-\mathcal{C}\mathcal{H}^T\mathcal{C}^{-1}\varphi_j=\omega_j\varphi_j$. Take the complex conjugate of both sides and use the Hermiticity of $\mathcal{H}$ and the fact $\mathcal{C}^{-1}=-\mathcal{C}$, we have $\mathcal{H}(\mathcal{C}\varphi_j^*)=-\omega_j(\mathcal{C}\varphi_j^*)$. This means that $\mathcal{C}\varphi_j^*$ is the eigenvector of $\mathcal{H}$ with the eigenvalue $-\omega_j$. Assume that $\omega_1,\omega_2,\ldots,\omega_N$ be the non-negative eigenvalues of $\mathcal{H}$, then other eigenvalues are given by $-\omega_1,\ldots,-\omega_N$. Therefore, if we set

$$\begin{pmatrix} U_1^\dagger\\ U_2^\dagger\end{pmatrix}\equiv(\varphi_1,\ldots,\varphi_N),\qquad U\equiv\begin{pmatrix} U_1 & U_2\\ U_2^* & -U_1^*\end{pmatrix}, \tag{31}$$

then

$$\mathcal{H}U^\dagger = U^\dagger \Lambda \quad \text{with} \quad \Lambda = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}.$$

We assume that U is unitary. Then we can see that iU satisfies the relation $\mathcal{C}=(iU)^T\mathcal{C}(iU)$. The set of such $2N\times 2N$ unitary matrices is called the symplectic group $\mathrm{Sp}(2N;\mathbf{C})$ [25,26], whose dimension is $\ell = 2N^2$.

The above consideration is summarized as follows: any $\mathcal{H}\in\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{C})$ can be diagonalized as

$$U\mathcal{H}U^\dagger = \Lambda = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}, \quad iU \in \mathrm{Sp}(2N;\mathbf{C}),$$

where $\omega = \mathrm{diag}(\omega_1, \ldots, \omega_N)$, $\omega_j \geq 0, 1\leq j \leq N$. We then consider the map

$$\mathcal{H} \overset{\varphi}{\mapsto} (\omega, U) = [(\omega_j)_{1\leq j\leq N}, \mathbf{p}=(p_\mu)_{1\leq \mu \leq \ell}], \tag{32}$$

where $\mathbf{p}$ denotes the ℓ-dimensional vector, whose elements are the independent variables of U. We have the Jacobian of this map as [23]

$$J(\varphi) = \left| \det\left(\frac{\partial \mathcal{H}}{\partial \omega_1}, \ldots, \frac{\partial \mathcal{H}}{\partial \omega_N}, \frac{\partial \mathcal{H}}{\partial p_1}, \ldots, \frac{\partial \mathcal{H}}{\partial p_\ell} \right) \right|$$
$$= C(\mathbf{p}) \prod_{j=1}^{N} \omega_j^2 \prod_{1\leq j<k\leq N} (\omega_j^2 - \omega_k^2)^2, \tag{33}$$

where $C(\mathbf{p})$ is a function independent of the eigenvalues ω.

Next we consider the set

$$\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{R}) = \left\{ \mathcal{H} = \begin{pmatrix} a & b \\ b & -a^T \end{pmatrix} : \ a \text{ and } b \text{ are } N\times N \text{ real symmetric matrices} \right\}.$$

In this case, since the dimensions of the spaces of a and b are both $N(N+1)/2$, the dimension of $\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{R})$ is $N(N+1)$. We can see that any $\mathcal{H}\in\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{R})$ can be diagonalized as

$$U'\mathcal{H}U'^T = \Lambda = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}, \quad iU' \in \mathrm{Sp}(2N;i\mathbf{R}),$$

where $\omega = \mathrm{diag}(\omega_1, \ldots, \omega_N)$, $\omega_j \geq 0, 1\leq j\leq N$. Here $\mathrm{Sp}(2N;i\mathbf{R})$ is the symplectic group of $2N\times 2N$ matrices whose elements are purely imaginary. The map

$$\mathcal{H} \overset{\varphi'}{\mapsto} (\omega, U') = [(\omega_j)_{1\leq j\leq N}, \mathbf{p}'=(p_\mu)_{1\leq \mu\leq \ell'}], \tag{34}$$

with $\ell' =$ dimension of $\mathrm{Sp}(2N;\mathbf{R}) = N^2$, is considered. Here $\mathbf{p}'$ denotes the ℓ'-dimensional vector with the elements of the independent variables of U'. The Jacobian of this map is determined as [23]

$$J(\varphi') = C'(\mathbf{p}') \prod_{j=1}^{N} |\omega_j| \prod_{1\leq j<k\leq N} |\omega_j^2 - \omega_k^2|, \tag{35}$$

where $C'(\mathbf{p}')$ is a function independent of the eigenvalues ω. The important point is that $J(\varphi)$ and $J(\varphi')$ are proportional to $|\hat{h}_N(\omega)|^2$ and $|\hat{h}_N(\omega)|$, respectively.

B. Gaussian ensembles and eigenvalue distributions

Altland and Zirnbauer introduced the Gaussian ensembles of the BdG Hamiltonians in $\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{C})$ and in $\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{R})$, in which the probability measures are in the form

$$\mu_N(\mathcal{H};\sigma^2)d\mathcal{H} \propto \exp\left(-\frac{1}{4\sigma^2}\,\mathrm{tr}\mathcal{H}^2 \right) d\mathcal{H}, \tag{36}$$

with variance $2\sigma^2$. For $\mathcal{H}\in\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{C})$, we write the complex variables as $a_{jk} = a_{jk}^{\mathrm{R}} + i a_{jk}^{\mathrm{I}}, b_{jk} = b_{jk}^{\mathrm{R}} + i b_{jk}^{\mathrm{I}}$ with $i = \sqrt{-1}, a_{jk}^{\mathrm{R}}, a_{jk}^{\mathrm{I}}, b_{jk}^{\mathrm{R}}, b_{jk}^{\mathrm{I}} \in \mathbf{R}$, and choose the independent variables as $\{a_{jk}^{\mathrm{R}}, b_{jk}^{\mathrm{R}}, b_{jk}^{\mathrm{I}} : 1\leq j\leq k\leq N\} \cup \{a_{jk}^{\mathrm{I}} : 1\leq j<k \leq N\}$. Since

$$\mathrm{tr}\mathcal{H}^2 = \sum_{j,k} |\mathcal{H}_{jk}|^2 = 2\sum_{j=1}^{N} \{(a_{jj}^{\mathrm{R}})^2 + (b_{jj}^{\mathrm{R}})^2 + (b_{jj}^{\mathrm{I}})^2\}$$
$$+ 4\sum_{1\leq j<k\leq N} \{(a_{jk}^{\mathrm{R}})^2 + (a_{jk}^{\mathrm{I}})^2 + (b_{jk}^{\mathrm{R}})^2 + (b_{jk}^{\mathrm{I}})^2\},$$

the probability measure (36) is rewritten as

$$\mu_N(\mathcal{H};\sigma^2) = \prod_{j=1}^{N} \frac{e^{-(a_{jj}^{\mathrm{R}})^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \frac{e^{-(b_{jj}^{\mathrm{R}})^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \frac{e^{-(b_{jj}^{\mathrm{I}})^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$$
$$\times \prod_{1\leq j<k\leq N} \frac{e^{-(a_{jk}^{\mathrm{R}})^2/\sigma^2}}{\sqrt{\pi\sigma^2}} \frac{e^{-(a_{jk}^{\mathrm{I}})^2/\sigma^2}}{\sqrt{\pi\sigma^2}}$$
$$\times \prod_{1\leq j<k\leq N} \frac{e^{-(b_{jk}^{\mathrm{R}})^2/\sigma^2}}{\sqrt{\pi\sigma^2}} \frac{e^{-(b_{jk}^{\mathrm{I}})^2/\sigma^2}}{\sqrt{\pi\sigma^2}} \tag{37}$$

with the integration measure

$$d\mathcal{H} = \prod_{j=1}^{N} da_{jj}^{\mathrm{R}} db_{jj}^{\mathrm{R}} db_{jj}^{\mathrm{I}} \prod_{1\leq j<k\leq N} da_{jk}^{\mathrm{R}} da_{jk}^{\mathrm{I}} db_{jk}^{\mathrm{R}} db_{jk}^{\mathrm{I}}. \tag{38}$$

The probability measure (36) is given for $\mathcal{H}\in\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{R})$ by setting all the imaginary parts $\{a_{jk}^{\mathrm{I}}, b_{jk}^{\mathrm{I}}\}$ as zeros in Eqs. (37) and (38).

On the other hand, following the maps (32) and (34), the integration measures are transformed as

$$d\mathcal{H} \propto J(\varphi)d\omega dU \propto \hat{h}_N(\omega)^2 d\omega dU \tag{39}$$

for $\mathcal{H}\in\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{C})$, $iU\in\mathrm{Sp}(2N;\mathbf{C})$, and

$$d\mathcal{H} \propto J(\varphi')d\omega dU' \propto |\hat{h}_N(\omega)| d\omega dU' \tag{40}$$

for $\mathcal{H}\in\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{R})$, $iU'\in\mathrm{Sp}(2N;i\mathbf{R})$, respectively. Since $\mathrm{tr}\mathcal{H}^2 = 2|\omega|^2 = 2\sum_{j=1}^{N}\omega_j^2$, integrating over the spaces $\mathrm{Sp}(2N;\mathbf{C})$ and $\mathrm{Sp}(2N;i\mathbf{R})$ gives the distribution functions of the non-negative eigenvalues $\omega=(\omega_1,\dots,\omega_N),\omega_i\geqslant 0$, in the form of Eq. (5) with the indices $\alpha=\beta=2$ (class C) and with $\alpha=\beta=1$ (class CI), respectively.

C. Harish-Chandra integral formula

The transition probability density (12) from the state **0** to the state **y** in time t is written as follows using Eq. (7), where the proportional constants are independent of the stochastic variables **y**,

$$\hat{g}_{N,T}(0,\mathbf{0};t,\mathbf{y})\propto e^{-|\mathbf{y}|^2/2t}\hat{h}_N(\mathbf{y})\int_{0\leqslant z_1<\dots<z_N} d\mathbf{z}\,\hat{f}_N(T-t,\mathbf{z}|\mathbf{y})$$

$$\propto \hat{h}_N(\mathbf{y})\int d\mathbf{z}\,\mathrm{sgn}[\hat{h}_N(\mathbf{z})]\times\det_{1\leqslant j,k\leqslant N}[e^{-y_j^2/2t-(y_j-z_k)^2/2(T-t)}-e^{-y_j^2/2t-(y_j+z_k)^2/2(T-t)}]$$

$$=\hat{h}_N(\mathbf{y})\int d\mathbf{z}\,\mathrm{sgn}[\hat{h}_N(\mathbf{z})]e^{-|\mathbf{z}|^2/2T}\times\det_{1\leqslant j,k\leqslant N}\left\{\exp\left[-\frac{T}{2t(T-t)}\left(y_j-\frac{t}{T}z_k\right)^2\right]-\exp\left(-\frac{T}{2t(T-t)}\left[y_j+\frac{t}{T}z_k\right)^2\right]\right\}.$$

Then we set $\omega_k'=tz_k/T,1\leqslant k\leqslant N$, and regard $\omega'=(\omega_1',\dots,\omega_N')$ as the non-negative eigenvalues of the BdG Hamiltonian $\mathcal{H}'=(\mathcal{H}_{jk}')\in\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{R})$. By Eq. (40), $d\mathbf{z}\propto d\omega'\propto d\mathcal{H}'/|\hat{h}_N(\omega')|$, and we have

$$\hat{g}_{N,T}(0,\mathbf{0};t,\mathbf{y})\propto\hat{h}_N(\mathbf{y})\int d\mathcal{H}'\frac{1}{\hat{h}_N(\omega')}e^{-T|\omega'|^2/2t^2}\times\det_{1\leqslant j,k\leqslant N}\left[\exp\left(-\frac{T}{2t(T-t)}(y_j-\omega_k')^2\right)-\exp\left(-\frac{T}{2t(T-t)}(y_j+\omega_k')^2\right)\right]. \tag{41}$$

The result recently reported by Nagao [24] will give a version of the Harish-Chandra (Itzykson-Zuber) integral formula in the present case,

$$\int dU\exp\left(-\frac{1}{4\sigma^2}\mathrm{tr}(U^\dagger\mathcal{H}U-\mathcal{H}')^2\right)\propto\frac{1}{\hat{h}_N(\omega)\hat{h}_N(\omega')}\det_{1\leqslant j,k\leqslant N}[e^{-(\omega_j-\omega_k')^2/2\sigma^2}-e^{-(\omega_j+\omega_k')^2/2\sigma^2}],$$

where the integral is taken over the unitary matrices U such that $iU\in\mathrm{Sp}(2N;\mathbf{C})$, and $\mathcal{H}$ and $\mathcal{H}'$ are Hermitian matrices in $\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{C})$ and $\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{R})$ having the non-negative eigenvalues $\omega=(\omega_1,\dots,\omega_N)$ and $\omega'=(\omega_1',\dots,\omega_N')$, respectively. Application of this identity to Eq. (41) gives

$$\hat{g}_{N,T}(0,\mathbf{0};t,\mathbf{y})\propto\hat{h}_N(\mathbf{y})^2\int dU\int d\mathcal{H}'\times\exp\left(-\frac{1}{2(\sigma')^2}\mathrm{tr}(\mathcal{H}')^2\right)\times\exp\left(-\frac{1}{2\sigma^2}\mathrm{tr}(U^\dagger YU-\mathcal{H}')^2\right),$$

with $\sigma^2=t(1-t/T)$ and $(\sigma')^2=t^2/T$, where $Y=\mathrm{diag}(y_1,\dots,y_N,-y_1,\dots,-y_N)$, $y_j\geqslant 0,1\leqslant j\leqslant N$. This can be regarded as a BdG version of the two-matrix model studied in [10] for the vicious-walker model without a wall. Since it is a convolution of two Gaussian distributions, we will arrive at the equality

$$\hat{g}_{N,T}(0,\mathbf{0};t,\mathbf{y})\propto\hat{h}_N(\mathbf{y})^2\int dU\hat{\mu}_{N,T}(t,U^\dagger YU), \tag{42}$$

where, for $\mathcal{H}\in\mathcal{M}^{\mathrm{BdG}}(2N;\mathbf{C})$,

$$\hat{\mu}_{N,T}(t,\mathcal{H})=\prod_{j=1}^{N}\left\{\frac{e^{-(a_{jj}^{\mathrm{R}})^2/2\sigma_{\mathrm{R}}^2}}{\sqrt{2\pi\sigma_{\mathrm{R}}^2}}\frac{e^{-(b_{jj}^{\mathrm{R}})^2/2\sigma_{\mathrm{R}}^2}}{\sqrt{2\pi\sigma_{\mathrm{R}}^2}}\frac{e^{-(b_{jj}^{\mathrm{I}})^2/2\sigma_{\mathrm{I}}^2}}{\sqrt{2\pi\sigma_{\mathrm{I}}^2}}\right\}\times\prod_{1\leqslant j<k\leqslant N}\left\{\frac{e^{-(a_{jk}^{\mathrm{R}})^2/\sigma_{\mathrm{R}}^2}}{\sqrt{\pi\sigma_{\mathrm{R}}^2}}\frac{e^{-(a_{jk}^{\mathrm{I}})^2/\sigma_{\mathrm{I}}^2}}{\sqrt{\pi\sigma_{\mathrm{I}}^2}}\right\}\times\prod_{1\leqslant j<k\leqslant N}\left\{\frac{e^{-(b_{jk}^{\mathrm{R}})^2/\sigma_{\mathrm{R}}^2}}{\sqrt{\pi\sigma_{\mathrm{R}}^2}}\frac{e^{-(b_{jk}^{\mathrm{I}})^2/\sigma_{\mathrm{I}}^2}}{\sqrt{\pi\sigma_{\mathrm{I}}^2}}\right\} \tag{43}$$

with

$$\sigma_{\mathrm{R}}^2=\sigma^2+(\sigma')^2=t,\quad \sigma_{\mathrm{I}}^2=\sigma^2=t\left(1-\frac{t}{T}\right).$$

Now the transition from class C to class CI is explicitly represented. The variance σ_{R}^2 increases linearly in t, but σ_{I}^2 increases in time only up to time $t=T/2$ and then decreases in time. As $t\to T$, $\sigma_{\mathrm{I}}^2\to 0$ making the imaginary parts of matrix elements zeros with probability one, and the symmetry class is changed.

V. MOMENTS OF VICIOUS WALKERS WITH A WALL

In this section, we study the moments of positions of vicious walkers with a wall in order to characterize the transition of distribution, as reported in Ref. [13] for the vicious walkers without a wall. The nth moment of the positions of walkers is defined as

$$m_{N,T}(t,n)=\left\langle \sum_{j=1}^{N} x_j^n \right\rangle_t$$
$$=\int_{0\le x_1<\dots<x_N} d\mathbf{x}\sum_{j=1}^{N} x_j^n \hat{g}_{N,T}(0,\mathbf{0};t,\mathbf{x})$$

for $n=1,2,\dots,$ where x_j denotes the position of the jth walker.

A. Wick's formula

By Eq. (39) and equality (42), if $n=2k$ is even, $k=1,2,\dots,$ we have the equality

$$m_{N,T}(t,2k)=\frac{1}{2}\langle \mathrm{tr}\mathcal{H}^{2k}\rangle, \tag{44}$$

where

$$\langle \mathrm{tr}\mathcal{H}^{2k}\rangle=\int \mathrm{tr}\mathcal{H}^{2k}\hat{\mu}_{N,T}(t,\mathcal{H})d\mathcal{H}.$$

Note that

$$\mathrm{tr}(\mathcal{H}^{2k})=\sum_{j_1,j_2,\dots,j_{2k}} \mathcal{H}_{j_1j_2}\mathcal{H}_{j_2j_3}\dots\mathcal{H}_{j_{2k-1}j_{2k}}\mathcal{H}_{j_{2k}j_1},$$

where the sum is taken over all N^{2k} combinations of indices $j_1,j_2,\dots,j_{2k}$.

Since Eq. (43) is a product of independent Gaussian integration kernels, we can apply the Wick formula with the variances

$$\langle (a_{j\ell}^{\mathrm{R}})^2\rangle=\frac{\sigma_{\mathrm{R}}^2}{2}(1+\delta_{j\ell}),\quad \langle (a_{j\ell}^{\mathrm{I}})^2\rangle=\frac{\sigma_{\mathrm{I}}^2}{2}(1-\delta_{j\ell}),$$

$$\langle (b_{j\ell}^{\mathrm{R}})^2\rangle=\frac{\sigma_{\mathrm{R}}^2}{2}(1+\delta_{j\ell}),\quad \langle (b_{j\ell}^{\mathrm{I}})^2\rangle=\frac{\sigma_{\mathrm{I}}^2}{2}(1+\delta_{j\ell})$$

for $1\le j\le \ell\le N$, where $\delta_{j\ell}$ is Kronecker's delta. These relations are rewritten as

$$\langle a_{j\ell}a_{mn}\rangle=\langle a_{j\ell}(a^T)_{nm}\rangle=\frac{c^2}{2}(\delta_{jn}\delta_{\ell m}+\gamma\delta_{jm}\delta_{\ell n}),$$

$$\langle b_{j\ell}b_{mn}\rangle=\frac{c^2}{2}\gamma(\delta_{jn}\delta_{\ell m}+\delta_{jm}\delta_{\ell n}),$$

$$\langle b_{j\ell}(b^\dagger)_{mn}\rangle=\frac{c^2}{2}(\delta_{jn}\delta_{\ell m}+\delta_{jm}\delta_{\ell n})$$

for $1\le j,\ell,m,n\le N,$ where

$$c^2=\frac{t(2T-t)}{T},\quad \gamma=\frac{t}{2T-t}.$$

Define

$$\delta_N(j,\ell;m,n)=\delta_{jn}\delta_{\ell m}-\delta_{j+N\,m}\delta_{\ell+N\,n}-\delta_{j-N\,m}\delta_{\ell-N\,n}+\delta_{j+N\,m}\delta_{\ell-N\,n}+\delta_{j-N\,m}\delta_{\ell+N\,n}. \tag{45}$$

Then we have the variance of the BdG-type Hamiltonian in our time-dependent ensemble as

$$\langle \mathcal{H}_{j\ell}\mathcal{H}_{mn}\rangle=\frac{c^2}{2}\{\delta_N(j,\ell;m,n)+\gamma\delta_N(j,\ell;n,m)\} \tag{46}$$

for $1\le j,\ell,m,n\le 2N$. The Wick formula for Eq. (44) is thus

$$m_{N,T}(t,2k)=\frac{1}{2}\sum_{j_1,j_2,\dots,j_{2k}}\sum_{\pi\in S_{2k}:\mathrm{R}} \times\langle \mathcal{H}_{j_{\pi(1)}j_{\pi(1)+1}}\mathcal{H}_{j_{\pi(2)}j_{\pi(2)+1}}\rangle \times\langle \mathcal{H}_{j_{\pi(3)}j_{\pi(3)+1}}\mathcal{H}_{j_{\pi(4)}j_{\pi(4)+1}}\rangle\dots \times\langle \mathcal{H}_{j_{\pi(2k-1)}j_{\pi(2k-1)+1}}\mathcal{H}_{j_{\pi(2k)}j_{\pi(2k)+1}}\rangle, \tag{47}$$

with the identification $j_{2k+1}=j_1$, where the first sum is taken over all N^{2k} combinations of indices $j_1,j_2,\dots,j_{2k}$, and the second one over the set of permutations S_{2k} of $\{1,2,\dots,2k\}$ with the restriction

$$\mathrm{R}: \pi(1)<\pi(3)<\dots<\pi(2k-1), \pi(2j-1)<\pi(2j), \quad 1\le j\le k.$$

The total number of the terms in the second summation is $(2k-1)!!$.

B. Möbius graph expansion

There are ten terms in variance (46) with Eq. (45). We will represent each of them by a pair of lines as shown in Fig. 3 by expressing Kronecker's delta δ_{jn} by a line without an arrow connecting j and n, $\delta_{j+N\,m}$ by a line with an arrow in the direction from j to m, and $\delta_{j-N\,m}$ by a line with an arrow in the direction from m to j. The weights, which should be multiplied to the factor $c^2/2$, are also listed. We regard these pairs of lines as the hems of ribbons. As shown in Fig. 3, in ten kinds of ribbons, half of them are twisted and others are untwisted. Two kinds of ribbons do not have any arrows on hems and other eight kinds of ribbons have arrows on hems. We will call the ribbons having arrows in the same direction *current ribbons* (c ribbons for short), the ribbons having arrows in the opposite directions *exchange ribbons* (e ribbons), and those not having any arrows *normal ribbons* (n ribbons).

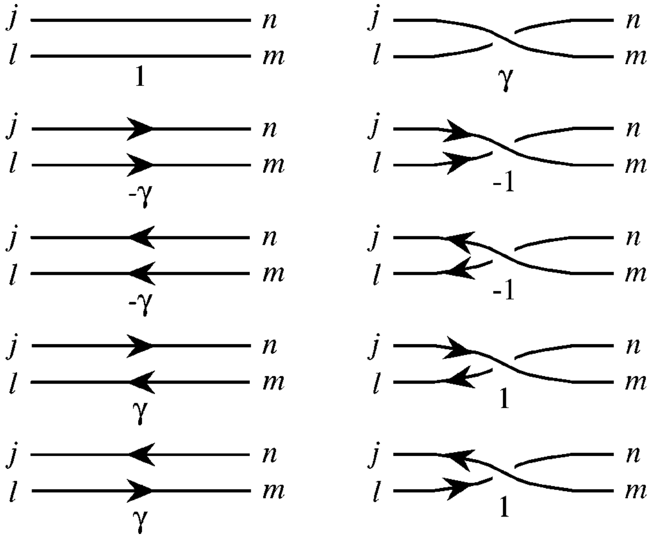

FIG. 3. Ten kinds of ribbons with and without arrows. Weights are also listed.

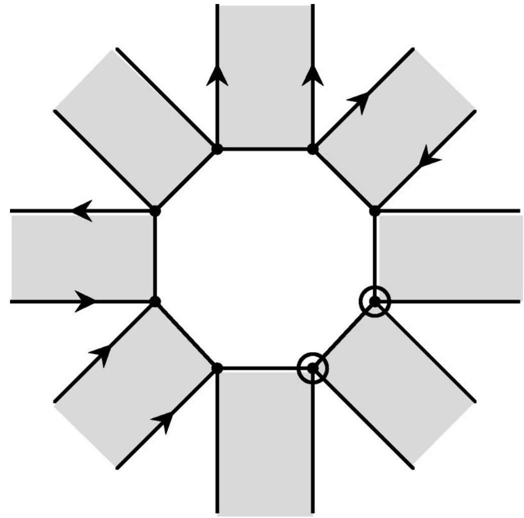

FIG. 4. An example of allowed ways of putting arrows on ribbons. The 0 vertices are marked by circles.

Inserting Eq. (46) into Eq. (47) gives a sum of the $K=(2k-1)!!\times 10^k$ terms in the form $m_{N,T}(t,2k)=(1/2)(c^2/2)^k\Sigma_{\ell=1}^{K}L_\ell$ with $L_\ell=\Sigma_{j_1,\ldots,j_{2k}}L_\ell(j_1,\ldots,j_{2k})$. As explained in Ref. [13], each term $L_\ell(j_1,\ldots,j_{2k})$ is represented by a graph which consists of a $2k$-gon with its edges $\overline{j_1j_2},\overline{j_2j_3},\ldots,\overline{j_{2k}j_1}$ connected by k ribbons to make k pairs (Wick pairs). Such graphs may be called *Möbius graphs having ribbons with and without arrows*. If we assume that there are k_n n ribbons, k_c c ribbons and k_e e ribbons, and among them φ_n n ribbons, φ_c c ribbons, and φ_e e ribbons are twisted, the weight of $L_\ell(j_1,\ldots,j_{2k})$ is $\gamma^{\varphi_n}(-\gamma)^{k_c-\varphi_c}(-1)^{\varphi_c}\gamma^{k_e-\varphi_e}=(-1)^{k_c}\gamma^{k-k_n-\varphi+2\varphi_n}$, where $k=k_n+k_c+k_e$ and $\varphi\equiv\varphi_n+\varphi_c+\varphi_e$ (the total number of twisted ribbons). For each vertex $j_s, 1\leqslant s\leqslant 2k$, we take the summation of the index over $1\leqslant j_s\leqslant 2N$ to calculate L_ℓ from $\{L_\ell(j_1,\ldots,j_{2k})\}$. Since each ribbon represents a product of two Kronecker's δ's in Eq. (45), any pair of indices j_s and j_s' connected by a line (a hem of ribbon) should be identified, or identified in modulus $\pm N$, and the free indices remaining after this "identification" of indices give N dependence to L_ℓ. We will find the following rules for the N dependence, where it should be noted that each vertex is the end point of two lines (two hems of two ribbons) with or without arrows. (i) If both of the lines connected to a vertex have no arrows, then we will call such a vertex a *0 vertex*. The summation over a free index on a 0 vertex gives $2N$. (ii) If at least one of the two lines connected to a vertex has an arrow and they are not a pair of lines with inward and outward arrows, then the summation over a free index on such a vertex gives N. (iii) Otherwise, the sum over free index becomes zero. Consider the equivalence classes of the Möbius graphs with n, c, and e ribbons. If a class is expressed by a representative graph, say Γ, the number of elements of the class Γ (i.e., the number of graphs topologically equivalent with Γ) is denoted by $|\Gamma|$. Let $V(\Gamma)$ be the total number of free indices of Γ and $V_0(\Gamma)$ be the number of free indices on 0 vertices. Set $\hat{\mathcal{G}}(k)$ as the collection of all such graphs $\{\Gamma\}$, then we have the combinatorial expression for the moments as

$$m_{N,T}(t,2k)=\frac{1}{2}\left(\frac{c^2}{2}\right)^k\sum_{\Gamma\in\hat{\mathcal{G}}(k)}|\Gamma|2^{V_0(\Gamma)}N^{V(\Gamma)}\times(-1)^{k_c(\Gamma)}\gamma^{k-k_n(\Gamma)-\varphi(\Gamma)+2\varphi_n(\Gamma)}.$$

In Ref. [13] the collection $\mathcal{G}$ of topologically distinct Möbius graphs, which consist of $2k$-gons and k normal ribbons, was introduced and the fact was used that each graph $\Gamma\in\mathcal{G}$ having only untwisted ribbons (having some twisted ribbons) defines an orientable (nonorientable) surface S_Γ by a *map* [34,35] to derive the $1/N^2$ ($1/N$) expansion. The number of distinct orientable (nonorientable) surfaces with genus g obtained from the graphs without any twisted ribbons (with m twisted ribbons) in $\mathcal{G}$ was denoted by $\varepsilon_g(k)$ [36] ($\tilde{\varepsilon}_{g,m}(k)$ [13]). We notice that the Möbius graphs in $\hat{\mathcal{G}}(k)$ introduced here are obtained by putting arrows on hems of some of the normal ribbons in $\Gamma\in\mathcal{G}(k)$. The only allowed ways to put them are such that there are no pairs of lines with inward and outward arrows connected to a vertex (see Fig. 4). We now introduce the following multiplicative factors to $\varepsilon_g(k)$ and $\tilde{\varepsilon}_{g,m}(k)$, $A^{g}_{v_0,\kappa_n,\kappa_c}(k)\equiv$ the number of allowed ways to put arrows on the hems of ribbons in a Möbius graph without twisted ribbons, which is mapped to a surface with genus g, so that the graph has κ_n n ribbons, κ_c c ribbons, and v_0 free indices on 0 vertices, and $\tilde{A}^{g,m}_{v_0,\kappa_n,\kappa_c,m_n}(k)\equiv$ the number of allowed ways to put arrows on the hems of ribbons in a Möbius graph with m twisted ribbons, which is mapped to a surface with genus g, so that the graph becomes to have κ_n n ribbons in which m_n are twisted, κ_c c ribbons, and v_0 free indices on 0 vertices. Here note that the total number of ribbons is fixed to be k, the number of e ribbons should be $k-\kappa_n-\kappa_c$, and that the total number of free vertices $V(\Gamma)$ is determined by k and g through the relations with the Euler characteristics $\chi\equiv V-k+1$ as $\chi=2-2g$ for $m=0$ and $\chi=2-g$ for $m\geqslant 1$, respectively. Then we have the following

$1/N$ expansion formula for the moments:

$$m_{N,T}(t,2k)=\frac{1}{2}\left(\frac{c^2}{2}\right)^k N^{k+1}\sum_{g=0}^{[k/2]}\varepsilon_g(k)\left(\frac{1}{N^2}\right)^g\sum_{v_0=0}^{k+1-2g}\sum_{\kappa_n=0}^{k}\times\sum_{\kappa_c}(-1)^{\kappa_c}2^{v_0}A^g_{v_0,\kappa_n,\kappa_c}(k)\gamma^{k-\kappa_n}+\frac{1}{2}\left(\frac{c^2}{2}\right)^k N^{k+1}\sum_{g=1}^{k}\left(\frac{1}{N}\right)^g\sum_{m=1}^{k}\tilde{\varepsilon}_{g,m}(k)\times\sum_{v_0=0}^{k+1-g}2^{v_0}\sum_{\kappa_n=0}^{k}\sum_{\kappa_c}\sum_{m_n=0}^{m}\times(-1)^{\kappa_c}\tilde{A}^{g,m}_{v_0,\kappa_n,\kappa_c,m_n}(k)\gamma^{k-\kappa_n-m+2m_n}. \quad (48)$$

Here we may prove that $\tilde{A}^{g,m}_{v_0,\kappa_n,\kappa_c,m_n}(k)=0$ for $k-\kappa_n-m+2m_n<0$, and Eq. (48) will give a series with non-negative powers of γ.

As an example, we consider the fourth moment. For $k=2$, we found [13]

$$\varepsilon_0(2)=2,\quad \varepsilon_1(2)=1,\quad \tilde{\varepsilon}_{1,1}(2)=4,\quad \tilde{\varepsilon}_{1,2}(2)=1,\quad \tilde{\varepsilon}_{2,1}(2)=2,\quad \tilde{\varepsilon}_{2,2}(2)=2.$$

As we can confirm easily that the factors $A^g_{v_0,\kappa_n,\kappa_c}(2)$ and $\tilde{A}^{g,m}_{v_0,\kappa_n,\kappa_c,m_n}(2)$ have nonzero values only in the following cases:

$$A^0_{3,2,0}(2)=1,\quad A^1_{1,2,0}(2)=1,$$

$$A^1_{0,1,1}(2)=4,\quad A^1_{0,0,0}(2)=2,$$

$$\tilde{A}^{1,1}_{1,1,0,0}(2)=2,\quad \tilde{A}^{1,1}_{2,2,0,1}(2)=1,\quad \tilde{A}^{1,2}_{0,0,1,0}(2)=4,$$

$$\tilde{A}^{1,2}_{2,2,0,2}(2)=1,\quad \tilde{A}^{2,1}_{0,1,1,0}(2)=2,\quad \tilde{A}^{2,1}_{1,2,0,1}(2)=1,$$

$$\tilde{A}^{2,1}_{0,0,1,0}(2)=2,\quad \tilde{A}^{2,1}_{0,1,0,1}(2)=2,\quad \tilde{A}^{2,2}_{0,0,0,0}(2)=2,$$

$$\tilde{A}^{2,2}_{0,1,0,1}(2)=4,\quad \tilde{A}^{2,2}_{1,2,0,2}(2)=1.$$

Then Eq. (48) gives

$$m_{N,T}(t,4)=\frac{c^4}{4}\{8N^3+(6+8\gamma+2\gamma^2)N^2+(1+2\gamma+5\gamma^2)N\}. \quad (49)$$

By definition we will see $A^0_{v_0,\kappa_n,\kappa_c}(k)=\delta_{v_0\,k+1}\delta_{\kappa_n k}\delta_{\kappa_c 0}$, and thus the leading term in Eq. (48) for large N is

$$m_{N,T}(t,2k)=\frac{1}{2}\left(\frac{c^2}{2}\right)^k(2N)^{k+1}\varepsilon_0(k)+\mathcal{O}(N^k).$$

Since $\varepsilon_0(k)$ is the Catalan number, Wigner's semicircle law will hold in $N\to\infty$ also in the BdG random matrices.

C. Calculation by density function

From Eq. (12) with Eqs. (7) and (11), it is easy to see that $\hat{g}_{N,T}(0,\mathbf{0};t,\mathbf{x})$ is symmetric in $x_1,\ldots,x_N$. Then we can define the density function as

$$\hat{\rho}(t,\mathbf{x})=\frac{1}{(N-1)!}\int_0^\infty\prod_{j=2}^{N}dx_j\hat{g}_{N,T}(0,\mathbf{0};t,\mathbf{x})$$

and the even moments (44) are calculated by it as

$$m_{N,T}(t,2k)=\int_0^\infty x^{2k}\hat{\rho}(t,x)dx. \quad (50)$$

Let $L_j^{(a)}(z)$ be the Laguerre polynomials with parameter a defined as

$$L_j^{(a)}(z)=\frac{e^z z^{-a}}{j!}\frac{d^j}{dz^j}(e^{-z}z^{j+a}).$$

Using them with $a=1/2$, we can define the monic polynomials $C_j(z)=(-1)^j j!L_j^{(1/2)}(z)$, which satisfy the orthogonality $\int_0^\infty z^{1/2}e^{-z}C_j(z)C_\ell(z)dz=h_j\delta_{j\ell}$ with $h_j=\Gamma(j+3/2)j!$. Quite recently Nagao gave the general expressions of dynamical correlations for the present system using $C_j(z)$ [24]. From his result, we can read the density function $\hat{\rho}(t,x)$ as

$$\hat{\rho}(t,x)=\frac{2x^2}{c^3}e^{-(x/c)^2}\sum_{j=0}^{N-1}\frac{C_j((x/c)^2)^2}{h_j}+\frac{2x^2}{c^3}e^{-(x/c)^2}\sum_{j=N}^{\infty}\sum_{\ell=0}^{N-1}\sum_{m=0}^{\ell}\beta_{j\ell}\alpha_{\ell m}\times\frac{C_j((x/c)^2)C_m((x/c)^2)}{h_j}\gamma^{j-m},$$

where

$$\alpha_{2j\ell}=(-1)^\ell\frac{(2j)!}{\sqrt{\pi}}\frac{\Gamma(2j-\ell+1/2)}{(2j-\ell)!\ell!},$$

$$\alpha_{2j+1\,\ell}=(-1)^\ell\frac{(2j+1)!}{\sqrt{\pi}}\left(2j-\ell+\frac{1}{4}\right)\frac{\Gamma(2j-\ell-1/2)}{(2j-\ell+1)!\ell!},$$

$$\beta_{j\,2\ell}=\frac{(-1)^{j+1}}{2\sqrt{\pi}}\frac{\Gamma(j-2\ell-1/2)}{(j-2\ell)!}\frac{j!}{(2\ell)!},$$

$$\beta_{j\,2\ell+1}=\frac{(-1)^j}{2\sqrt{\pi}}\frac{j!}{(2\ell+1)!}\sum_{m=0}^{[(j-2\ell-1)/2]}\frac{\Gamma(j-2\ell-2m-3/2)}{(j-2\ell-2m-1)!}.$$

Let $H_j(z)$ be the jth Hermite polynomial, $H_j(z)=(-1)^j e^{z^2}(d/dz)^j e^{-z^2}$. If we use the following equalities [37,38]:

$$L_j^{(a-1)}(z)=L_j^{(a)}(z)-L_{j-1}^{(a)}(z),$$

$$zL_j^{(a+1)}(z)$$
$$=(j+a+1)L_j^{(a)}(z)-(j+1)L_{j+1}^{(a)}(z),$$

$$\sum_{m=0}^{j}\frac{m!}{\Gamma(m+a+1)}(L_m^{(a)}(z))^2$$
$$=\frac{(j+1)!}{\Gamma(j+a+1)}\{L_j^{(a)}(z)L_j^{(a+1)}(z)-L_{j+1}^{(a)}(z)L_{j-1}^{(a+1)}(z)\},$$

$$L_j^{(-1/2)}(z)=\frac{(-1)^j}{2^{2j}j!}H_{2j}(\sqrt{z}),$$

$$\sqrt{z}L_j^{(1/2)}(z)=\frac{(-1)^j}{2^{2j+1}j!}H_{2j+1}(\sqrt{z}) \quad \text{for } z\geq 0,$$

we will have the following expression for the density function,

$$\hat{\rho}(t,x)=\frac{N}{2^{4N-3}(N-1)!\Gamma(N+1/2)}\frac{1}{c}e^{-(x/c)^2}\Bigg\{[H_{2N-1}(x/c)]^2-\frac{(2N-1)(N-1)}{N}\frac{c}{x}H_{2N}(x/c)H_{2N-3}(x/c)$$
$$-\frac{N-1}{2N}\frac{c}{x}H_{2N}(x/c)H_{2N-1}(x/c)\Bigg\}$$
$$+\frac{1}{c}\sum_{j=N}^{\infty}\sum_{\ell=0}^{N-1}\sum_{m=0}^{\ell}\beta_{j\ell}\alpha_{\ell m}\gamma^{j-m}\frac{1}{2^{2m}(2j+1)!\sqrt{\pi}}e^{-(x/c)^2}H_{2j+1}(x/c)H_{2m+1}(x/c). \tag{51}$$

Substituting Eq. (51) into Eq. (50) and using the integral formulas of Hermite polynomials used in Ref. [13] and the relation $\Sigma_{\ell=m}^{n}\beta_{n\ell}\alpha_{\ell m}=\delta_{nm}$ for $n\geq m$ [24], we will arrive at the expression

$$m_{N,T}(t,2k)=\left(\frac{c^2}{2}\right)^k N(2k-1)!!\sum_{j=0}^{k}\binom{2N-1}{k-j}2^{k-j}\left[\binom{k}{j}-\binom{k-1}{j}\left\{\frac{k-j-1}{2(k-j+1)}+\frac{N-1}{2N-k+j}\right\}\right]$$
$$-\left(\frac{c^2}{2}\right)^k\sum_{j=0}^{k-1}\sum_{\ell=0}^{N-1}\sum_{m=\ell}^{k-j-1}\gamma^{m+1}\sum_{n=N}^{m-\ell+N}\beta_{m-\ell+N\,n}\alpha_{n\,N-\ell-1}$$
$$\times\frac{(2k)!(2N-2\ell-1)!2^{m-j+1}}{j!(j+2N-2\ell+m-k)!(k-j-m-1)!(k-j+m+1)!}. \tag{52}$$

As a matter of course, when we set $k=2$, Eq. (52) gives the fourth moment (49). The large-N behavior discussed below Eq. (49) is obtained also from Eq. (52).

VI. CONCLUDING REMARKS

In the present paper we have considered the solvability of one-dimensional vicious-walker models, which are generalizations of the model studied in earlier papers [10,12]. We are interested in these systems as the nonequilibrium statistical models, since they provide in general spatially and temporally inhomogeneous systems. The temporally inhomogeneous one-particle systems have been extensively studied by Yor by constructing them from Brownian motions and Bessel processes [17]. Our present work may be regarded as an attempt to construct many-particle systems in one dimension using such temporally inhomogeneous processes as elementary processes, so that we can also discuss the spatial inhomogeneity.

We have imposed the noncolliding condition between particles, since we have considered them as *vicious walkers*. This condition introduces the effective long-ranged repulsive interactions among particles and it may make the models nontrivial many-particle systems. Our strategy to analyze the models is to map these interacting particle systems to multimatrix models defined in the appropriate matrix spaces, which have in general higher dimensions than the original phase spaces of particle systems. The particle positions are expressed by the statistics of eigenvalues [28] or the radial coordinates [18–21,30] of random matrices.

We have proved the following equivalences in probability distributions.

(i) The noncolliding system of Brownian motions with a wall, defined as the continuum limit of the vicious-walker model with a wall, and the noncolliding system of Brownian meanders, which are constructed from the three-dimensional Bessel process.

(ii) The noncolliding N generalized meanders constructed from the d-dimensional Bessel processes with $d=2,4,6,\ldots$ and the radial coordinate of the $\{N+(d-2)/2\}\times N$ rectangular random matrices in the Gaussian ensemble with time-dependent variances (a two-matrix model of the chiral GUE and chiral GOE).

(iii) The noncolliding N Brownian meanders and the non-negative eigenvalues of $2N\times 2N$ BdG-type random Hamiltonians in the Gaussian ensemble with time-dependent variances (a two-matrix model of the BdG-type random matrices in classes C and CI of Altland and Zirnbauer).

We have discussed the noncolliding system of generalized meanders made from the general odd-dimensional Bessel processes. So a natural question is what is the corresponding random matrix theory for the noncolliding system of generalized meanders associated with the d-dimensional Bessel processes with $d=5,7,9,\ldots$. As far as we know, it is an open problem.

We have claimed that the above equivalence between the interacting particle systems and the Gaussian multimatrix models implies the solvability of the systems. This statement may be true, but to obtain exact expressions of general correlation functions for the multimatrix models [39,40] is far from trivial and one has to use a series of techniques developed in the random matrix theory. Exact expressions of general dynamical correlation functions enable us to discuss the infinite particle limit $N\to\infty$ of the nonequilibrium system as reported in Refs. [12,41] for the vicious-walker model (noncolliding Brownian motions) without a wall, and in Ref. [24] for the system with a wall. We expect that the infinite particle limits of dynamical correlations for the noncolliding generalized meanders can be generally evaluated by following the strategy employed in Refs. [24,42].

As the simplest case of the correlation functions, the density function can be determined. As reported in Ref. [13], calculation of the moments of the positions of vicious walkers is related with an enumeration problem of orientable and nonorientable surfaces with a fixed number of genus, which are obtained from the Möbius graphs with a fixed number of twisted ribbons by *map* [34,35]. In the present paper, we showed that a graphical problem arises from the vicious-walker model with wall restriction; an enumeration problem of the ways of assigning arrows on the ribbons of Möbius graphs following some rules. It should be noted that, roughly speaking, there were two kinds of ribbons in our expansion formula: with and without arrows. These ribbons are thinned into lines and they are drawn on surfaces. Then this enumeration problem may provide, if we consider the lines as ''world lines'' of particles, the statistical mechanics of composite particles on random surfaces.

Recently, a variety of problems associated with the conditional random walks/Brownian motions have been proposed and intensively studied in statistical physics, e.g., first passage problem [43], ''lion-lamb'' problem [44,45], diffusion particle systems with mobile traps [46], families of vicious walkers [7], leader and laggard problem [47], system of stochastic Loewner evolutions [48], friendly walker models [49–53], and so on. Solvability and unsolvability of these models will be important topics in statistical physics far from equilibrium.

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